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Logarithmic Divergence Measure for Fuzzy Matrix and Application

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Abstract:- This paper introduces a new divergence measure for a fuzzy matrix with proof of its validity. In addition, the properties are proved for the new fuzzy divergence measure. A method to solve decision making problem is developed by using the proposed fuzzy divergence measure. Finally, the application of this fuzzy divergence measure to decision making is shown using real-life example.

Keywords: Fuzzy Sets, Fuzzy Matrices, Divergence Measures, Decision Making Problem.

I. Introduction

A general problem in all fields like Mathematics, Science and Engineering is to distinguish two probability distributions $P = (p_1, p_2, \dots p_n)$ and $Q = (q, q_2, \dots q_n)$. Measurement of this distance in Information theory is called divergence measure. Information theoretic measures (entropy, similarity measures and divergence measures) are consequential tools to manage the precarious information and successfully applied in different directions. Fuzziness or degree of diceyness is measured by the tool called Entropy. De Luca and Termini (1972) commenced the fuzzy entropy measures to define the uncertainty between fuzzy sets. Subsequently, entropy measures of fuzzy sets received great curiosity from researchers in different areas like analysis of contingency table by Gokhle and Kullback(1978), in approximation of probability distributions by Chao and Liu (1968), Kazakos and Cotsidas (1980), Lin and Wong (1988), in signal processing Kadota ansd Shepp (1967) and in pattern recognition by Ben (1978), Chen (1973) and in many other areas. To measure the distinct information between fuzzy sets, essentially Bhandari and Pal (1993) explored fuzzy divergence measure. In due course, various divergence measures developed and modified by kapur (1997), Parkash (2000), Rosenfeld (1985) for fuzzy sets. Bhandari and Pal(1993) introduced divergence measure for fuzzy sets, defined $D(A, A^{near})$ ratio to $D(A, A^{far})$ an ambiguity measure and α-order fuzzy entropy using Renyi's α-order probabilistic entropy. Afterwards, various scholars have consciousness on divergence measure for fuzzy sets. Corresponding to information measure of Bhandari and Pal (1993), Fan and Xie (1999) introduce exponential fuzzy divergence measure based on operations. They also studied its relation with divergence measure introduced by Bhandari and Pal (1993). Montes et al. (2002) studied the special classes of divergence measures and used the link between fuzzy and probabilistic uncertainty. Bajaj and Hooda (2010) generalized the measure of fuzzy directed divergence. Fuzzy directed divergence measure has wide range of applications in different areas. Sharma et al. (2020) proposed a non-probabilistic divergence measure for fuzzy matrices and applied the same in decision making problem and in feature selection problem. Rani et al. (2020) studied an information

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measure and found application of fuzzy soft matrices. Vlachos and Sergiadis (2007) introduced the divergence measure for IFSs, studied the relationship between divergence measure and entropy measures and found the application in image segmentation, pattern recognition and medical diagnosis.

The remaining part of paper is organized as follows. Section 2 is devoted to introduce some conventional concepts, and notions related to fuzzy set theory and fuzzy matrix theory. In section 3, we proposed a new fuzzy divergence measure for fuzzy matrix corresponding to Bhandari and Pal (1993). Section 4 provides more dignified properties of the proposed measure in form of theorems. It is followed by the applications of the proposed divergence measure for the IFM to multi-criteria decision making (MCDM) and multi-attribute decision making (MADM) with a numerical example in section 5. Finally, some concluding remarks are drawn in section 6.

II. Preliminaries

This section is completely devoted to explain some basic concepts and the assumptions of fuzzy set theory and fuzzy matrix theory. Then, we recall axiomatic definition of fuzzy divergence measure.

A. Fuzzy set:

(Rani et al. (2020)) The linguistic values of the alternatives assessment are usually represented by fuzzy sets to deal with uncertainty of real –world problems. Fuzzy sets are the sets having degree of membership as an element of the set. Zadeh (1965) acquainted the fuzzy sets as the expansion of the crisp sets.

Definition: (2020) A fuzzy set ḿ(ỳ) on Ủ is defined by a membership function ḿ(ỳ) : Ủ → [0,1] . For ỳ ∈ Ủ, ḿ (ỳ) the membership function denotes the degree to which ỳ belongs to fuzzy set ḿ.

