Managing the uncertainty for student understanding of the infinity

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Abstract— The concept of infinity plays a fundamental role for the adequate learning of many mathematical topics, like the limits of sequences and functions, the differential and integral calculus, etc. However, from the instructor's point of view, and not only, the student understanding of the infinity is characterized by a degree of uncertainty. Therefore fuzzy logic, due to its property of assigning multiple values to the ambiguous cases, could help for a more effective study of the student difficulties to deal with the infinity. In this paper the fuzzy system possibilistic/probabilistic uncertainty is utilized as an assessment tool in an experimental study on the effects that the instruction to the basic philosophical / epistemological aspects about the infinity could have for the improvement of student abilities to deal successfully with it in their mathematical courses.

Keywords— Potential/Actual Infinity, Unattainable Infinity, Fuzzy Assessment Methods, Possibilistic and Probabilistic Uncertainty

I. INTRODUCTION

The concept of infinity is involved in many mathematical topics, like set and number theory, limits of sequences and functions, differential and integral calculus, fractals etc., playing a fundamental role for their understanding. However, the majority of students face considerable difficulties in dealing with the infinity. For example, Tsamir [1] found that prospective teachers erroneously attribute properties of finite to infinite sets, Mamona-Downs [2] found that many students consider that the limit of a sequence is its last term and, given the sequence (a_n), $n \in \boldsymbol{N}$, they write a_∞ for its limit Furthermore, it is well known the student confusion caused by the cardinalities of the infinite sets, etc.

Fuzzy Logic (FL), due to its property of characterizing the uncertain and ambiguous situations with multiple values, offers rich resources for the evaluation of such kind of situations. Consequently, since from the instructor's point of view, and not only, the understanding of the infinity by students' is characterized by a degree of uncertainty, the application of *fuzzy assessment methods* (e.g. [3]-[6], etc.) could help for a more effective study of student skills to deal successfully in their mathematical courses with situations in which the infinity is involved.

In this work we utilize a fuzzy system's *possibilistic/probabilistic uncertainty* as a tool for assessing the degree of student understanding of the infinity. The rest of the paper is formulated as follows:

In Section II the basic philosophical / epistemological aspects about the infinity are introduced. In Section III a brief account for a fuzzy system's probabilistic and total possibilistic uncertainty is given. In Section IV a classroom experiment performed with first year university students is described, and the corresponding fuzzy system's uncertainty is used for the assessment of the student skills to deal successfully with the infinity. The creditability of our fuzzy uncertainty model is validated through the parallel use of a traditional assessment method of the bi-valued logic, the calculation of the *Grade Point Average (GPA)* index. Finally, Section V is devoted to our conclusions and a brief discussion on the perspectives of future research on the subject.

II. CONCEPTIONS OF THE INFINITY

Philosophers, mathematicians, mathematical historians and educators, students and many others have struggled for centuries to resolve the various issues and paradoxes regarding conceptions of the infinity. Aristotle's (384-322 BC) *potential/actual dichotomy* dominated these conceptions for centuries. According to the Aristotle's view, the potential infinity could be understood as the infinity presented over time, while the actual infinity as the infinity present at a moment in time. For Aristotle the actual infinity is incomprehensible, because the underlying process of such an actuality would require the whole of time. This distinction of the concept of infinity allowed Aristotle to acknowledge the existence of the infinity, provided that it was not present "all at once" ([7], p. 39). Further, the actual infinity explains all the paradoxes connected to the infinity.

However, views also appeared disputing the ideas of Aristotle, mainly expressed by the *rationalists*, who believed that we can invoke the pure logic for the understanding of the real world in general and the actual infinity in particular. Bolzano (1741-1848) advanced, against the empiricist Aristotle's negative assertion, the idea of the existence of an *infinite collection* as a completed

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whole. His main argument to support this view was the existence of the *large finite numbers*, like the grains of sand in a desert, a set with $10^{10^{10^{10^{10}}}}$ elements, etc, which, although they doubtlessly exist, they cannot been enumerated by human beings. A concern with Bolzano's view is that the examples he used are finite sets. For instance, in case of enumerating the set of the first $10^{10^{10^{10^{10}}}}$ natural numbers one can reflect on the last counting number as indicating its cardinality, a fact which cannot occur in an infinite set, where there is no such number.

