Fuzzy and Grey Assessment Methods

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Abstract - In this work two innovative methods are developed for evaluating a system's mean performance by linguistic expressions, which utilize Triangular Fuzzy Numbers and Grey Numbers respectively. Examples are also presented on student, athlete and CBR systems' evaluation to illustrate their applicability to real life problems. The outcomes of those methods are compared to the corresponding outcomes of the traditional assessment methods of calculating the mean value of scores and the GPA index respectively. Although it is finally shown that the two methods provide the same assessment outcomes, it turns out that the use of the Grey Numbers reduces significantly the required computational burden.

Key-Words - Fuzzy Set (FS), Membership Function (MF), Triangular Fuzzy Number (TFN), Grey System (GS), Grey Numbers (GN), Grade Point Average Index (GPA), Case-Based Reasoning (CBR)

I INTRODUCTION

The traditional method for evaluating a system's **mean performance** when numerical scores are used is the calculation of the average of those scores. However, in order to comfort the reviewer's existing uncertainty about the exact value of the numerical scores corresponding to the performance of each of the system's components, frequently in real world applications the assessment is made not by numerical scores but by linguistic expressions (grades), like excellent, very good, good, etc. This makes the traditional calculation of the mean value of those grades impossible.

A popular in such cases method for evaluating the overall system's performance is the calculation of the **Grade Point Average (GPA) index** ([1], Chapter 6, p.125). However, GPA is a weighted average in which greater coefficients (weights) are assigned to the higher grades, thus reflecting not the mean but the **quality performance** of the system.

To overcome such difficulties, we have utilized in earlier works the system's **total uncertainty** under fuzzy conditions as a measure of its effectiveness ([1], Chapter 5). This is based on the fundamental principle of the classical Information Theory that the reduction of a system's uncertainty is connected to the increase of information obtained by a certain activity. In other words, lower uncertainty indicates a greater amount of information and therefore a better system's performance with respect to the corresponding activity. However, this method needs laborious calculations, it does not give a precise qualitative characterization of the system's performance and, most importantly, it is applicable for comparing performances only under the assumption that the existing uncertainty is the same in the compared systems before their common activity.

For this reason we have also used **Fuzzy Numbers (FNs)** for assessing a system's mean performance under fuzzy conditions (e.g. see [2]). On applying this method it was observed that, although the calculation of three components is needed for expressing the mean value of the qualitative grades in the form of a **Triangular FN (TFN)**, only the middle component is used for its defuzzification. This suggests the search for an analogous method that possibly reduces the required computational burden. As a result we have utilized **Grey Numbers (GNs)** as an alternative tool for assessing a system's mean performance with qualitative grades [3]

In the paper at hands we present the above two innovative assessment methods and we prove that they provide the same assessment outcomes. More explicitly, the paper is formulated as follows: In Section II the background information about TFNs and GNs is presented, which is necessary for the good understanding of the present work. The assessment methods with the TFNs and GNS are presented in Sections III and IV respectively and their equivalence is proved in Section IV. Examples illustrating the applicability of those methods to real life problems are presented in Section V and the paper closes with the final conclusion presented in Section VI.

II. PRELIMINARIES

A. Triangular Fuzzy Numbers (TFNs)

Roughly speaking, a **Fuzzy Set (FS)** on the universe U, initiated by Zadeh in 1965 [4], is a map m: U $\rightarrow$ [0, 1], called the **membership function (MF)** of A. For readers not familiar with the basic principles of the FS theory, the book [5] is proposed as a general reference.

A FN, say A, is a FS on the set $\boldsymbol{R}$ of the real numbers, which is *normal* (i.e. there exists x in $\boldsymbol{R}$ such that $m(x) = 1$) and *convex* (which in practice means that all its *a-cuts* $A^a = \{x \in U: m(x) \geq a\}$, a in [0, 1] are closed real intervals) and whose MF $y = m(x)$ is

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a piecewise continuous function. For general facts on FNs we refer to [6]

A TFN (*a, b, c*), with *a, b, c* real numbers such that $a < b < c$ is the simplest form of FN representing mathematically the fuzzy statement that "the value of *b* lies in the interval [*a, c*]". The MF $y = m(x)$ of (*a, b, c*) is zero outside the interval [*a, c*], while its graph in [*a, c*] consists of two straight line segments forming a triangle with the OX axis (Fig. 1).