B. Fuzzy Matrix Theory:

The basic idea behind the fuzzy matrix theory is very elementary and an easily applicable in all types of circumstances. The algorithms and algebra of fuzzy matrix theory are applicable for data related problems. Social scientists apply this approach to analyze interactions between attributes and to analyze other analytical tools.

Definition: Fuzzy matrix (2020): A fuzzy matrix A of order $m \times n$ is defined as $A = [< a_{ij}, >]_{m\times n}$ where a_{ij} is the membership value of the element a_{ij} in A. We may write A as $A = [a_{ij}]_{m\times n}$. For example,

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.5 \\ 0.2 & 0.1 & 0.9 \end{bmatrix}$$

is 3×3 fuzzy matrix.

Definition: Let $A = [a_{ij}] \in [F(\mathrm{M})]_{m\times n}$, If $m \neq n$, then A is called a fuzzy rectangular matrix.

Definition: Let $A = [a_{ij}] \in [F(\mathrm{M})]_{m\times n}$, If $m = n$, then A is called a fuzzy square matrix.

Definition: Let $A = [a_{ij}] \in [F(\mathrm{M})]_{m\times n}$, If $n = 1$, then A is called a fuzzy column matrix. For example,

$$A = \begin{bmatrix} 0.2 \\ 07 \\ 0.5 \end{bmatrix}$$

is 3×1 column fuzzy matrix.

Definition: Let A$= [a_{ij}] \in [F(\mathrm{M})]_{m\times n}$, If $m = 1$, then A is called a fuzzy row matrix. For example,

$$A = [0.1 \quad 0.5 \quad 0.8]$$

is 1×3 row matrix.

Operations on Two Fuzzy Matrices

Here we performed some operation on fuzzy matrices. Two fuzzy matrices A and B of order 3×3 are taken for further operations as follows.

$$A = \begin{bmatrix} 0.7 & 0.5 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.3 & 0.7 & 0.6 \end{bmatrix} \quad (2.2.1)$$

$$B = \begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.5 & 0.2 & 0.6 \\ 0.8 & 0.7 & 0.2 \end{bmatrix} \quad (2.2.2)$$

Union (Addition) Operation of Two Fuzzy Matrices

Definition (2020): Let $A = [a_{ij}]$, $B = [b_{ij}] \in [F(\mathrm{M})]_{m\times n}$.Then Union of fuzzy matrix A, B is defined by $(A_{m\times n} \cup B_{m\times n}) = C_{m\times n} = [c_{ij}]_{m\times n}$, where $c_{ij} = \max(a_{ij}, b_{ij})$ for all i and j.

Example: The union operation of two matrices given by (2.2.1) and (2.2.2) is

$$(A_{3\times 3} \cup B_{3\times 3}) = C_{3\times 3} = \begin{bmatrix} 0.7 & 0.5 & 0.2 \\ 0.5 & 0.2 & 0.6 \\ 0.8 & 0.7 & 0.6 \end{bmatrix}$$

Intersection Operation of Two Fuzzy Matrices

Definition (2020): Let $A = [a_{ij}]$, $B = [b_{ij}] \in [F(\mathrm{M})]_{m\times n}$. Then intersection of A, B is defined by $(A_{m\times n} \cap B_{m\times n}) = C_{m\times n} = [c_{ij}]_{m\times n}$, where $c_{ij} = min(a_{ij}, b_{ij})$ for all i and j.

Example: The minimum operation of two matrices given by (2.2.1) and (2.2.2) is

$$(A_{3\times 3} \cap B_{3\times 3}) = C_{3\times 3} = \begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.1 \\ 0.3 & 0.7 & 0.2 \end{bmatrix}$$

Some researchers used suprimum and infimum operation as an analogous to our usual maximum and minimum.

Minimum-Maximum Operation of Two Fuzzy Matrices

Definition: Let $A = [a_{ij}] \in [F(\mathrm{M})]_{m\times n}$ & $B = [b_{ji}] \in [\mathrm{F(M)}]_{n\times m}$.

Then Min-Max operation of A, B is defined by $min - max(A_{m\times n}, B_{m\times n}) = C_{m\times m} = [c_{ij}]_{m\times m}$, where $c_{ij} = \min\{\max[(a_{ij}, b_{ji})\ for\ j = 1\ to\ n]\}\ for\ i = 1\ to\ m$.

Conjugate (Complement) of Fuzzy Matrix

Definition (2020): Let $A = [a_{ij}]$, $B = [b_{ij}] \in [\mathrm{F(M)}]_{m\times n}$, then A is conjugate (complement) of B denoted by $B^c = A = [a_{ij}]$, where $a_{ij} = 1 - b_{ij}$ for all i and j.

