Characterizations of Fuzzy W – Compactness and Fuzzy W-Closed Spaces in Fuzzy Topological Spaces

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Abstract— The concepts of W – compactness and W – closed spaces in the fuzzy setting are defined and investigated. Fuzzy filter bases are used to characterize these concepts.

Keywords— Fuzzy topological space, fuzzy W-open set, quasi-coincident, fuzzy W-compact space, fuzzy W-closed space, fuzzy filter base.

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I. Introduction

The concept of fuzzy set operation was first introduced by Zadeh [15] and subsequently, several authors including Zadeh [15] have discussed various aspects of the theory and applications of fuzzy sets. Fuzzy topological spaces were introduced by Chakrabarty and Ahsanullah (1992) and Chang (1968). Compactness occupies a very important place in fuzzy topology and so do some of its other forms including closed space, countably compactness and Lindelof space. In [7], Talal Al – Hawary introduced the concepts of fuzzy W – closed sets, fuzzy W – open sets and Fuzzy W – generalized closed sets as well as fuzzy W – g – continuous and fuzzy W – g – irresolute functions and investigated their some basic properties. The objective of this paper is devoted to introduce and study the concepts of W – compactness and W – closed spaces in the fuzzy setting. We use fuzzy filter bases to characterize fuzzy W – compactness and fuzzy W – closed spaces. We will also explore several basic properties and characterizations of these concepts.

II. Preliminaries

Let X be a nonempty set and I = [0,1]. A fuzzy set on X is a mapping from X into I. The null fuzzy set 0 X is the mapping from X into I which assumes only the value is 0 and whole fuzzy sets 1 X is a mapping from X into I which takes the values I only. The union (resp. intersection) of a family {Aα : α ∈ Λ} of fuzzy sets of X is defined to be the mapping Sup{Aα : α ∈ Λ} (resp. Inf{Aα : α ∈ Λ}). A fuzzy set A of X is contained in a fuzzy set B of X if A(x) ≤ B(x) for each x ∈ X. A fuzzy point xβ in X is a fuzzy set defined by xβ(y) = β for y = x and xβ(y) = 0 for y ≠ x, β ∈ (0,1] and y ∈ X. A fuzzy point xβ is said to be quasi – coincident with the fuzzy set A denoted by xβqA if and only if β + A(x) > 1. A fuzzy set A is quasi – coincident (not quasi – coincident ) with a fuzzy set B denoted by AqB (AqB) if and only if there exists a point x ∈ X such that A(x) + B(x) > 1 (A(x) + B(x) ≤ 1). A family T of fuzzy sets of X is called a fuzzy topology on X if X, φ belong to T and T is closed with respect to arbitrary union and finite intersection. The members of T are called fuzzy open sets and their complements are fuzzy closed sets. For any fuzzy set A of X, the closure of A (denoted by Cl(A)) is the intersection of all the fuzzy closed supersets of A and the interior of A (denoted by Int(A)) is the union of all fuzzy open subsets of A. Throughout this paper X and Y will mean fuzzy topological spaces. The complement and the support of a fuzzy set U are denoted by Uc and S(U), respectively.

Definition 2.1. Let A be a fuzzy subset of a fuzzy topological space (X, T). The fuzzy W – interior of A is the union of all fuzzy open subsets of X whose closures are contained in Cl(A), and is denoted by W – Int(A). A is called fuzzy W – open if A = W – Int(A). The complement of a fuzzy W – open subset is called fuzzy W – closed. Alternatively, a fuzzy subset A of X is fuzzy W – closed if and only if A = W – Cl(A), where W – Cl(A) = ∩α∈Λ Aα, Aα is FC – set in X. Cl
early \( \text{Int}(A) \leq W - \text{Int}(A) \leq \text{Cl}(A) \) and \( A \leq \text{Cl}(A) \leq W - \text{Cl}(A) \) and hence every fuzzy \( W - \) closed set is a fuzzy closed set, but the converse needs not be true.