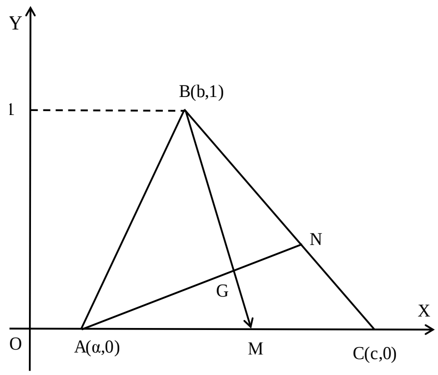

Fig. 1. Graph and Centre of Gravity of the TFN (a, b, c)

Therefore we have:

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

Using elementary methods of Analytic Geometry it is straightforward to check that the coordinates (*X*, *Y*) of the **Centre of Gravity (COG)** of the graph of the TFN A = (*a*, *b*, *c),* being the intersection point G of the medians of the triangle ABC (Fig. 1), are calculated by the formulas

$$X(\text{A}) = \frac{a+b+c}{3},\ Y(\text{A}) = \frac{1}{3} \quad (1)$$

In fact, since M($\frac{a+c}{2}$. 0) and N($\frac{b+c}{2}, \frac{1}{2}$), the proof is easily obtained by calculating the equations of the medians AN and BM and by solving their linear system.

According to the **COG defuzzification technique** [7] the first of formulas (1) can be used to represent a TFN by a crisp number.

The two equivalent to each other methods in use for defining **arithmetic operations** on FNs [6] lead to the following simple rules for the *addition* and *subtraction* of TFNs:

Let A = (*a, b, c*) and B = (a_1, b_1, c_1) be two TFNs. Then one defines:

- The **sum** A + B = ($a+a_1$, $b+b_1$, $c+c_1$).
- The **difference** A - B = A + (-B) = ($a-c_1$, $b-b_1$, $c-a_1$), where –B = ($-c_1$, $-b_1$, $-a_1$) is defined to be the *opposite* of B.

On the contrary, the **product** and the **quotient** of A and B are FNs which are not TFNs in general, apart from certain special cases. Further, the following two **scalar operations** can be also defined:

- k + A= (k+*a*, k+*b*, k+*c*), $k \in \boldsymbol{R}$
- kA = (k*a*, k*b*, k*c*), if $k \in \boldsymbol{R}$, k>0 and kA = (k*c*, k*b*, k*a*), if $k \in \boldsymbol{R}$, k<0.

B. Grey Numbers

Frequently in the everyday life and in many applications of science and engineering, a system's data cannot be easily determined precisely and in practice estimates of them are used. Apart from the FS theory, an alternative tool for dealing with such approximate data is the **Grey System (GS)** theory, initiated by Deng in 1982 [8].

A GS is defined to be any system that lacks information, such as structure message, operation mechanism or behaviour document. On the grounds of existing grey relations and elements one usually can identify where "grey" means poor, incomplete, uncertain, etc.

The use of GNs is the tool for handling the approximate data of a GS. A GN is an indeterminate number whose probable range is known, but which has unknown position within its boundaries. More explicitly, if ***R*** denotes the set of real numbers, a GN, say A, can be expressed mathematically by $A \in [a, b] = \{x \in \boldsymbol{R} : a \le x \le b\}$.

If $a = b$, then A is called a **white number** and if $A \in (-\infty, +\infty)$, then it is called a **black number.**

Compared to the interval [*a, b*], the GN $A \in [a, b]$ may enrich its uncertainty representation with a *whitening function* g: $[a, b] \rightarrow [0, 1]$ defining the **degree of greyness** g(x), for each x in [*a, b*]. The closer is g(x) to 1, the greater the probability of x to be the representative crisp value of the corresponding GN. For general facts on GNs we refer to [9].

The well known arithmetic of the real intervals [10] has been used to define the basic arithmetic operations on GNs. More explicitly, if $A \in [a_1, a_2]$ and $B \in [b_1, b_2]$ are given GNs and *k* is a real number, one defines:

- **Addition** by $A + B \in [a_1 + b_1, a_2 + b_2]$
- **Subtraction** by: $A - B = A + (-B) \in [a_1 - b_2, a_2 - b_1]$, where $-B \in [-b_2, -b_1]$ is defined to be the **opposite** of B.
- **Multiplication** by: $A \times B \in [min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$
- **Division** by: $A : B = A \times B^{-1} \in [min\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\}, max\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\}]$, where $0 \notin [b_1, b_2]$ and $B^{-1} \in [\frac{1}{b_2}, \frac{1}{b_1}]$ is defined to be the **inverse** of B.
- **Scalar multiplication** by: $kA \in [ka_1, ka_2]$, if *k* is a positive real number and by $kA \in [ka_2, ka_1]$, if *k* is a negative real number.