Cantor (1845-1918) extended Bolzano's thinking. His theory of *transfinite numbers* is connected to his view that infinite sets to which a cardinality or order can be assigned "enjoy a kind of finitude" or are "really finite". Cantor thus suggests three cognitive categories, the *finite*, the *attainably infinite* and the *unattainably infinite*. The last one, termed by Moore [7] as the "really infinite", refers to immeasurably large collections to which no cardinality or order can be assigned, like the collection of everything thinkable, the set of all the sets, etc. According to Cantor, actual infinite entities are considered to be attainably infinite, while potentially infinite collections that cannot be actualized are considered to be unattainably infinite.

Nowadays, the best way for connecting the potential to the actual infinity is probably the use of *fractals* [8], which are obtained by infinite processes characterized by a kind of self – similarity. Consider, for example, the *ternary set* discovered by Henry John Stephen Smith in 1874, but better known as *the Cantor's comb or dust*. This set, through the consideration of which Cantor (1883) and others were helped for laying the foundations of the modern point-set Topology, is created by removing repeatedly the open middle thirds of a line segment [9]. The first five steps of this construction are represented in Figure 1.

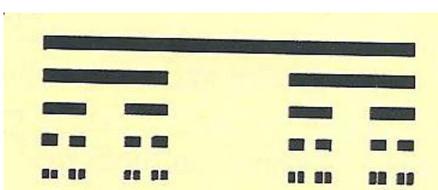

Fig. 1: Graph of the ternary set

Figure 1 does not represent the set's final image, the creation of which requires an infinite number of such steps (actual infinity); it gives however a very precise approximation of it. In fact, it is easy to observe that the left and right parts of Figure 1 are similar, containing equal lengths. Further, each of these parts is similar to the whole figure and it also contains its own left and right parts. Therefore we have 4, 8, 16, ,,,,,, and so on smaller subsets similar to the original set. As the process continues, it becomes evident that the ternary set contains an infinite number of smaller and smaller subsets, all of which are similar to the original set (self-similarity). Cantor's comb is probably the first fractal discovered in the history of mathematics.

Dubinsky et al. [10] analyzed the difficulties appearing to individuals for understanding the concept of infinity in terms of their *APOS theory* for teaching/learning mathematics, developed during the 1990's in the USA [11-14]. According to this theory, an individual deals with a mathematical situation by using the mental mechanisms of *interiorization* and *encapsulation* to build cognitive structures that applied to the situation. The related structures involve ***A**ctions*, ***P**rocesses*, ***O**bjects* and ***S**chemas* and the word APOS is an acronym formed by the initial letters of these words.

According to the APOS theory [10], one's ability to perform isolated steps of an infinite process is an action, while the interiorization of this action to a process implies the individual's ability of repeating mentally this action for an unlimited number of steps (potential infinity). Further, the actual infinity involves the understanding of an infinite process as a totality (Bolzano) and the encapsulation of this totality to a cognitive object (Cantor), i.e. the actual infinity is an attainable form of the infinite.

However, the understanding of a process as a totality and therefore its encapsulation to an object is not always possible, which means that the unattainable infinite is a form of potential infinity that cannot be understood as a totality. Conclusively the potential and actual infinity are two different cognitive conceptions of the infinite, which, in an advanced phase of the individual's cognitive progress, are embodied together in his/her corresponding cognitive schema. Obviously the existence of the one does not deny the existence of the other, neither is a wrong conception of the other.

The relationship between them can be better understood through the transformation from an infinite process (e.g. a sequence) to the final result obtained by the encapsulation of this process to an object (e.g. limit of the sequence). This result *transcends* in general the corresponding process, in the sense that it is not connected, neither is obtained by any of its steps. This is the characteristic difference between the large finite numbers and the infinite, which explains why the former can be more easily understood than the latter one.

