Use of TFNs and TpFNs for evaluating the effectiveness of CBR systems

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***Abstract*—** The fuzzy numbers play an important role in fuzzy mathematics analogous to the role played by the ordinary numbers in crisp mathematics. In the present paper we utilize a combination of the Centre of Gravity defuzzification technique and of triangular (TFNs) / trapezoidal (TpFNs) fuzzy numbers as tools for evaluating the effectiveness of Case-Based Reasoning (CBR) systems. The CBR approach for problem solving and learning (usually with the help of computers) has got a lot of attention over the last few years, because as an intelligent-systems method enables information managers to increase efficiency and reduce cost by substantially automating processes like diagnosis, scheduling and design.

***Keywords*—** Analogical Reasoning (AR), Case-Based Reasoning (CBR), Triangular (TFNs) and Trapezoidal (TpFNs) Fuzzy Numbers (FNs), Center of Gravity (COG) Defuzzification Technique, Grade Point Average (GPA) Index.

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I. Introduction

One of the most popular problem solving strategies is the strategy of the *analogous problem*: When the solver is not sure of the appropriate procedure to solve a given problem, a good hint would be to look for a similar problem solved in the past, and then try to adapt the solution procedure of this problem for use with the new problem.

The way of thinking by analogy, usually referred as *Analogical Reasoning* (AR) is a special case of the general class of the *transfer of knowledge*, i.e. of the use of already existing knowledge to produce new knowledge [1]. The importance of AR in human thinking has been recognized years ago. In fact, there is a considerable number of studies developed and many experiments performed on individuals by mathematicians, psychologists and other scientists about the AR process ([2], Section 2).

However, it is the *Case-Based Reasoning* (CBR) approach for PS and learning (usually with the help of computers) that has got a lot of attention over the last few years, because as an intelligent-systems method enables information managers to increase efficiency and reduce cost by substantially automating processes such as diagnosis, scheduling and design ([3], Section 3 of [2], etc).

Note that the term AR is sometimes used as a synonymous of the typical CBR approach [4]. However, is often used also to characterize methods, that solve new problems based on past cases of *different domains* [5, 6], while typical CBR methods focus on single-domain cases (a form of intra-domain analogy).

As a general PS methodology intended to cover a wide range of real-world applications, CBR must face the challenge to deal with uncertain, incomplete and vague information. Correspondingly, recent years have witnessed an increased interest in formalizing parts of the CBR methodology within frameworks of reasoning under uncertainty, and in building hybrid approaches by combining CBR with methods of uncertain and approximate reasoning. In an earlier work [7] we have developed a mathematical framework for the CBR process by introducing a finite Markov Chain on it main steps, while in [8] we have represented those steps as fuzzy subsets of a set of linguistic labels characterizing the degree of success of the CBR process and we have utilized the corresponding CBR system's *total possibilistic uncertainty* for measuring its effectiveness. Also, in [9] we have applied the *Trapezoidal Fuzzy Assessment Model* (TpFAM), which is a recently developed variation of the *Center of Gravity* (COG) *defuzzification technique* [10], for assessing a CBR system's performance.

In the present paper *Fuzzy Numbers* (FNs) are utilized as an alternative assessment tool for a CBR system's effectiveness. The rest of the paper is formulated as follows: In Section II we give a brief account of the CBR process. In Section III we present the basic concepts of FNs, which are necessary for the development of our new fuzzy assessment method, which is presented in Section IV. Further, in this section the outcomes of our examples are compared with the corresponding outcomes of two traditional assessment methods of the bi-valued logic, the calculation of the *mean values* and of the *Grade Point Average (GPA)* index. Finally, Section V is devoted to our conclusion and to some hints for future research.