The white number with the highest probability to be the representative real value of the GN A$\in$[*a, b*] is denoted by *w(*A*)*. The technique of determining the value of *w*(A) is called **whitening** of A.

When the distribution of A is unknown (i.e. no whitening function has been defined for A), one usually takes

$$w(\mathrm{A}) = \frac{a+b}{2} \quad (2).$$

III. THE ASSESSMENT METHOD WITH TFNS

Let A_i, i = 1, 2,…, *n* be given TFNs, where *n* is a non negative integer, $n \geq 2$. Then, we define the *mean value* of the A_i's to be the TFN

$$\mathrm{A} = \frac{1}{n}(A_1 + A_2 + \ldots + A_n).$$

Assume that the qualitative grades A = excellent, B = very good, C = good, D = satisfactory and F = failed, are used for the evaluation of a system's performance. Some times the grade E = almost satisfactory is added between D and F, but this does not change our method; it simply needs some more calculations. Our method using TFNs involves the following steps:

- A scale of numerical scores from 1 – 100 is assigned to the above grades as follows: A (85 - 100), B (75 - 84), C (60 – 74), D (50 - 59) and F (0 - 49). The above choice, although compatible to the common logic, is not unique, depending on the user's personal goals. For example, in a more strict assessment one may take A (90 - 100), B (80 - 89), etc.

- Each of the above grades is represented with the help of a TFN, denoted for simplicity by the same letter, as follows: A = (85, 92.5, 100), B = (75, 79.5, 84), C (60, 67, 74), D (50, 54.5, 59) and F (0, 24.5, 49). Observe that the middle entry of each of the above TFNs is equal to the average of its two extreme entries.

- The performance of each of the system's components is evaluated by one of the above five qualitative grades, which means that one of the TFNs A, B, C, D, F can be assigned to each component.

- Let *n* be the total number of the system's components and let n_X denote the number of the components corresponding to the TFN X, X = A, B, C, D, F. Then the mean value M of all those TFNs is equal to the TFN

$$\mathrm{M}(a, b, c) = \frac{1}{n}(n_A\mathrm{A} + n_B\mathrm{B} + n_C\mathrm{C} + n_D\mathrm{D} + n_F F) \quad (3).$$

- Since the calculation in the traditional way of the mean value of the qualitative grades is not possible, it looks logical to consider the TFN M as the fuzzy representative of the system's mean performance. It is straightforward to check that the components *a, b* and *c* of M are equal to

$$a = \frac{85n_A + 75n_B + 60n_C + 50n_D + 0n_F}{n}$$

$$b = \frac{92.5n_A + 79.5n_B + 67n_C + 54.5n_D + 24.5n_F}{n}$$

$$c = \frac{100n_A + 84n_B + 74n_C + 59n_D + 49n_F}{n}$$

- The defuzzification of M by replacing the values of *a, b* and *c* to equations (1) gives that

$$\mathrm{X(M)} = \frac{92.5n_A + 79.5n_B + 67n_C + 54.5n_D + 24.5n_F}{n} = b \quad (4).$$

Therefore, for the defuzzification of M (*a, b, c*) it is enough to calculate only its middle component b. The value of X (M) provides a crisp representation of the TFN M that can be used for evaluating the system's mean performance.

IV. THE ASSESSMENT METHOD WITH GNS

The fact that only the middle component *b* of the TFN M(*a, b, c*) is needed for evaluating a system's mean performance by qualitative grades, gave us the impulse to search for a "formal" assessment method, analogous to that of Section 3 with TFNs, that possibly reduces the required computational burden. The result of this search was the utilization of GNs for the system's assessment.

For this, we correspond to each of the numerical scores A (100-85), B(84-75), C (74-60), D(59-50), F(49-0) attached to the qualitative grades A, B, C, D, F a GN (instead of a TFN), denoted for simplicity by the same letter, as follows: A$\in$ [85, 100], B$\in$[75, 84], C$\in$ [60, 74], D$\in$[50, 59] and F$\in$[0, 49].