III. TYPES OF UNCERTAINTY IN FUZZY SYSTEMS

Uncertainty is the shortage of precise knowledge and of complete information on data, which describe together the state of the corresponding system. One of the key problems of artificial intelligence is the modelling of the uncertainty for solving real life problems and several models have been proposed for this purpose.

The amount of information obtained by an action can be measured by the reduction of the uncertainty resulting from this action. Therefore, a system's uncertainty is connected to its capacity for obtaining relevant information. Accordingly a measure of uncertainty could be adopted as a measure of a system's effectiveness in solving related problems. The greater is the decrease of the uncertainty resulting from the action (i.e. the difference of the existing uncertainty before and after the action), the better the system's performance with respect to the action. Obviously, the measurement of the uncertainty focuses on the system's *mean performance*.

In classical probability theory a system's uncertainty and the information connected to it are measured by the Shannon's formula, better known as the *Shannon's entropy* [15]. This term comes from the mathematical definition of the information, say I, by I= $-\frac{\Delta(\log P)}{\log 2}$, where P is the probability of appearance of each of the equally probable cases of the evolution of the corresponding real situation. This expression appears to be analogous to the well known from Physics formula $\Delta S = \frac{\Delta Q}{T}$, where ΔS is the increase of a physical system's entropy caused by an increase of the heat by ΔQ, when the absolute temperature T remains constant.

Let U denote the universal set of the discourse. It is recalled here that a *fuzzy set (FS)* A on U (or otherwise a fuzzy subset of U), , is defined in terms of its *membership function* m_A that assigns to each element of U a real value from the interval [0,1]. More explicitly, A can be written a set of ordered pairs of the form A = {(x, m_A(x)): x ∈U}, where m_A(x) is called the *membership degree* of the element *s* of U in A. For general facts on fuzzy sets and the uncertainty connected to them we refer to the book [16] of Klir and Folger.

For use in a fuzzy environment Shannon's formula has been adapted ([17], p. 20) to the form:

$$H = -\frac{1}{\ln n}\sum_{s=1}^{n} m_s \ln m_s \qquad (1).$$

In formula (1) $m_s = m(s)$ denotes the membership degree of the element *s* of U in the corresponding fuzzy set and *n* denotes the total number of the elements of U. The sum is divided by the natural logarithm of *n* in order to be normalized. Thus H takes values within the real interval [0, 1]. Formula (1) measures a fuzzy system's *probabilistic uncertainty*.

It is recalled that the *fuzzy probability* of an element *s* of U is defined in a way analogous to the crisp probability, i.e. by $P_s = \frac{m_s}{\sum_{s \in U} m_s}$ (2).

However, according to the British economist Shackle [18] and many other researchers after him, human reasoning can be formulated more adequately by the possibility rather, than by the probability theory. The *possibility*, say r_s, of an element *s* of U is defined by $r_s = \frac{m_s}{\max\{m_s\}}$ (3), where max$\{m_s\}$ denotes the maximal value of m_s, for all *s* in U. In other words, the possibility of *s* expresses the relative membership degree of *s* with respect to max$\{m_s\}$.

Within the domain of possibility theory uncertainty consists of *strife (or discord)*, which expresses conflicts among the various sets of alternatives, and of *non-specificity (or imprecision)*, which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives ([17], p. 28). For a better understanding of the above two types of uncertainty we give the following simple example:

EXAMPLE: Let U be the set of integers from 0 to 120 representing human ages and let *Y* = young, *A* = adult and *O* = old be fuzzy subsets of U defined by the membership functions m_Y, m_A and m_O respectively. People are considered as young, adult or old according to their outer appearance. Then, given *x* in U, there usually exists a degree of uncertainty about the values that the membership degrees $m_Y(x)$, $m_A(x)$ and $m_O(x)$ could take, resulting to a conflict among the fuzzy subsets *Y*, *A* and *O* of U. For instance, if *x* = 18, values like $m_Y(x)$ = 0.8 and $m_A(x)$ = 0.3 are acceptable, but they are not the only ones. In fact, values like $m_Y(x)$ = 1 and $m_A(x)$ = 0.5 are also acceptable, etc. The existing conflict becomes even greater if *x* =50. In fact, it is not reasonable in this case to take $m_Y(x)$ =0, because sometimes people being 50 years old look much younger than others aged 40 or even 30 years. But, there exist also people aged 50 who look older from others aged 70, or even 80 years! All the above are examples of the type of uncertainty that we have termed as strife.