II. Case-Based Reasoning

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CBR is often used when experts find it hard to articulate their thought processes when solving problems. This is because knowledge acquisition for a classical knowledge-based system would be extremely difficult in such domains, and is likely to produce incomplete or inaccurate results. When using CBR the need for knowledge acquisition can be limited to establishing how to characterize *cases*, i.e. the analogous problems. A *case-library* can be a powerful corporate resource allowing everyone in an organization to tap in the corporate library when handling a new problem. A CBR *system,* usually designed and functioning with the help of computers, allows the case-library to be developed incrementally, while its maintenance is relatively easy and can be carried out by domain experts.

There are two styles of CBR, the *Problem Solving (PS) style* and the *interpretive style*. The PS style can support a variety of tasks including planning, diagnosis and design in Medicine [11], Industry [12], Robotics [13], etc. The interpretive style is useful for classification, evaluation or justification of a solution, argumentation and for the projection of effects of a decision. For example, lawyers and managers making strategic decisions use the interpretive style [14, 15].

CBR has been formalized for purposes of computer and human reasoning as a four steps process, often referred as the "*four R's*". These steps involve:

- **R_1:** *Retrieve* the most similar to the new problem past case.
- **R_2:** *Reuse* the information and knowledge of the retrieved case for the solution of the new problem.
- **R_3:** *Revise* the proposed solution.
- **R_4:** *Retain* the part of this experience likely to be useful for future problem solving.

The first three of the above steps are not linear, characterized by a backward - forward flow among them. A simplified flow - chart of the CBR process, adequate for the purposes of the present paper, is presented in Figure 1:

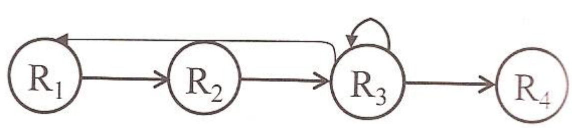

Fig. 1: A simplified flow-chart of the CBR process

More details about the CBR methodology, history and applications can be found in [2, 3] and in the relevant references given in the above two papers. A detailed functional diagram illustrating the four steps of the CBR process is also available [2, Figure 1].

III. FUZZY NUMBERS

Basic Definitions

It is recalled that a *fuzzy set (FS),* initiated by Zadeh [16] in 1965, is defined as follows:

Definition 1: Let U denote the universal set of the discourse. Then a FS A on U (or otherwise a fuzzy subset of U), , is defined in terms of the *membership function* m_A that assigns to each element of U a real value from the interval [0,1]. More explicitly, A is a set of ordered pairs of the form $A = \{(x, m_A(x)): x \in U\}$.

The definition of a FN is given as follows:

Definition 2: A FN is a FS A on the set $\boldsymbol{R}$ of real numbers with membership function m_A: $\boldsymbol{R} \rightarrow [0, 1]$, such that:

- A is *normal*, i.e. there exists x in $\boldsymbol{R}$ such that $m_A(x) = 1$,
- A is *convex*, i.e. all its a-cuts $A^a = \{x \in U: m_A(x) \geq a\}$, a in [0, 1], are closed real intervals, and
- Its membership function $y = m_A(x)$ is a *piecewise continuous* function.

For general facts on FNs we refer to the book of Kaufmann and Gupta [17].

Triangular Fuzzy Numbers (TFNs)

TFNs is the simplest form of FNs. Roughly speaking a TFN (a, b, c), with a, b, c in $\boldsymbol{R}$, states that "the value of b lies in the interval [a, c]". The membership function of (a, b, c) is zero outside the interval [a, c], while its graph in [a, c] consists of two straight line segments forming a triangle with the OX axis (Figure 2).