Then, assigning to each of the system's components one of the above GNs evaluating its performance and using an analogous to the TFNs definition for the *mean value* of GNs, we find that the mean value of all those GNs is equal to the GN:

$$\mathrm{M} = \frac{1}{n}[n_A\mathrm{A} + n_B\mathrm{B} + n_C\mathrm{C} + n_D\mathrm{D} + n_F\mathrm{F}] \quad (5)$$

The GN M is considered as the grey representative of the system's mean performance. But $n_A\mathrm{A} \in [85n_A, 100n_A]$, $n_B\mathrm{B} \in [75n_B, 84n_B]$, $n_C\mathrm{C} \in [60n_C, 74n_C]$, $n_D\mathrm{D} \in [50n_D, 59n_D]$ and $n_F\mathrm{F} \in [0n_F, 49n_F]$, therefore it turns out that $\mathrm{M} \in [m_1, m_2]$, with

$$m_1 = \frac{85n_A + 75n_B + 60n_C + 50n_D + 0n_F}{n}$$

$$m_2 = \frac{100n_A + 84n_B + 74n_C + 59n_D + 49n_F}{n}.$$

Since the distributions of the GNs A, B, C, D and F are unknown, the same happens with the distribution of M. Therefore, one takes

$$W(\mathrm{M}) = \frac{m_1 + m_2}{2} \quad (6).$$

The value of *w*(M) is the crisp outcome used for the quantitative estimation of the system's mean performance.

On comparing the crisp outcomes obtained by equations (3) and (6) deffuzifying and whitening the mean values M (*a, b, c*) and M $\in$ [m_1, m_2] respectively, it turns out that the assessment methods of a system's mean performance using TFNs and GNs respectively are *equivalent*, since they provide the same assessment outcomes.

Also, one observes that in the extreme case where the maximum possible numerical score corresponds to each component for each grade, i.e. the n_A scores corresponding to A are 100, the n_B scores corresponding to B are 84, etc., the mean value of all those scores is equal to *b* or m_2 respectively. Also, in the other extreme case, where the minimum possible numerical score corresponds to each system's component for each grade, i.e. the n_A scores corresponding to A are 85, the n_B scores corresponding to B are 75, etc., the mean value of all those scores is equal to *a* or m_1 respectively. Consequently, the assessment methods with the TFNs and the GNs give a *reliable approximation* of the system's mean performance and therefore they are useful when no numerical scores are used, but the system's performance is assessed by qualitative grades.

V. EXAMPLES

In this section three examples on student, athlete and CBR computer systems' assessment are presented illustrating the applicability of our assessment methods with GNs and TFNs to real life problems. The outcomes of those methods are compared to the corresponding outcomes of the traditional methods of calculating the mean value of scores and the GPA index respectively.

Example 1: The performance of four athletes was assessed by three different experts using a numerical scale from 0 – 100 as follows: $\mathbf{A_1}$ (athlete 1): 48, 51, 52, $\mathbf{A_2}$: 87, 91, 93, $\mathbf{A_3}$: 82, 89, 94 and $\mathbf{A_4}$: 35, 40, 44. It is asked to evaluate the four athletes' mean performance.

Calculation of the mean value of scores: The mean value of the 4 X 3 = 12 in total scores assigned to the athletes is approximately equal to 67.17, a score demonstrating a good (C) mean performance of the four athletes.

Solution using GNs: One observes that 5 of the scores assigned to the athletes correspond to the qualitative grade A, 1 corresponds to B, 2 to D and 4 scores correspond to F. Therefore, by assigning to the above qualitative grades the corresponding GNs one finds that their mean value is

$$M = \frac{1}{12}\,(5A+B+2D+4F) \in [50, 74.5].$$

Therefore $W(M) = \frac{50+74.5}{2} = 62.25$, which demonstrates again a good (C) mean performance.

Solution using TFNs: In this case one finds the mean value

$$M = \frac{1}{12}\,(5A+B+2D+4F) = (50, 62.25, 74.5).$$

The defuzzification of M provides the same assessment outcomes with the method using the TFNs.

In concluding, when a system's performance is assessed with numerical scores, the calculation of the mean value of those scores gives the exact outcome for the assessment of its mean performance, whereas the methods using TFNs and GNs give an approximate value of it. Therefore the last two methods are useful in practice only when the assessment is made by qualitative grades.