**Example 2.2.** Suppose that \( X = \{a, b, c\} \) and \( T = \{0, 1, \chi_{[a]}, \chi_{[b]}, \chi_{[a,b]}\} \). Then the set \( \chi_{[a,b]} \) is a fuzzy closed set but not a fuzzy \( W - \) closed set since \( W - \text{Cl}(\chi_{[a,b]}) = 1 \).

The intersection of two fuzzy \( W - \) open subsets need not be fuzzy \( W - \) open.

**Example 2.3.** Let \( X = \{a, b, c, d\} \) and \( T = \{0, 1, \chi_{[a]}, \chi_{[b]}, \chi_{[c]}, \chi_{[a,b]}, \chi_{[a,c]}, \chi_{[b,c]}, \chi_{[a,b,c]}\} \). Then \( \chi_{[a,b]} \) and \( \chi_{[a,c]} \) are fuzzy \( W - \) open sets, but \( \chi_{[a,b]} \cap \chi_{[a,c]} = \chi_{[a]} \) is not a fuzzy \( W - \) open set. The set \( \chi_{[a,b]} \) is a fuzzy closed set but not a fuzzy \( W - \) closed set since \( W - \text{Cl}(\chi_{[a,b]}) = 1 \).

In classical topology, the interior of a set is a subset of the set itself. But this is not the case for fuzzy \( W - \) open sets. Next we show that \( A \leq W - \text{Int}(A) \) and \( W - \text{Int}(A) \leq A \) need not be true.

**Example 2.4.** Consider the space in Example 2.3. Then
\[
W - \text{Int}(\chi_{[c]}) = 0 < \chi_{[c]} \cdot \chi_{[a,b]} < W - \text{Int}(\chi_{[a,b]}) = 1.
\]

Next, we state the result as proved in [7] that arbitrary unions of fuzzy \( W - \) open subsets are fuzzy \( W - \) open.

**Theorem 2.5.** If \( (X, T) \) is a fuzzy topological space, then arbitrary unions of fuzzy \( W - \) open subsets are fuzzy \( W - \) open.

**Corollary 2.6.** The arbitrary intersection of fuzzy \( W - \) closed subsets are fuzzy \( W - \) closed, while finite unions of fuzzy \( W - \) closed subsets need not be fuzzy \( W - \) closed.

**Corollary 2.7.** If \( A \) is a fuzzy \( W - \) dense subset of \( X \), \( (W - \text{Cl}(A) = 1) \), then \( W - \text{Int}(A) = 1 \).

**Definition 2.8.** A collection \( \xi \) of fuzzy subsets of a fuzzy topological space \( (X, T) \) is said to form a fuzzy filterbases if and only if for every finite subcollection \( \lambda \) of \( \xi \), \( \bigcap A \neq 0_X \).

**Definition 2.9.** A collection \( \mu \) of fuzzy sets in a fuzzy topological space \( (X, T) \) is said to be a cover of a fuzzy set \( U \) of \( X \) if and only if \( \left( \bigcup_{\alpha \in \mu} A \right)(x) = 1_X \), for every \( x \in S(U) \).

**Definition 2.10.** A fuzzy cover \( \mu \) of a fuzzy set \( U \) in a fuzzy topological space \( (X, T) \) is said to have a finite subcover if and only if there exists a finite subcollection \( \eta \) of \( \mu \) such that \( \left( \bigcup_{\alpha \in \eta} A \right)(x) \geq U(x) \), for every \( x \in S(U) \).

**III. Fuzzy \( W - \) Compact Spaces**

**Definition 3.1.** A fuzzy topological space \( (X, T) \) is said to be fuzzy \( W - \) compact if and only if for every family \( \mu \) of fuzzy \( W - \) open sets such that \( \bigcup_{\alpha \in \mu} A = 1_X \) there is a finite subfamily \( \eta \subseteq \mu \) such that \( \bigcup_{\alpha \in \eta} A = 1_X \).

**Definition 3.2.** A fuzzy set \( U \) in a fuzzy topological space \( (X, T) \) is said to be fuzzy \( W - \) compact relative to \( X \) if and only if for every family \( \mu \) of fuzzy \( W - \) open sets such that \( \bigcup_{\alpha \in \mu} A \geq S(U) \) there is a finite subfamily \( \eta \subseteq \mu \) such that \( \bigcup_{\alpha \in \eta} A \geq U(x) \) for every \( x \in S(U) \).