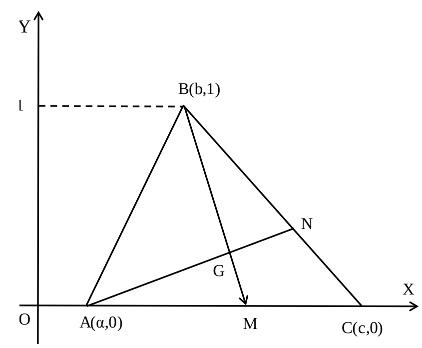

Fig. 2: Graph and COG of the TFN (a, b, c)

Therefore the analytic definition of a TFN is given as follows:

Definition 3: Let a, b and c be real numbers with $a < b < c$. Then the *TFN* (a, b, c) is a FN with membership function:

$$y = m(x) = \begin{cases} \dfrac{x-a}{b-a}, & x \in [a,b] \\ \dfrac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

The following Proposition uses the *Center of Gravity (COG)* technique for the *defuzzification* of a given TFN:

Proposition 4: The coordinates (X, Y) of the COG of the graph of the TFN *(a, b, c)* are calculated by the formulas

$X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$.

Proof: The graph of the TFN (*a*, *b*, *c)* is the triangle ABC of Figure 2, with A (*a*, 0), B(*b*, 1) and C (*c*, 0). Then, the COG, say G, of ABC is the intersection point of its medians AN and BM. The proof of the Proposition is easily obtained by calculating the equations of the straight lines of AN and BM and by solving the linear system of these two equations.

Trapezoidal Fuzzy Numbers

Another simple form of FNs that are frequently used in applications are the TpFNs. Roughly speaking, a TpFN (*a, b, c, d*) with *a, b, c, d* in ***R*** states that "a certain real value lies in the interval [*b, c*], which is a sub-interval of [*a, d*]". Its membership function *y=m(x)* is constantly 0 outside the interval [*a, d*], while its graph in this interval is the union of three straight line segments forming a trapezoid with the X-axis (Figure 3).

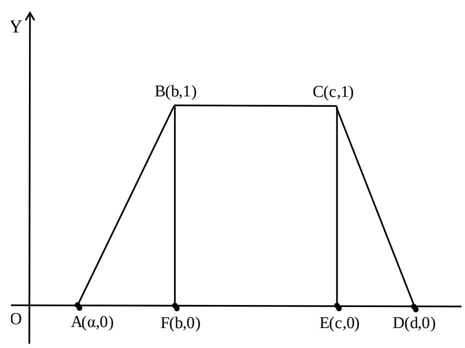

Fig. 3: Graph of the TpFN (a, b, c, d)

Therefore, the analytic definition of a TpFN is given as follows:

Definition 5: Let $a < b < c < d$ be given real numbers. Then the TpFN (*a*, *b*, *c*, *d*) is the FN with membership function:

$$y = m(x) = \begin{cases} \frac{x-a}{b-a} \quad , & x \in [a,b] \\ x = 1, \quad , & x \in [b,c] \\ \frac{d-x}{d-c}, & x \in [c,d] \\ 0, & x < a \text{ and } x > d \end{cases}$$

A TFN (*a*, *b*, *d*) can be considered as a special case of the TpFN (*a*, *b*, *c*, *d)* with *b*=*c*, i.e. the TpFNs are generalizations of the TFNs.

Te following Proposition utilizes the COG technique for defuzzifying TpFNs:

Proposition 6: The coordinates (*X*, *Y*) of the COG of the graph of the TpFN (*a, b, c, d*) are calculated by the formulas

$$X = \frac{c^2+d^2-a^2-b^2+dc-ba}{3(c+d-a-b)},\ Y = \frac{2c+d-a-2b}{3(c+d-a-b)}.$$

Proof: We divide the trapezoid forming the graph of the TpFN (*a*, *b*, *c*, *d*) in three parts, two triangles and one rectangle (Figure 3). The coordinates of the three vertices of the triangle ABE are (*a*, 0), (*b*, 1) and (*b*, 0) respectively. Therefore, by Proposition 4 the COG of this triangle is the point C_1 ($\frac{a+2b}{3}$, $\frac{1}{3}$). Similarly one finds that the COG of the triangle FCD is the point C_2 ($\frac{d+2c}{3}$, $\frac{1}{3}$). Also, it is easy to check that the COG of the rectangle BCFE, being the intersection of its diagonals, is the point C_3 ($\frac{b+c}{2}$, $\frac{1}{2}$). Further, the areas of the two triangles are equal to $S_1 = \frac{b-a}{2}$ and $S_2 = \frac{d-c}{2}$ respectively, while the area of the rectangle is equal to S_3 = c - b (in all cases the corresponding height is 1, since the TpFN (*a*, *b*, *c*, *d*) is a normal fuzzy set on ***R***).