On the other hand, non - specificity is connected to the question: How many *x* in U should have non zero membership degrees in *Y*, *A* and *O* respectively? In other words, the existing in this case uncertainty creates a conflict among the cardinalities (sizes) of the fuzzy subsets of U. It is recalled that the *cardinality* of a fuzzy subset, say *B*, of U is defined to be the sum $\sum_{x \in U} m_B(x)$ of all membership degrees of the elements of U in *B*.-

Strife is measured ([17]; p.28) by the function *ST(r)* on the ordered possibility distribution *r:* $r_1=1 \geq r_2 \geq \ldots\ldots \geq r_n \geq r_{n+1}$ of a group of a system's entities defined by

$$ST(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}\right] \qquad (4).$$

Under the same conditions non-specificity is measured ([17]; p.28) by the function

$$N(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log i\right] \qquad (5).$$

The sum *T(r) = ST(r) + N(r)* measures the fuzzy system's *total possibilistic uncertainty*.

IV. The Classroom Experiment

One can find in the literature reflections of the development of the concept of infinity in students of today ([3, 19, 20], etc). Doubtlessly, the pioneer of this study was *E. Fischbein*, whose empirical researches revealed many conflicting intuitional student perceptions of the infinity [21-25]. His last article [25] was published just after his death, in 2001, together with six articles of other authors [1, 2, 26-29] in a special issue of the "Educational Studies of

Mathematics" on the concept of infinity, dedicated to his memory.

The impulsion to perform the following classroom experiment was given by the concern to study the effects that an instructor's lecture to students on the basic philosophical/epistemological aspects about the infinity could have for the improvement of their abilities to deal successfully in their mathematical courses with situations involving this concept. For this, we selected two equivalent - according to the marks obtained in their first term course "Higher Mathematics I"- student groups from the School of Technological Applications (prospective engineers) of the Graduate Technological Educational Institute (T. E. I.) of Western Greece (in the city of Patras) being at their second term of studies. A two hours lecture was delivered separately to the students of both groups. The lecture to the first (experimental) group G_1 was focused mainly on the basic philosophical/epistemological aspects of the infinite (see Section II), while the attention of the lecture for the second (control) group G_2 was turned to examples related to the topics of the course "Higher Mathematics I" involving, directly or indirectly, the concept of infinity. Note that this course involves an introductory chapter on the basic sets of numbers, Differential and Integral Calculus in one variable and elements of Analytic Geometry and of Linear Algebra.

Next, a written test was performed for both groups in terms of the questionnaire presented in the Appendix at the end of this paper together with some representative wrong student answers. The student answers were ranked in terms of the following *linguistic labels (grades)*: A = excellent, B = very good, C = good, D = fair and F = unsatisfactory. The student results are depicted in Table 1.

Table 1: Student results

Grade	G_1	G_2
A	1	10
B	13	6
C	4	3
D	3	0
F	0	1
Total	21	20

The performance of the two student groups was evaluated by calculating the GPA index first and second by measuring the two group total possibilistic and probabilistic uncertainty.

i) GPA index: It is recalled that the *Grade Point Average (GPA)* index is a weighted mean, in which more importance is given to the higher scores, by assigning greater coefficients (weights) to them. In other words, the GPA index measures the *quality performance* of a student group. For calculating the GPA index let us denote by n_A, n_B, n_C, n_D and n_F the numbers of students whose performance was characterized by A, B, C, D and F respectively and by n the total number of students of each group. .It is well known then [30] that the GPA index is calculated by the formula

$$\text{GPA}=\frac{0n_F+n_D+2n_C+3n_B+4n_A}{n} \quad (1).$$

Formula (1) gives that, GPA=0, if $n_F = n$ (worst case) and GPA=4, if $n_A = n$ (ideal case). Therefore $0 \leq \text{GPA} \leq 4$, which implies that values of GPA greater than the half of its maximal value, i.e. greater than 2, could be considered as being connected to a more than satisfactory group's performance.