**Theorem 3.3.** A fuzzy topological space \( (X, T) \) is fuzzy \( W - \) compact if and only if for every collection \( \{A_j : j \in J\} \) of fuzzy \( W - \) closed sets of \( X \) having the finite intersection property, \( \bigcap_{j \in J} A_j \neq 0_X \).

**Proof.** Let \( \{A_j : j \in J\} \) be a collection of fuzzy \( W - \) closed sets with the finite intersection property. Suppose that \( \bigcap_{j \in J} A_j = 0_X \). Then \( \bigcup_{j \in J} A_j^c = 1_X \). Since \( \{A_j^c : j \in J\} \) is a collection of fuzzy \( W - \) open sets cover of \( X \), then from the \( W - \) compactness of \( X \) it follows that there exists a finite subset \( F \subseteq J \) such that \( \bigcup_{j \in F} A_j^c = 1_X \).

Then \( \bigcap_{j \in F} A_j = 0_X \) which gives a contradiction and therefore \( \bigcap_{j \in J} A_j \neq 0_X \).

Conversely, Let \( \{A_j : j \in J\} \) be a collection of fuzzy \( W - \) open sets cover of \( X \). Suppose that for every finite subset \( F \subseteq J \), we have \( \bigcup_{j \in F} A_j \neq 1_X \). Then
\[ \bigcup_{x \in A} A^c_x \neq 0 \]. Hence \( \{ A^c_x : j \in J \} \) satisfies the finite intersection property. Then from the hypothesis, we have \( \bigcap_{j \in J} A^c_x \neq 0 \) which implies \( \bigcap_{j \in J} A_j \neq 1 \) and this contradicts the fact that \( \{ A_j : j \in J \} \) is a fuzzy \( W \)-open open cover of \( X \). Thus \( X \) is fuzzy \( W \)-compact.

Now, we give some results of fuzzy \( W \)-compactness in terms of fuzzy filterbases

**Theorem 3.4.** A fuzzy topological space \( (X, T) \) is fuzzy \( W \)-compact if and only if every filterbases \( \xi \) in \( X \),

\[ \bigcap_{G \in \xi} W - \text{Cl}(G) \neq 0 \]  

**Proof.** Let \( \mu \) be a fuzzy \( W \)-open set cover of \( X \) and \( \mu \) has no finite subcover. Then for every finite subcollection \( \lambda \) of \( \mu \), there exists \( x \in X \) such that \( A(x) < 1 \) for every \( A \in \lambda \). Then \( A^c(x) > 0 \), so that \( \bigcap_{\lambda \in \lambda} A^c(x) \neq 0 \).

Thus \( \{ A^c : A \in \mu \} \) forms a filterbases in \( X \). Since \( \mu \) is fuzzy \( W \)-open set cover of \( X \), then

\[ \left( \bigcup_{A \in \mu} A \right)(x) = 1 \]  

for every \( x \in X \) and hence

\[ \left( \bigcap_{A \in \mu} W - \text{Cl}(A^c) \right)(x) = \left( \bigcap_{A \in \mu} A^c \right)(x) = 0 \]  

which is a contradiction. Then every fuzzy \( W \)-open set cover of \( X \) has a finite subcover and hence \( X \) is fuzzy \( W \)-compact.

Conversely, suppose there exists a filterbases \( \xi \) such that

\[ \bigcap_{G \in \xi} W - \text{Cl}(G)(x) \neq 0 \]  

so that

\[ \bigcap_{G \in \xi} \left( \left( W - \text{Cl}(G) \right)^c \right)(x) = 1 \]  

for every \( x \in X \) and hence \( \mu = \left( \left( W - \text{Cl}(G) \right)^c : G \in \xi \right) \) is a fuzzy \( W \)-open set cover of \( X \). Since \( X \) is fuzzy \( W \)-compact, then \( \mu \) has a finite subcover. Then there exists a finite subset \( \lambda \subseteq \mu \) such that