It is well known then [18] that the coordinates of the COG of the trapezoid, being the resultant of the COGs C_i (x_i, y_i), i=1, 2, 3, are calculated by the formulas:

$$X = \frac{1}{S}\sum_{i=1}^{3} S_i x_i,\ Y = \frac{1}{S}\sum_{i=1}^{3} S_i y_i \quad (1),$$

where $S = S_1 + S_2 + S_3 = \frac{c+d-b-a}{2}$ is the area of the trapezoid.

The proof of the Proposition is completed by replacing the above found values of S, S_i, x_i and y_i, i = 1, 2, 3, in formulas (1) and by performing the corresponding calculations.

Arithmetic Operations on TFNs/TpFNs

Let A = (*a*, *b*, *c*) and B = (a_1, b_1, c_1) be two TFNs. Then one can define [17]:

- The sum A + B = ($a+a_1$, $b+b_1$, $c+c_1$).
- The difference A - B = ($a-c_1$, $b-b_1$, $c-a_1$).

Consequently, the sum and the difference of two TFNs are always TFNs. The *product* and the *quotient* of two TFNs can be also defined, but, although they are FNs, they are not always TFNs [17].

One can further define the following two *scalar operations*:

- k + A= (k+*a*, k+*b*, k+*c*), $k \in \mathbf{R}$
- kA = (k*a*, k*b*, k*c*), if k>0 and kA = (k*c*, k*b*, k*a*), if k<0.

It can be shown [17] that the same rules can be applied for the corresponding arithmetic operations between TpFNs.

Mean value of TFNs/TpFNs

The following definition is introduced to be used in Section IV for assessing the performance of CBR systems with the help of TFNs/TpFNs:

Steps	F	D	C	B	A
R_1	0	18	45	27	0
R_2	18	24	48	0	0
R_3	36	27	27	0	0

Definition 7: Let A_i , i = 1, 2,…, *n* be TFNs/TpFNs, where *n* is a non negative integer, $n \geq 2$. Then we define the *mean value* of the A_i's to be the TFN/TpFN:

$$A = \frac{1}{n}(A_1 + A_2 + \ldots + A_n).$$

V. Assessing the Performance of CBR Systems using TFNs/TpFNs

In this section two examples are presented in which the TFNs/TpFNs are used as tools for assessing the effectiveness of CBR systems. For this, ranking in a range from 0 to 100 the effectiveness of a CBR system's past cases when used with new similar problems, we consider the following *linguistic labels (*or *grades* or *degrees)* for their performance: A (85-100) = excellent, B (75-84) = very good, C (60-74) = good, D (50-59) = fair and F (0- 49) = unsatisfactory.

Note that the scores attached to the above linguistic labels are not standard, depending on the designer's personal criteria. For example, in a more strict evaluation one could consider A (90 - 100), B (80 – 89), C (70 – 79), D (60 – 69), F (0 – 59), etc.

Our new fuzzy assessment approach is validated by comparing its outcomes in our examples with the corresponding outcomes of two traditional assessment methods of the bi-valued logic, the calculation of the mean values and of the GPA index.