Example 2: The following Table depicts the performance of two student groups, say G_1 and G_2, in a common mathematical test

Table 1. Student performance

Grade	G_1	G_2
A	20	20
B	15	30
C	7	15
D	10	10
F	8	10
Total	60	85

It is asked to estimate the mean performance of the two groups on the test.

Solution using GNs: Assigning to each student the corresponding GN and calculating the mean values M_1 and M_2 of those GNs for the groups G_1 and G_2 respectively one finds that:

$$M_1 = \frac{1}{60}(20A+15B+7C+10D+8F) \in [62.42, 79.33]$$

$$M_2 = \frac{1}{85}(20A+30B+15C+10D+10F) \in [62.94, 78.94]$$

Therefore, equation (6) gives that

$$W(M_1) \approx \frac{62.42+79.33}{2} \approx 70.88,$$

$$W(M_2) \approx \frac{62.94+78.94}{2} \approx 70.94.$$

Consequently both groups demonstrated a good (C) mean performance, with the mean performance of the second group being slightly better.

Solution using TFNs: In this case the calculation of the mean values: gives that

$$M_1 = \frac{1}{60}(20A+15B+7C+10D+8F)$$
$$= (62.42, 70.88, 79.33)$$

$$M_2 = \frac{1}{85}(20A+30B+15C+10D+10F)$$
$$=(62.94, 70.94, 78.94)$$

Therefore, using equation (3) for defuzzifying those mean values one obtains the same assessment outcomes as in the case of GNs. However, the use of TFNs requires the calculation of three components for obtaining the mean values in contrast to the use of GNs, which requires the calculation of two components only.

In concluding, although the above two methods provide the same assessment outcomes, the use of GNs reduces the required computational burden.

Remark: The GPA index is calculated ([1], Chapter 6, p.125) by the formula

$$\text{GPA} = \frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n} \quad (7).$$

Therefore, one finds that

GPA= $\frac{4.20+3.15+2.7+1.10}{60} \approx 2.48$ for the first and

GPA= $\frac{4.20+3.30+2.15+1.10}{85} \approx 2.47$ for the second group.

Thus, in contrast to their mean performance, the first group demonstrated a slightly better quality performance than the second one.

In concluding, the assessment of a system's quality performance could lead to different outcomes than the assessment of its mean performance.

Example 3 (Assessment of CBR Systems)**:**

Background: Case-Based Reasoning (CBR) is the process of solving new problems by adapting the solution of similar (analogous) problems solved in the past, which are usually referred as *past cases*. A case - library can be a powerful corporate resource allowing everyone in an organization to tap in it when handling a new problem. A CBR system, usually designed and functioning with the help of computers, allows the case-library to be developed incrementally, while its maintenance is relatively easy and can be carried out by domain experts. The CBR approach has got a lot of attention over the last 30-40 years, because as an intelligent – systems' method enables information managers to increase efficiency and reduce cost by substantially automating processes.

CBR has been formalized for purposes of computer and human reasoning as a four step process involving:

- R_1: **Retrieving** the most similar to the new problem past case.
- R_2: **Reusing** the information and knowledge of the retrieved case for the solution of the new problem.
- R_3: **Revising** the proposed solution for solving the new problem.
- R_4: **Retaining** the part of this experience likely to be useful for future problem solving.

The first three of the above steps are not linear, characterized by a backward - forward flow. A simplified flow - chart of the CBR process, which is adequate for the purposes of the present example, is presented in Fig. 2:

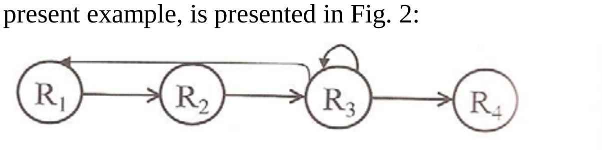

Fig. 2. A simplified flow-chart of the CBR process

More details about the CBR process and a detailed functional diagram illustrating the flow among its four steps can be found in [11]

The Problem: Consider two CBR systems designed for help desk applications with their libraries containing 105 and 90 past cases respectively. Assume that the two systems' designers have supplied them with the same mechanism (software) that enables the assessment of the degree of success of each one of their past cases at each step of the CBR process, when used for the solution of new similar problems. Table 2 depicts the degree of success of their past cases in each of the three first steps of the CBR process

Table 2. Assessment of the past cases of the two CBR systems

FIRST SYSTEM

Steps	F	D	C	B	A
R_1	0	0	51	24	30
R_2	18	18	48	21	0
R_3	36	30	39	0	0

SECOND SYSTEM

Steps	F	D	C	B	A
R_1	0	18	45	27	0
R_2	18	24	48	0	0
R_3	36	27	27	0	0

Here we shall compare the quality performance of the two systems by calculating the GPA index and their mean performance by applying our assessment methods with TFNs and GNs.