Example: The conjugate (complement) operation of matrix given by (2.2.1) is

$$A^c = B = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0.7 & 0.8 & 0.9 \\ 0.7 & 0.3 & 0.4 \end{bmatrix}$$

Maximum-Minimum Operation of Two Fuzzy Matrices

Definition (2020): Let $A = [a_{ij}] \in [F(\mathrm{M})]_{m\times n}$ & $B = [b_{ji}] \in [\mathrm{F(M)}]_{n\times m}$

Then Max-Min operation of A, B is defined by $max - min(A_{m\times n}, B_{n\times m}) = C_{m\times m} = [c_{ij}]_{m\times m}$, where $c_{ij} = \max\{\min[(a_{ij}, b_{ji})\ for\ j = 1\ to\ n]\}\ for\ i = 1\ to\ m$.

Example: The max-min operation of two matrices given by (2.2.1) and (2.2.2) is

$$max - min(A_{3\times 3}, B_{3\times 3}) = C_{3\times 3} = \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0.2 & 0.3 & 0.2 \\ 0.6 & 0.6 & 0.6 \end{bmatrix}$$

where

$a_{11} = max\{\min(0.7, 0.2), \min(0.5, 0.5), \min(0.2, 0.8)\} = max\{0.2, 0.5, 0.2\} = 0.5$

$a_{12} = max\{\min(0.7, 0.3), \min(0.5, 0.2), \min(0.2, 0.7)\} = max\{0.3, 0.2, 0.2\} = 0.3$.

and so on

Example: The min-max operation of two matrices given by (2.2.1) and (2.2.2) is

$$Min - Max(A_{3\times 3}, B_{3\times 3}) = C_{3\times 3} = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 0.3 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$$

where, $a_{11} = min\{\max(0.7, 0.2), \max(0.5, 0.5), \max(0.2, 0.8)\} = max\{0.7, 0.5, 0.8\} = 0.5$

a_{12}
$= min\{\max(0.7, 0.3), \max(0.5, 0.2), \max(0.2, 0.7)\}$
$= min\{0.7, 0.5, 0.7\} = 0.5.$
and so on.

III. Fuzzy Divergence Measure

Zadeh (1968) defined a measure of information for a fuzzy set known as fuzzy entropy, which is different from the classical Shannon (1948) entropy, as follows

$$Z_P(A) = -\sum_{i=1}^{n} \mu_A(x_i) p_i \log p_i \qquad (3.1)$$

Later on, De Luca and Termini (1972) provided the following measure of fuzzy entropy:

$$H_{DT}(A) = -\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i) log \mu_A(x_i) + (1-\mu_A(x_i)) log(1-\mu_A(x_i))\right] \qquad (3.2)$$

They also pointed out that it is quite distinctive from Shannon's information measures because no probabilistic concept is desirable to characterize it. After the revolutionary work of De Luca and Termini (1972), various non-probabilistic information measures have been insinuated by researchers and gave applications in various fields. For a random variable X with probability distribution $P = (p_1, p_2, \ldots \ldots, p_n)$, Pal and Pal [1989] introduced the exponential entropy as:

$$H_e(P) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n} p_i (e^{1-p_i} - 1) \qquad (3.3)$$

Later on, Pal and Pal (1992) defined an exponential information measure for fuzzy entropy, analogous to equation (3.3) as:

$$H_e(A) = \frac{1}{n\sqrt{e}-1}\sum_{i=1}^{n}\left(\mu_A(x_i) e^{1-\mu_A(x_i)} + (1-\mu_A(x_i)) e^{\mu_A(x_i)} - 1\right) \qquad (3.4)$$

Bhandari and Pal (1993) described the idea of fuzzy directed divergence considering the elementary properties of Kullback and Leibler (1951) directed divergence measure. Fuzzy divergence measure as developed by Bhandari and Pal, gives a fuzzy information measure for distinction of a two fuzzy set.
Bhandari and Pal (1993) defined the fuzzy information for discrimination in favour of fuzzy set A against B by

$$I(A:B) = \sum_{i=1}^{n}\left[\mu_A(x_i) log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i)) log \frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right] \qquad (3.5)$$