Fxample 1 (GPA – TFNs)

Consider two CBR systems designed for help desk applications, with their libraries containing 105 and 90 past cases respectively. The designers of both systems have supplied them with the same mechanism (software) for assessing the degree of success of their past cases when used with new similar problems. The outcomes of this mechanism are depicted in Table 1 for each of the three first steps of the CBR process:

Table 1: Degrees of success for the CBR systems

FIRST SYSTEM

Steps	F	D	C	B	A
R_1	0	0	51	24	30
R_2	18	18	48	21	0
R_3	36	30	39	0	0

SECOND SYSTEM

Here we shall use the GPA index and the TFNs as assessment methods:

(i) GPA index: We recall that the *Great Point Average* (*GPA*) index is a weighted mean in which more importance is given to the higher scores by attaching greater coefficients (weights) to them [18]. In other words, the GPA index focuses on the *quality performance* of a system.

Denote by y_i , i = 1, 2, 3, 4, 5 the frequencies of the CBR system's cases whose performance is characterized by F, D, C, B and A respectively, then the GPA index is calculated by the formula GPA = $y_2 + 2y_3 + 3y_4 + 4y_5$ (2). In case of the ideal performance (y_5 = 1) we have GPA = 4, while in the worst case (y_1 = 1) we have GPA = 0; therefore $0 \leq$ GPA ≤ 4. Consequently, values of GPA greater than the half of its maximal value (4 : 2 = 2) correspond to a more than satisfactory system's performance

In our case, the data of Table 1 give the following frequencies:

Table 2: Frequencies of success for the CBR systems

FIRST SYSTEM

Steps	Y_1	Y_2	Y_3	y_4	y_5
R_1	0	0	$\frac{51}{105}$	$\frac{24}{105}$	$\frac{30}{105}$
R_2	$\frac{18}{105}$	$\frac{18}{105}$	$\frac{48}{105}$	$\frac{21}{105}$	0
R_3	$\frac{36}{105}$	$\frac{30}{105}$	$\frac{39}{105}$	0	0

SECOND SYSTEM

Steps	Y_1	Y_2	Y_3	y_4	y_5
R_1	0	$\frac{18}{90}$	$\frac{45}{90}$	$\frac{27}{90}$	0
R_2	$\frac{18}{90}$	$\frac{24}{90}$	$\frac{48}{90}$	0	0
R_3	$\frac{36}{90}$	$\frac{27}{90}$	$\frac{27}{90}$	0	0

Replacing the values of frequencies from Table 2 in formula (2) one finds the following values for the GPA index:

First System: **R₁**: $\frac{294}{105}$ = 2.8, **R₂**: $\frac{177}{105} \approx 1.69$, **R₃**: $\frac{108}{105} \approx 1.03$**.**

Second System: **R₁**: $\frac{189}{90}$ = 2.1, **R₂**: $\frac{168}{90} \approx 1.87$, **R₃**: $\frac{81}{90} = 0.9$.

The above values of the GPA index show that the first system demonstrated a better quality performance at steps $\mathbf{R_1}$ and $\mathbf{R_3}$ (Retrieve, Revise),, while the second one demonstrated a better performance at $\mathbf{R_2}$ (Reuse). Further, the two systems' performance was proved to be more than satisfactory at $\mathbf{R_1}$ and less than satisfactory at the other two steps, being worse at $\mathbf{R_3}$. This was logically expected, since the success in each step depends on the success in the previous steps.

Note that the two systems' performance at the last step $\mathbf{R_4}$ was not examined, since all past cases, even the unsuccessful ones, are retained in a system's library for possible use in future with related new problems; the unsuccessful ones to help for exploring possible reasons of failure to find a solution for a new problem.

Finally, the mean values of the GPA indices for the two systems at the steps $\mathbf{R_1}$, $\mathbf{R_2}$ and $\mathbf{R_3}$ are approximately equal to 1.84 and 1.62 respectively, showing that the first system demonstrated a better overall performance.

(iii) Use of the TFNs: We assign to each assessment grade a TFN (denoted for simplicity by the same letter) as follows: A= (85, 92.5, 100), B = (75, 79.5, 84), C = (60, 67, 74), D= (50, 54.5, 59) and F = (0, 24.5, 49). The left and right entries of each of the above TFNs are equal to the minimal and maximal score respectively assigned to the corresponding degree, whereas the middle entry is equal to the mean value of the other two entries..