\[ \left( \bigcup_{G \in \lambda} W - \text{Cl}(G) \right)(x) = 1 \]  

and hence

\[ \left( \bigcap_{G \in \lambda} G^c \right)(x) = 1 \]  

so that \( \bigcap_{G \in \lambda} G = 0 \) which is a contradiction since \( \lambda \) is a finite subset of filterbases \( \xi \). Therefore

\[ \bigcap_{G \in \xi} W - \text{Cl}(G) \neq 0 \]  

for every filterbases \( \xi \), such that every finite set of members of \( \xi \) is quasi-coincident with \( U \), and

\[ \left( \bigcap_{G \in \xi} W - \text{Cl}(G) \right)(x) \neq 0 \]  

for some \( x \in S(U) \), then for every \( x \in S(U) \),

\[ \left( \bigcap_{G \in \xi} W - \text{Cl}(G) \right)(x) \neq 0 \]  

and hence \( \xi = \{ A^c : A \in \mu \} \) forms a filterbases and

\[ \left( \bigcap_{A \in \lambda} W - \text{Cl}(A^c) \right)(x) = 0 \]  

for every \( x \in S(U) \).

Thus \( \mu = \left( \left( W - \text{Cl}(G) \right)^c : G \in \xi \right) \) is fuzzy \( W \)-open cover of \( U \). Since \( U \) is fuzzy \( W \)-compact relative to \( X \), then there exists a finite subcover, say

\[ \left( \left( W - \text{Cl}(G_j) \right)^c : j = 1, \ldots, n \right) \], such that

\[ \left( \bigcup_{j=1}^n \left( W - \text{Cl}(G_j) \right)^c \right)(x) \geq U(x) \]  

for every \( x \in S(U) \). Hence

\[ \left( \bigcap_{a=1}^n \left( W - \text{Cl}(G_j) \right)^c \right)(x) \leq U(x) \]  

for every \( x \in S(U) \), so that \( \left( \bigcap_{a=1}^n \left( W - \text{Cl}(G_j) \right)^c \right)(x) \leq U(x) \) which is a contradiction. Therefore for every filterbases \( \xi \), such that
every finite set of members of $\xi$ is quasi–coincident with $U$, $(\bigcap_{A \in \nu} W - \text{Cl}(G)) \cap U \neq 0_X$.

The following theorem proves that hereditary property for fuzzy $W$–compact spaces.

**Theorem 3.6.** Every fuzzy $W$–closed subset of a fuzzy $W$–compact space $(X, T)$ is fuzzy $W$–compact relative to $X$.

**Proof.** Let $\xi$ be a fuzzy filterbases in $X$ such that $Uq \cap \{G : G \in \lambda\}$ holds for every finite subcollection $\lambda$ of $\xi$ and a fuzzy $W$–closed set $U$. Consider $\xi^* = \{U\} \cup \xi$. For any finite subcollection $\lambda^*$ of $\xi^*$, if $U \not\in \lambda^*$, then $\bigcap_{\lambda^*} \neq 0_X$. If $U \in \lambda^*$ and since $Uq \cap \{G : G \in \lambda^* - \{U\}\}$, then $\bigcap_{\lambda^*} \neq 0_X$. Hence $\lambda^*$ is a fuzzy filterbases in $X$. Since $X$ is fuzzy $W$–compact, then $\bigcap_{A \in \nu} W - \text{Cl}(G) \neq 0_X$, so that $\left(\bigcap_{A \in \nu} W - \text{Cl}(G)\right) \cap U = \left(\bigcap_{A \in \nu} W - \text{Cl}(G)\right) \cap \text{Cl}(U) \neq 0_X$.

Hence by Theorem 3.5, we have $U$ is fuzzy $W$–compact relative to $X$.

**IV. Fuzzy $W$–Closed Spaces**

**Definition 4.1.** A fuzzy topological space $(X, T)$ is said to be fuzzy $W$–closed space if and only if for every family $\mu$ of fuzzy $W$–open sets such that $\bigcup_{A \in \nu} A(x) = 1_X$ there is finite subfamily $\eta \subseteq \mu$ such that $\left(\bigcup_{A \in \nu} W - \text{Cl}(A)\right)(x) = 1_X$, for every $x \in X$.