Bhandari and Pal (1993) provide the fuzzy entropy by using $I(A, A_F)$ as follows:

$$H_{DT}(A) = 1 - \frac{1}{n\, ln2} I(A, A_F) \qquad (3.6)$$

Bhandari and Pal (1993) defined a divergence measure between two fuzzy sets R & S as follows:

$$D(R,S) = I(R,S) + I(S,R) = \sum_{i=1}^{n}\left[(\mu_R(x_i) - \mu_S(x_i)) log \frac{\mu_R(x_i)}{\mu_S(x_i)} + (\mu_S(x_i) - \mu_R(x_i)) log \frac{1-\mu_R(x_i)}{1-\mu_S(x_i)}\right] \qquad (3.7)$$

Definition: Shang and Jiang (1997) pointed out that the measure in Equation (3.7) has a drawback, i.e., when $\mu_S(x_i)$ approaches 0 or 1 for any value of x_i, its value tends toward infinity. Therefore, using the idea of Lin (1991) divergence measure as in equation (3.4), for $R, S \in FS(X)$, Shang and Jiang extended a fuzzy divergence measure (3.7), as:

$$D_{SJ}(R,S)\sum_{i=1}^{n}\left[\begin{array}{c}\left((\mu_R(x_i)-\mu_S(x_i))ln\frac{\mu_R(x_i)}{\left(\frac{\mu_R(x_i)+\mu_S(x_i)}{2}\right)}\right)\\+\left((\mu_S(x_i)-\mu_R(x_i))ln\frac{1-\mu_R(x_i)}{\left(\frac{2-\mu_R(x_i)-\mu_S(x_i)}{2}\right)}\right)\end{array}\right] \quad (3.8)$$

Some other measure of fuzzy directed divergence measures have been introduced and studied by Pal and Bezdek (1994), Kapur (1997), Fan and Xie (1999), Hooda (2004), Bajaj and Hooda (2010), Kumar et al. (2011), Bhatia and Singh (2013), Tomar and Ohlan (2014), Verma and Sharma (2014), Gupta and Santosh (2014), Ohlan (2015) and etc. These fuzzy divergence measures have application in various fields including decision making problems, fuzzy clustering, artificial intelligence etc.

IV. LOGARITHMIC DIVERGENCE MEASURE FOR FUZZY MATRIX

In this section, a divergence measure for fuzzy matrices R & S is proposed which is of logarithmic form. The validity of the proposed divergence measure is verified.

Here we define a fuzzy divergence measure for fuzzy matrices:

Definition (2020): Let F_M be the set of all fuzzy matrices having m rows and n columns and X & $Y \in F_M$. Then a mapping $J : F_M \times F_M \rightarrow A$ is called non-probabilistic divergence measure of fuzzy matrices if and only if

a. $J(X:Y) \geq 0$

b. $J(X:Y) = 0$ when X & Y are equal fuzzy matrix

c. $J(X:Y) = J(Y:X)$ i.e. divergence measure is symmetric in nature.

d. $J(X:Y)$ is convex in X and Y.

Thus we say that a measure is information theoretic divergence measure for fuzzy matrices if it satisfies axioms (a) to (d).

Here we proposed a divergence measure for fuzzy matrices X and Y of order $m \times n$ which is logarithmic in nature as follows:

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{\left((1-x_{ij})-(1-y_{ij})\right)log\frac{1-x_{ij}}{1-y_{ij}}\right\}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right] \quad (4.1)$$

where $x_{ij} \in X$ & $y_{ij} \in Y$

To show that the proposed measure is a valid measure since it satisfies all the above four axioms which are proving by following theorem:

Theorem 4.1: $J(X:Y) \geq 0$ if $X\ and\ Y \in [F_M]_{m\times n}$.

Proof: It is trivial that the measure is non-negative for each a & b (where $a = x_{ij} \in X$ & $b = y_{ij} \in Y$).

Theorem 4.2: $J(X:Y) = 0$ if $X = Y$ or $x_{ij} = y_{ij}$.

Proof: We have

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

And if $X = Y$ or $x_{ij} = y_{ij}$ then

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-x_{ij})log\frac{x_{ij}}{x_{ij}}+(x_{ij}-x_{ij})log\frac{1-x_{ij}}{1-x_{ij}}\right]$$

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}[0+0] = \sum_{i=1}^{m}\sum_{j=1}^{n}[0]$$

$$J(X:Y) = 0$$

Hence proved.

Theorem 4.3: $J(X:Y) = J(Y:X)$.