In our case, replacing to formula (1) the data of Table 1, one finds that the GPA index is equal to $\frac{54}{21} \approx 2.57$ for G_1 and to $\frac{64}{20} = 3.2$ for G_2. Thus, the two groups demonstrated a more than satisfactory quality performance with the performance of the control group being better.

(ii) *Total possibilistic uncertainty:* Defining the membership function in terms of the frequencies of the student grades one can represent the two student groups as fuzzy sets on the set U = {A, B, C, D. F} of the linguistic grades in the form $\{(x, \frac{n_x}{n}): x \in U\}$, where n_x and is the number of students who received the grade x and n is the total number of the students in each group. In other words we can write

$G_1 = \{(A, \frac{1}{21}), (B, \frac{13}{21}), (C, \frac{4}{21}), (D, \frac{3}{21}), (F, 0)\}$ and

$G_2 = \{(A, \frac{10}{20}), (B, \frac{6}{20}), (C, \frac{3}{20}), (D, 0), (F, \frac{1}{20})\}$.

The maximal membership degree in G_1 is equal to $\frac{13}{21}$ and therefore the possibilities of the elements of U in G_1 are: $r(A) = \frac{1}{13}$, $r(B) = 1$, $r(C) = \frac{4}{13}$, $r(D) = \frac{3}{13}$, $r(F) = 0$. Therefore the ordered possibility distribution defined on G_1 is $\boldsymbol{r}: r_1 = 1 > r_2 = \frac{4}{13} > r_3 = \frac{3}{13} > r_4 = \frac{1}{13} > r_5 = 0$ (6) .

Working in the same way one finds that the ordered possibility distribution on G_2 is:

$$\boldsymbol{r}: r_1 = 1 > r_2 = \frac{6}{10} > r_3 = \frac{3}{10} > r_4 = \frac{1}{10} > r_5 = 0 \quad (7) .$$

Formula (4) gives in our case that $ST(r) = \frac{1}{\log 2}[(r_2-r_3)\log\frac{2}{r_1+r_2} + (r_3-r_4)\log\frac{3}{r_1+r_2+r_3} + (r_4-r_5)\log\frac{4}{r_1+r_2+r_3+r_4}]$,

Replacing the values of the possibility distribution $\boldsymbol{r}$ from

(6) one finds for G_1 that $ST(r) = \frac{1}{\log 2}[\frac{1}{13}\log(\frac{26}{17}) + \frac{2}{13}\log(\frac{39}{20}) + \frac{1}{13}\log(\frac{42}{21}) \approx 0.27$.

Also, formula (5) gives for G_1 that

$N(r)= \frac{1}{\log 2}[\frac{1}{13}\log 2+\frac{2}{13}\log 3+\frac{1}{13}\log 4] \approx 0.48$. Therefore, the total possibilistic uncertainty for G_1 is $T(r) \approx 0.27+ 0.48 =0.75$.

In the same way, replacing the values of ***r*** from (7) one finds that the total possibilistic uncertainty for G_2 is $T(r) \approx 0.33 + 0.82 =1.15$.

Since the two student groups were chosen to be equivalent, the existing uncertainty before the two hours' lectures was the same for both of them. Thus, the reduction of the uncertainty was greater for the experimental group G_1, which therefore demonstrates a better mean performance than the control group G_2.

iii) Probabilistic uncertainty: Replacing the membership degrees of G_1 to formula (1) one finds that the probabilistic uncertainty for the experimental group is equal to

$$H=-\frac{1}{\ln 5}(\frac{1}{21}\ln\frac{1}{21}+\frac{13}{21}\ln\frac{13}{21}+\frac{4}{21}\ln\frac{4}{21}+\frac{3}{21}\ln\frac{3}{21}) \approx 0.64.$$

In the same way one finds that for G_2 the probabilistic uncertainty is approximately equal to 0.71. Therefore the experimental group demonstrates again a better performance.