**Theorem 4.2.** A fuzzy topological space $(X, T)$ is fuzzy $W$–closed if and only if for every fuzzy $W$–open filterbases $\xi$ in $X$, $\bigcap_{A \in \nu} W - \text{Cl}(G) \neq 0_X$.

**Proof.** Let $\mu$ be a fuzzy $W$–open set cover of $X$ and let for every finite subfamily $\eta$ of $\mu$, $\left(\bigcup_{A \in \nu} W - \text{Cl}(A)\right)(x) < 1_X$, for some $x \in X$. Then $\left(\bigcap_{A \in \nu} W - \text{Cl}(A)\right)(x) > 0_X$ for some $x \in X$. Thus $\xi = \left\{\left(W - \text{Cl}(A)\right)^c : A \in \mu\right\}$ forms a fuzzy $W$–open filterbases in $X$. Since $\mu$ is a fuzzy $W$–open set cover of $X$, then $\bigcap_{A \in \nu} A^c = 0_X$ which implies $\bigcap_{A \in \nu} W - \text{Cl}(W - \text{Cl}(A))^c = 0_X$, which is a contradiction. Then every fuzzy $W$–open set cover of $X$ has a finite subfamily $\eta$ such that $\left(\bigcup_{A \in \nu} W - \text{Cl}(A)\right)(x) = 1_X$ for every $x \in X$. Hence $X$ is fuzzy $W$–closed.

Conversely, suppose there exists a fuzzy $W$–open filterbases $\xi$ in $X$ such that $\bigcap_{A \in \nu} W - \text{Cl}(G) = 0_X$, so that $\left(\bigcup_{A \in \nu} \left(W - \text{Cl}(A)\right)^c\right)(x) = 1_X$ for every $x \in X$ and hence $\mu = \left\{(W - \text{Cl}(A))^c : G \in \xi\right\}$ is a fuzzy $W$–open set cover of $X$. Since $X$ is fuzzy $W$–closed, then $\mu$ has a finite subfamily $\eta$ such that $\left(\bigcup_{A \in \nu} W - \text{Cl}(W - \text{Cl}(A))^c\right)(x) = 1_X$ for every $x \in X$, and hence $\bigcap_{G \in \eta} (W - \text{Cl}(W - \text{Cl}(G))^c) = 0_X$.

Thus $\bigcap_{G \in \eta} G = 0_X$ which is a contradiction, since all the $G$ are members of filterbases.

**Definition 4.3.** A fuzzy set $U$ in a fuzzy topological space $(X, T)$ is said to be fuzzy $W$–closed relative to $X$ if and only if for every family $\mu$ of fuzzy $W$–open sets such that $\bigcup_{A \in \nu} A = U$, there is a finite subfamily $\eta \subseteq \mu$ such that $\left(\bigcup_{A \in \nu} W - \text{Cl}(A)\right)(x) = U(x)$ for every $x \in S(U)$.

**Theorem 4.4.** A fuzzy subset $U$ in a fuzzy topological space $(X, T)$ is fuzzy $W$–closed relative to $X$ if and only if for every fuzzy $W$–open filterbases $\xi$ in $X$, $\left(\bigcap_{A \in \nu} W - \text{Cl}(G)\right) \cap U = 0_X$, there exists a finite subfamily $\lambda$ of $\xi$ such that $\left(\bigcap_{A \in \lambda} G\right) \cap U$.