Proof: To prove we show $J(X:Y) - J(Y:X) = 0$

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$J(Y:X) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(y_{ij}-x_{ij})log\frac{y_{ij}}{x_{ij}}\right\}+\left\{(x_{ij}-y_{ij})log\frac{1-y_{ij}}{1-x_{ij}}\right\}\right]$$

$$J(X:Y)-J(Y:X) = \sum_{i=1}^{m}\sum_{j=1}^{n}\begin{bmatrix}(x_{ij}-y_{ij})\left(log\frac{x_{ij}}{y_{ij}}+log\frac{1-y_{ij}}{1-x_{ij}}\right)\\+(y_{ij}-x_{ij})\left(log\frac{y_{ij}}{x_{ij}}+log\frac{1-x_{ij}}{1-y_{ij}}\right)\end{bmatrix}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\begin{bmatrix}(x_{ij}-y_{ij})\left(log\frac{x_{ij}}{y_{ij}}+log\frac{1-y_{ij}}{1-x_{ij}}\right)\\-(x_{ij}-y_{ij})\left(log\frac{y_{ij}}{x_{ij}}+log\frac{1-x_{ij}}{1-y_{ij}}\right)\end{bmatrix}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\left(\left(log\frac{x_{ij}}{y_{ij}}+log\frac{1-y_{ij}}{1-x_{ij}}\right)-\left(log\frac{y_{ij}}{x_{ij}}+log\frac{1-x_{ij}}{1-y_{ij}}\right)\right)\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\left(\left(log\frac{x_{ij}}{y_{ij}}\frac{1-y_{ij}}{1-x_{ij}}\right)-\left(log\frac{y_{ij}}{x_{ij}}\frac{1-x_{ij}}{1-y_{ij}}\right)\right)\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\left(log\frac{x_{ij}}{y_{ij}}\frac{1-y_{ij}}{1-x_{ij}}\frac{y_{ij}}{x_{ij}}\frac{1-x_{ij}}{1-y_{ij}}\right)\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\log(1)\right]$$

$$J(X:Y)-J(Y:X) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\ (0)\right] = \sum_{i=1}^{m}\sum_{j=1}^{n}[0]$$

$$J(X:Y)-J(Y:X) = 0$$

Theorem 4.4: $J(X:Y)$ is convex in X and Y.

Proof: First we prove $J(X:Y)$ convex in X.

$$J(X:Y)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$\frac{\partial J(X:Y)}{\partial x_{ij}}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\left(\frac{1}{x_{ij}}\right)+log\frac{x_{ij}}{y_{ij}}+(y_{ij}-x_{ij})\left(\frac{-1}{1-x_{ij}}\right)+log\frac{1-x_{ij}}{1-y_{ij}}(-1)\right]$$

When $X=Y\ or\ x_{ij}=y_{ij}$, then

$$\frac{\partial J(X:Y)}{\partial x_{ij}}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-x_{ij})\left(\frac{1}{x_{ij}}\right)+log\frac{x_{ij}}{x_{ij}}+(x_{ij}-x_{ij})\left(\frac{-1}{1-x_{ij}}\right)+log\frac{1-x_{ij}}{1-x_{ij}}(-1)\right]$$

$$\frac{\partial J(X:Y)}{\partial x_{ij}}=\sum_{i=1}^{m}\sum_{j=1}^{n}[0+\log(1)+0-\log(1)]=0$$

$$\frac{\partial^2 J(X:Y)}{\partial {x_{ij}}^2}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\left(\frac{-1}{x_{ij}^2}\right)+\left(\frac{1}{x_{ij}}\right)+\left(\frac{1}{x_{ij}}\right)+(y_{ij}-x_{ij})\left(\frac{-1}{(1-x_{ij})^2}\right)\right.$$
$$\left.+\left(\frac{1}{1-x_{ij}}\right)+\left(\frac{1}{1-x_{ij}}\right)\right]$$

$$\frac{\partial^2 J(X:Y)}{\partial {x_{ij}}^2}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})\left(\frac{-1}{x_{ij}^2}\right)+\left(\frac{2}{x_{ij}}\right)+(y_{ij}-x_{ij})\left(\frac{-1}{(1-x_{ij})^2}\right)+\left(\frac{2}{1-x_{ij}}\right)\right]$$