GPA index: Denote by y_i , i = 1, 2, 3, 4, 5 the frequencies of the CBR systems' cases whose performance is characterized by F, D, C, B and A respectively, i.e. $y_1 = \frac{n_F}{n}$, $y_2 = \frac{n_D}{n}$, etc. Then equation (7) can be written as:

$$\text{GPA} = y_2 + 2y_3 + 3y_4 + 4y_5 \quad (8).$$

In case of the ideal performance ($y_5 = 1$, $y_4 = y_3 = y_2 = y_1 = 0$) we have GPA = 4, while in case of the worst performance ($y_1 = 1$, $y_5 = y_4 = y_3 = y_2 = 0$) we have GPA = 0; therefore $0 \leq \text{GPA} \leq 4$. Consequently, values of GPA greater than 2 could be considered as demonstrating a more than satisfactory performance. In our case, the data of Table 2 give the following frequencies presented in Table 3:

Table 3. Frequencies of the past cases of the two CBR systems

FIRST SYSTEM

Steps	Y_1	y_2	y_3	Y_4	Y_5
R_1	0	0	$\frac{51}{105}$	$\frac{24}{105}$	$\frac{30}{105}$
R_2	$\frac{18}{105}$	$\frac{18}{105}$	$\frac{48}{105}$	$\frac{21}{105}$	0

R_3	$\frac{36}{105}$	$\frac{30}{105}$	$\frac{39}{105}$	0	0

SECOND SYSTEM

Steps	Y_1	y_2	y_3	Y_4	Y_5
R_1	0	$\frac{18}{90}$	$\frac{45}{90}$	$\frac{27}{90}$	0
R_2	$\frac{18}{90}$	$\frac{24}{90}$	$\frac{48}{90}$	0	0
R_3	$\frac{36}{90}$	$\frac{27}{90}$	$\frac{27}{90}$	0	0

Replacing the values of frequencies from Table 3 to equation (8) one finds the following values for the GPA index:

First System: R_1: $\frac{294}{105} = 2.8$, R_2: $\frac{177}{105} \approx 1.69$, R_3: $\frac{108}{105} \approx 1.03$.

Second System: R_1: $\frac{189}{90} = 2.1$, R_2: $\frac{168}{90} \approx 1.87$, R_3: $\frac{81}{90} = 0.9$.

The above values show that the first system demonstrated a better quality performance at steps R_1 and R_3 (Retrieve, Revise), whereas the second one demonstrated a better performance at R_2 (Reuse). Further, the two systems' performance was proved to be more than satisfactory in R_1 and less than satisfactory in the other two steps, being worse at R_3. This was logically expected, since the success in each step depends on the success in the previous steps. Notice that the two systems' performance at the last step R_4 was not examined, since *all* the past cases, even the unsuccessful ones, are retained in a system's library for possible use in future applications with new problems. The unsuccessful cases help for exploring possible reasons of failure to find a solution for the new problem.

The mean values of the GPA index for the two systems at the three steps R_1, R_2 and R_3 are approximately equal to 1.84 and 1.62 respectively, showing that the first system demonstrated a better overall quality performance.

Use of the TFNs: From the data of Table 2 one finds that for the first system and in step R_1 we have 51 TFNs equal to C(60, 67, 74), 24 TFNs equal to B(75, 79.5, 85) and 30 TFNs equal to A(85, 92.5, 100). The mean value of all those TFNs, denoted for simplicity by the same letter R_1, is equal to $R_1 = \frac{1}{105}(51C + 24B + 30A)$

$= \frac{1}{105}[(3060, 3417, 3774) + (1800, 1908, 2016) + (2550, 2775. 3000)$

Therefore, equation (3) gives that $X(R_1) = 77.14$, which shows that the first system demonstrated a very good (B) performance at step R_1.