If T is one of the TFNs A, B, C, D, F then b $= \frac{a+c}{2}$.

Therefore, Proposition 4 gives that

$$X(T) = \frac{a + \frac{a+c}{2} + c}{3} = \frac{3(a+c)}{6} = b .$$

But, by Definition 7 the mean value M of a linear combination of the TFNS A, B, C, D and F with coefficients non zero integers is of the form M=k_1A+k_2B+k_3C+k_4D+k_5F, with k_i non negative rational numbers, i=1, 2, 3, 4, 5. Consequently, if A (a_1, b_1, c_1), B (a_2, b_2, c_2),…., F(a_5, b_5, c_5) and M(a, b, c), then

$$M = \sum_{i=1}^{5} k_i(a_i, b_i, c_i) = (\sum_{i=1}^{5} k_i a_i, \sum_{i=1}^{5} k_i b_i, \sum_{i=1}^{5} k_i c_i) .$$

Therefore

$$X(M) = \frac{\sum_{i=1}^{5} k_i a_i + \sum_{i=1}^{5} k_i b_i + \sum_{i=1}^{5} k_i c_i}{3} = \sum_{i=1}^{5} k_i \frac{a_i + b_i + c_i}{3}$$

$$= \sum_{i=1}^{5} k_i b_i = b \quad (3).$$

Inspecting the data of Table 1 one finds that for the step $\mathbf{R_1}$ of the first system we have 51 TFNs equal to C, 24 TFNs equal to B and 30 TFNs equal to A. Then, by Definition 7, the mean value of these TFNs, denoted for simplicity by the same letter $\mathbf{R_1}$, is equal to $\mathbf{R_1}$ = $\frac{1}{105}$(51C + 24B + 30A) = $\frac{1}{105}$[(3060, 3417, 3772) + (1800, 1908, 2016) + (2550, 2775. 3000) = $\frac{1}{105}$(7410, 8100, 8788) $\approx$(70.57, 77.14, 83.7). Therefore, equation (3) gives that $X(\mathbf{R_1}) \approx 77.14$, which shows that the first system demonstrated a very good *mean performance* at the step of retrieval.

In the same way one calculates the mean values $\mathbf{R_2}$ = $\frac{1}{105}$(18F + 18D+ 48C + 21B) $\approx$ (51, 60.07, 69.14) and $\mathbf{R_3}$ = $\frac{1}{105}$(36F + 30D + 39C) $\approx$ (36.57, 48.86, 61.14). Therefore, $X(\mathbf{R_2}) \approx 60.07$ and $X(\mathbf{R_2}) \approx 48.86$, showing that the first system demonstrated a good performance at $\mathbf{R_1}$ **and** an unsatisfactory performance at $\mathbf{R_3}$.

The overall system's performance can be assessed by the mean value **R** = $\frac{1}{3}$($\mathbf{R_1}$ + $\mathbf{R_2}$ + $\mathbf{R_3}$) $\approx$ (52.71, 62.02, 71.33), which shows that the first system demonstrated a good (C) mean performance.

A similar argument gives for the second system the mean values $\mathbf{R_1}$ = (62.5, 68.25, 74), $\mathbf{R_2}$ $\approx$(45.33, 55.17, 65) , $\mathbf{R_3}$ = (33, 46.25, 59.5) and **R** $\approx$ (46.94, 56.56, 66.17). Observing the middle entries of the above TFNs one concludes that the second system demonstrated good performance at $\mathbf{R_1}$, fair performance at $\mathbf{R_2}$, unsatisfactory performance at $\mathbf{R_3}$ and a fair overall performance. Therefore, the first system demonstrated a clearly better performance than the second one.