When $X=Y\ or\ x_{ij}=y_{ij}$, then

$$\frac{\partial^2 J(X:Y)}{\partial {x_{ij}}^2}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-x_{ij})\left(\frac{-1}{x_{ij}^2}\right)+\left(\frac{2}{x_{ij}}\right)+(x_{ij}-x_{ij})\left(\frac{-1}{(1-x_{ij})^2}\right)+\left(\frac{2}{1-x_{ij}}\right)\right]$$

$$\frac{\partial^2 J(X:Y)}{\partial {x_{ij}}^2}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[0+\left(\frac{2}{x_{ij}}\right)+0+\left(\frac{2}{1-x_{ij}}\right)\right]=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(\frac{2}{x_{ij}}\right)+\left(\frac{2}{1-x_{ij}}\right)\right]$$

$$\frac{\partial^2 J(X:Y)}{\partial {x_{ij}}^2}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{2(1-x_{ij}+x_{ij})}{x_{ij}(1-x_{ij})}\right]=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{2(1)}{x_{ij}(1-x_{ij})}\right]>0$$

$$\frac{\partial^2 J(X:Y)}{\partial {x_{ij}}^2}=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{2(1)}{x_{ij}(1-x_{ij})}\right]>0\ where\ x_{ij}>0$$

The proof of the theorem shows $J(X:Y)$ is a convex function of X.

Similarly, we can show that $J(X:Y)$ is a convex function of Y.

The proposed divergence measure is a valid divergence measure of fuzzy matrices, satisfies all the four axioms.

V. Properties of Fuzzy Divergence Measure

In this section, some properties of proposed measure are proved in the form of theorems.

Theorem 5.1: Let $X\ and\ Y \in [F_M]_{m\times n}$, then the following properties are satisfied by $J(X:Y)$.

a. $J(X:Y)=J(X^c:Y^c)$
b. $J[(X\cup Y):(X\cap Y)]=J(X:Y)$
c. $J(X:(X\cup Y))=J(Y:(X\cap Y))$

d. $J(X:(X\cap Y)) = J(Y:(X\cup Y)$

Proof: To prove these theorems, we define both fuzzy matrices into two sets as given below:

$$\left.\begin{aligned} S_1 &= \{x_{ij}\ or\ y_{ij}\ ;\ x_{ij} \in X\ or\ y_{ij} \in Y;\ x_{ij}\ \geq\ y_{ij}\} \\ S_2 &= \{x_{ij}\ or\ y_{ij}\ ;\ x_{ij} \in X\ or\ y_{ij} \in Y;\ x_{ij}\ <\ y_{ij}\} \end{aligned}\right\}$$

(a) We have

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$J(X^c:Y^c) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left((1-x_{ij})-(1-y_{ij})\right)log\frac{1-x_{ij}}{1-y_{ij}}+\left\{\left((1-y_{ij})-(1-x_{ij})\right)log\frac{1-(1-x_{ij})}{1-(1-y_{ij})}\right\}\right]$$

$$J(X^c:Y^c)\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}+(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right]$$

$J(X^c:Y^c) = J(X:Y)$

(b) We have

$$J[(X\cup Y):(X\cap Y)] = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}\left(\max(x_{ij},y_{ij})-\min(x_{ij},y_{ij})log\frac{\max(x_{ij},y_{ij})}{\min(x_{ij},y_{ij})}\right)\\ +\left(\min(x_{ij},y_{ij})-\max(x_{ij},y_{ij})\right)log\frac{1-\max(x_{ij},y_{ij})}{1-\min(x_{ij},y_{ij})}\end{array}\right]$$

$$= \sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right] + \sum_{x_{ij},y_{ij}\in S_2}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$= \left\{\sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}\right] + \sum_{x_{ij},y_{ij}\in S_2}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}\right]\right\} + \left\{\sum_{x_{ij},y_{ij}\in S_1}\left[(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right]+\sum_{x_{ij},y_{ij}\in S_2}\left[(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right]\right\}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right] + \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right] = J(X:Y)$$

Hence proved.