In the same way one calculates for the first system the mean values

$R_2 = \frac{1}{105}(18F + 18D + 48C + 21B) \approx (51, 60.07, 69.14)$ and

$R_3 = \frac{1}{105}(36F + 30D + 39C) \approx (36.57, 48.86, 61.14)$, thus obtaining the analogous conclusions for the system's performance at the steps R_2 and R_3 of the CBR process.

The overall system's performance can be assessed by the mean value

$$R = \frac{1}{3}(R_1 + R_2 + R_3) \approx (52.71, 62.02, 71.33).$$

Therefore, since $X(R) = 62.02$, the system demonstrated a good (C) mean performance.

A similar argument gives for the second system the values $R_1 = (62.5, 68.25, 74)$, $R_2 \approx (45.33, 55.17, 65)$, $R_3 = (33, .46.25, 59.5)$ and $R \approx (46.94, 56.56, 66.17)$, thus obtaining the analogous conclusions for its mean performance at each step of the CBR process and its overall mean performance.

Use of the GNs: For the first system and in step R_1 we have 51 GNs equal to $C \in [60,74]$, 24 GNs equal to $B \in [75, 84]$ and 30 GNs equal to $A \in [85, 100]$. The mean value of all those GNs, denoted by R_1^*, is equal to

$R_1^* = \frac{1}{105}(51C + 24B + 30A) \in [70.57, 83.71]$. Therefore,

$W(R_1^*) = \frac{70.57+83.71}{2} = 77.14$, etc.

As we have seen in Section 4 this approach provides in general the same assessment outcomes with the use of TFNs, but, as it becomes more evident here, it reduces significantly the required computational burden.

V. CONCLUSION

An inevitable consequence of the enormous development of the technology during the last years is the continuous development of more and more complicated artificial systems, whose function is characterized by vagueness and/ or uncertainty and it is frequently assessed by using approximate data. As a result the use of methods and principles of FS and GS theory has become recently very popular for treating, explaining and improving the performance of such kind of systems. The main tools for the application of the above two approaches are the FNs and of the GNs respectively.

In the present work we have developed two innovative methods using TFNs and GNs for assessing a system's mean performance. Those methods are useful when qualitative grades and not numerical scores are used for the assessment purposes, because in this case the calculation of the mean value of those grades in the traditional way is impossible. Although the two methods have been proved to be equivalent, the use of the GNs reduces significantly the required computational burden.

Examples have been also presented on student, athlete and CBR systems assessment illustrating the applicability of our methods in real life problems and showing that the system's quality performance, calculated by the traditional GPA index, may lead to different assessment outcomes.

REFERENCES

[1] Voskoglou, M. Gr., *Finite Markov Chain and Fuzzy Logic Assessment Models: Emerging Research and Opportunities*, Createspace.com. – Amazon, Columbia, SC, USA, 2017.

[2] Voskoglou, M. Gr., "An Application of Fuzzy Numbers to Assessment of Human Activities", *International Journal of Control Systems and Robotics*, 2, 52-60, 2017

[3] Voskoglou, M. Gr. & Theodorou, Y., "Application of Grey Numbers to Assessment Processes", *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, 7, 273-280, 2017.

[4] Zadeh , L.A., "Fuzzy Sets", *Information and Control*, 8, 338-353, 1965.

[5] Klir, G. J. & Folger, T. A., *Fuzzy Sets, Uncertainty and Information*, Prentice-Hall, London, 1988.

[6] Kaufmann, A. & Gupta, M., *Introduction to Fuzzy Arithmetic*, Van Nostrand Reinhold Company, New York, 1991.

[7] van Broekhoven, E. & De Baets, B., "Fast and accurate centre of gravity defuzzification of fuzzy systems outputs defined on trapezoidal fuzzy partitions", *Fuzzy Sets and Systems*, 157(7), 904-918, 2006.

[8] Deng, J., "Control Problems of Grey Systems", *Systems and Control Letters*, 288-294, 1982.

[9] Liu, S. F. & Lin, Y. (Eds.), *Advances in Grey System Research*, Springer, Berlin – Heidelberg, 2010.

[10] Moore, R.A., Kearfort, R. B. & Clood, M.J., *Introduction to Interval Analysis*, 2nd Printing, SIAM, Philadelphia, 1995.

[11] Voskoglou, M. Gr. & Salem A.-B.M., "Analogy-Based and Case-Based Reasoning: Two Sides of the Same Coin", *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, 4, 7-18, 2014

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