**Proof.** Let $U$ be a fuzzy $W$–closed relative to $X$, suppose $\xi$ is a fuzzy $W$–open filterbases in $X$ such that for every finite subfamily $\lambda$ of $\xi$, $\left(\bigcap_{A \in \lambda} G\right) \cap U$, but
\[
\left( \bigcap_{G \in \xi} W - \text{Cl}(G) \right) \cap U = 0_x. \quad \text{Then for every } x \in S(U), \quad \left( \bigcap_{G \in \xi} W - \text{Cl}(G) \right)(x) = 0_x \quad \text{and hence}
\]
\[
\left( \bigcup_{G \in \xi} \left( W - \text{Cl}(G) \right)^c \right)(x) = 1_x \quad \text{for every } x \in S(U).
\]
Then \(\mu = \left\{ \left( W - \text{Cl}(G) \right)^c : G \in \xi \right\}\) is a fuzzy \(W\)-open set cover of \(U\) and hence there exists a finite subfamily \(\lambda \subseteq \xi\) such that \(\bigcup_{G \in \lambda} W - \text{Cl}(W - \text{Cl}(G))^c \supseteq U\), so that
\[
\bigcap_{G \in \lambda} \left( W - \text{Cl}(W - \text{Cl}(G))^c \right)^c = \bigcap_{G \in \lambda} W - \text{Int}(W - \text{Cl}(G)) \leq U^c \quad \text{and hence}
\]
\[
\bigcap_{G \in \lambda} G \leq U^c. \quad \text{Then } \left( \bigcap_{G \in \lambda} G \right) \bar{q} U \quad \text{which is a contradiction.}
\]
Conversely, let \(U\) not be a fuzzy \(W\)-closed set relative to \(X\), then there exists a fuzzy \(W\)-open set \(\mu\) cover of \(U\) such that every finite subfamily \(\eta \subseteq \mu\),
\[
\left( \bigcup_{A \in \eta} W - \text{Cl}(A) \right)(x) \leq U(x) \quad \text{for some } x \in S(U)
\]
and hence \(\left( \bigcap_{A \in \eta} W - \text{Cl}(A)^c \right)(x) > U^c(x) \geq 0\) for some \(x \in S(U)\). Thus
\[
\xi = \left\{ \left( W - \text{Cl}(A)^c \right)^c : A \in \mu \right\} \quad \text{forms a fuzzy}
\]
\(W\)-open filter bases in \(X\). Let there exists a finite subfamily \(\left\{ \left( W - \text{Cl}(A)^c \right)^c : A \in \eta \right\}\) such that
\[
\left( \bigcap_{A \in \eta} W - \text{Cl}(A)^c \right)^c \bar{q} U. \quad \text{Then}
\]
\[
U \leq \bigcup_{A \in \eta} W - \text{Cl}(A). \quad \text{So there exists a finite subfamily}
\]
\(\eta \subseteq \mu\) such that \(\bigcup_{A \in \eta} W - \text{Cl}(A) \supseteq U\) which is a contradiction. Then for each finite subfamily \(\lambda = \left\{ \left( W - \text{Cl}(A)^c \right)^c : A \in \eta \right\}\) of \(\xi\), we have
\[
\left( \bigcap_{A \in \eta} W - \text{Cl}(A)^c \right)^c \bar{q} U. \quad \text{Hence by the given condition}
\]
\[
\left( \bigcap_{A \in \mu} W - \text{Cl}(W - \text{Cl}(A))^c \right) \cap U \neq 0_x, \quad \text{so there exists}
\]
\(x \in S(U)\) such that
\[
\left( \bigcap_{A \in \mu} W - \text{Cl}(W - \text{Cl}(A))^c \right)(x) > 0_x. \quad \text{Then}
\]
\[
\left( \bigcup_{A \in \mu} \left( W - \text{Cl}(W - \text{Cl}(A))^c \right)^c \right)(x) =
\]
\[
\left( \bigcup_{A \in \mu} W - \text{Int}(W - \text{Cl}(A)) \right)(x) < 1_x, \quad \text{and hence}
\]
\[
\left( \bigcup_{A \in \mu} A \right)(x) < 1_x \quad \text{which contradicts the fact that } \mu \quad \text{is a fuzzy } W\quad \text{-open set cover of } U. \quad \text{Therefore } U \quad \text{is fuzzy}
\]
\(W\)-closed relative to \(X\).

**Conclusion**

In this paper, we have introduced the concepts of fuzzy \(W\)-compactness and fuzzy \(W\)-closed spaces and have investigated their several properties by making use of fuzzy filter bases. We can extend this concept to introduce the notions of fuzzy \(W\)-generalized compactness and fuzzy \(W\)-generalized closed spaces and may also investigate several characterizations of these new notions via fuzzy filter bases.

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