V. DISCUSSION AND CONCLUSIONS

In the present paper we calculated the GPA index and the group possibilistic/probabilistic uncertainty for assessing the student understanding of the infinity. Since the students of the control group were exposed to more examples involving, the concept of infinity and related to the topics of their first term mathematics course, it was normally expected that the control group will demonstrate a better performance in the written test. However, according to the outcomes of our classroom experiment the experimental group demonstrated a better mean performance and only the quality performance (GPA index) of the control group was proved to be better. This means that the two hours' lecture on the philosophical / epistemological aspects of the infinity was beneficial only for the mediocre students (lower scores), but it didn't affect the good students (higher scores), who had already acquired a good understanding of the infinity.

Since the good understanding of the concept of infinity is fundamental for the student progress in mathematics, more empirical research on the subject, with possible use of other fuzzy assessment methods as well (center of gravity defuzzification technique, fuzzy numbers, etc.) and the comparison of their outcomes with the corresponding outcomes of the traditional assessment methods of the bi-valued logic, will be very useful.

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APPENDIX:

Questionnaire of the Experiment and Representative Wrong Answers

I. Questionnaire

1. a) Compare the numbers 4.9999….. and 5.

b) Are there any fractions between $\frac{1}{10}$ and $\frac{1}{11}$? If yes, write one of them.

2. Compare the cardinalities of the sets *N* of natural numbers, N_E of the even natural numbers, *Z* of the integers, *Q* of the rational and *R* of the real numbers. Justify your answers.

3. Examine if there exist the limits: a) $\lim_{x\to 2}\sqrt{x^2-9}$,

b) $\lim_{x\to a} f(x)$, with $f(x)=\begin{cases}1, & x\in Q\\ 0, & x\in R-Q\end{cases}$, $a\in R$,

where *Q* is the set of rational and *R* is the set of real numbers.

4. Given the line segment AB with length 1 m we add to it the line segments BC of length $\frac{1}{2}$ m, CD of length $\frac{1}{4}$ m, DE of length $\frac{1}{8}$ m, EG of length $\frac{1}{16}$ m,…. and so on. Find the total length of AB + BC + CD + DE + EG +….. (This problem was retrieved from [4]).

5. Starting from the interval [0, 1] we delete first its middle third ($\frac{1}{3}$, $\frac{2}{3}$), then the middle thirds ($\frac{1}{9}$,$\frac{2}{9}$) and ($\frac{7}{9}$,$\frac{8}{9}$) of the two remaining intervals [0, $\frac{1}{3}$] and [$\frac{2}{3}$, 1] respectively, and so on (Cantor's comb: See Section 1).

a) Find the total length of the removed intervals when the above process is repeated infinitely many times (the lengths of the removed intervals form a geometric progression with first term equal to $\frac{1}{3}$ and ratio $\frac{2}{3}$, therefore their infinite sum is 1).

b) Are there any points left behind in this case?

II. Wrong Answers

1. a) 5 is greater than 4.9999……

b) No, because $\frac{1}{11}$ is the fraction next to $\frac{1}{10}$.

2. Since $N_E \subset N$, *N* has a greater cardinality, etc. Also: All these sets are infinite and therefore they have the same cardinality, which is equal to ∞, or they have no cardinality, which, in case of existence, should be a real number.

3. a) The limit does not exist, because $2^2 - 9 < 0$ and the negative numbers have not real square roots.

b) There are two limits equal to 0 and 1 respectively.

4. The total length is infinite, since the successive additions are repeated infinitely many times.

5. a) The total length removed is less than 1, because there are some points of the initial interval [0, 1] left behind, like $\frac{1}{3}$, $\frac{2}{3}$, etc.

b) There are no points left behind, since the total length removed is equal to 1.

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