Example 2 (TFNs – TpFNs)

Six different users of a CBR system ranked with scores from 0-100 the effectiveness of its following five past cases for solving new related problems:

C₁ (Case 1)**:** 43, 48, 49, 49, 50, 52, **C₂:** 81, 83. 85, 88, 91, 95, **C₃:** 76, 82, 89, 95, 95, 98, **C₄:** 86, 86, 87, 87, 87, 88, **C₅:** 35, 40, 44, 52, 59, 62.

Here we shall evaluate the system's effectiveness with respect to the above five cases by calculating the mean value of the scores assigned to them and by using the TFNs and the TpFNs:

(i) Mean value: The mean value of the 30 in total scores assigned by the six users to the five cases is approximately equal to 72.07 demonstrating a good performance.

(ii) TFNs: We consider again the TFNs A, B, C, D and F used in our previous example. Observing the given scores

one finds that in the present example we have 14 TFNs equal to A, 4 equal to B, 1 equal to C, 4 equal to D and 7 TFNs equal to F characterizing the five cases' performance.

The mean value of the above TFNs is equal to

M = $\frac{1}{30}$(14A + 4B + C + 4D + 7F) $\approx$(60.33, 68.98, 79.63). Therefore, the system demonstrates a good (68.98) mean performance

(ii) TpFNs: We assign to each case $\mathbf{C_i}$, I = 1, 2, 3, 4, 5 a TpFN (denoted, for simplicity, with the same letter) as follows: $\mathbf{C_1}$ = (0, 43, 52, 59), $\mathbf{C_2}$ = (75, 81, 95, 100), $\mathbf{C_3}$ = (75, 76, 98, 100), $\mathbf{C_4}$ = (85, 86, 88, 100) and $\mathbf{C_5}$ **=** (0**,** 35, 62, 74). Each of the above TpFNs characterizes the performance of the corresponding case in the form (*a*, *b*, *c*, *d*), where *a* is the lower bound of its performance with respect to the corresponding linguistic grades, *b* and *c* are the lower and higher scores respectively assigned to the case by the six system's users and *d* is the upper bound of its performance with respect to the linguistic grades.

For assessing the overall system's performance with respect to the given five past cases we calculate the mean value of the TpFNs $\mathbf{C_i}$, i =1, 2, 3, 4, 5, which is equal to the

TpFN $\mathbf{C} = \frac{1}{5}\sum_{i=1}^{5} C_i$ = (47, 64.2, 79, 86.6).

Then, by Proposition 6 one finds that $X(\mathbf{C})$ is equal to $\frac{79^2+(86.6)^2-(64.2)^2-47^2+79*86.6-47*(64.2)}{3(79+86.6-47-64.2)} \approx 68.84$, which shows that the system demonstrated a good mean performance.

The mean value **M** of the TFNs determines, before its defuzzification, the score corresponding to the system's performance to be in the interval [60.33, 79.63], whereas the mean value **C** of the TpFNs determines it to be in the interval [64.2, 79]. Obviously, the use of the TpFNs gives more accurate results in general. However, in many cases, like in Example1, it is practically difficult to use the TpFNs as assessment tools due to the great number of the existing past cases.

V. DISCUSSION AND CONCLUSIONS

From the discussion performed in this paper the following conclusions can be drawn:

- Using TFNs or TpFNs one can assess the *mean performance* of the CBR systems. The use of the TpFNs gives in general more accurate results, but in many cases it is difficult to be applied in practice due to the great number of the existing past cases in the system's library..
- The differences appearing to the outcomes when using the traditional assessment methods of the bi-valued logic (mean value, GPA) instead of the FNs is mainly due to the different philosophy of FL (multiple values) with respect to the traditional logic.
- Another reason is that the GPA index focuses on the system's *quality performance* by assigning greater coefficients to the higher scores.
- The use of the TFNs/TpFNs as assessment tools seems to have the potential of a *general assessment method* that could be used for assessing a variety of other human or machine activities [19, 20]. This gives a good hint for more future research on the subject.

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