(c) $J\big(X:(X\cup Y)\big) = J(Y:(X\cap Y))$

Taking left hand side, we have

$$J\big(X:(X\cup Y)\big)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}\left(x_{ij}-\max(x_{ij},y_{ij})\right)log\dfrac{x_{ij}}{\max(x_{ij},y_{ij})}\\+\left(\max(x_{ij},y_{ij})\right)-x_{ij}\big)log\dfrac{1-x_{ij}}{1-\max(x_{ij},y_{ij})}\end{array}\right]$$

$$=\sum_{x_{ij},y_{ij}\in S_1}\left[\left(x_{ij}-x_{ij}\right)log\frac{x_{ij}}{x_{ij}}+\left(x_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-x_{ij}}\right]$$

$$+\sum_{x_{ij},y_{ij}\in S_2}\left[\left\{\left(x_{ij}-y_{ij}\right)log\frac{x_{ij}}{y_{ij}}\right\}+\left\{\left(y_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$=\sum_{x_{ij},y_{ij}\in S_2}\left[\left\{\left(x_{ij}-y_{ij}\right)log\frac{x_{ij}}{y_{ij}}\right\}+\left\{\left(y_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

Taking R.H.S,

$$J(Y:(X\cap Y))=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{\left(\min(x_{ij},y_{ij})-y_{ij}\right)log\frac{\min(x_{ij},y_{ij})}{y_{ij}}\right\}\right.$$

$$\left.+\left\{\left(y_{ij}-\min(x_{ij},y_{ij})\right)log\frac{1-\min(x_{ij},y_{ij})}{1-y_{ij}}\right\}\right]$$

$$=\sum_{x_{ij},y_{ij}\in S_1}\left[\left(y_{ij}-y_{ij}\right)log\frac{y_{ij}}{y_{ij}}+\left(y_{ij}-y_{ij}\right)log\frac{1-y_{ij}}{1-y_{ij}}\right]$$

$+\sum_{x_{ij},y_{ij}\in S_2}\left[\left\{\left(x_{ij}-y_{ij}\right)log\frac{x_{ij}}{y_{ij}}\right\}+\left\{\left(y_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$,$=\sum_{x_{ij},y_{ij}\in S_2}\left[\left\{\left(x_{ij}-y_{ij}\right)log\frac{x_{ij}}{y_{ij}}\right\}+\left\{\left(y_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$

L.H.S = R.H.S.

(d) $J(X:(X\cap Y))=J(Y:(X\cup Y))$

Evaluating L.H.S. ,

$$J(X:(X\cap Y))=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}\left(x_{ij}-\min(x_{ij},y_{ij})\right)log\dfrac{x_{ij}}{\min(x_{ij},y_{ij})}\\+\left(\min(x_{ij},y_{ij})-x_{ij}\right)log\dfrac{1-x_{ij}}{1-\min(x_{ij},y_{ij})}\end{array}\right]$$

$$=\sum_{x_{ij},y_{ij}\in S_1}\left[\left\{\left(x_{ij}-y_{ij}\right)log\frac{x_{ij}}{y_{ij}}\right\}+\left\{\left(y_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$+\sum_{x_{ij},y_{ij}\in S_2}\left[\left(x_{ij}-x_{ij}\right)log\frac{x_{ij}}{x_{ij}}+\left(x_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-x_{ij}}\right]$$

$$=\sum_{x_{ij},y_{ij}\in S_1}\left[\left\{\left(x_{ij}-y_{ij}\right)log\frac{x_{ij}}{y_{ij}}\right\}+\left\{\left(y_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

Taking R.H.S. ,

$$J(Y:(X \cup Y)) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}(\max(x_{ij},y_{ij})-y_{ij})log\frac{\max(x_{ij},y_{ij})}{y_{ij}} \\ +(y_{ij}-\max(x_{ij},y_{ij}))log\frac{1-\max(x_{ij},y_{ij})}{1-y_{ij}}\end{array}\right]$$

$$= \sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$+ \sum_{x_{ij},y_{ij}\in S_2}\left[(x_{ij}-x_{ij})log\frac{x_{ij}}{x_{ij}}+(x_{ij}-x_{ij})log\frac{1-x_{ij}}{1-x_{ij}}\right]$$

$$= \sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

Thus L.H.S = R.H.S.

Corollary 5.1: If $X \; and \; Y \in [F_M]_{m\times n}$ then we have

$J(X:(X\cup Y)) + J(X:(X\cap Y)) = J(X:Y)$

Proof: From the part c and d of theorem (5.1), we have

As we know.

$$J(X:(X\cup Y)) = \sum_{x_{ij},y_{ij}\in S_2}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

And also,

$$J(X:(X\cap Y)) = \sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

And now,

$$J(X:(X\cup Y)) + J(X:(X\cap Y)) = \left[\begin{array}{l}\sum_{x_{ij},y_{ij}\in S_2}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right] \\ +\sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]\end{array}\right]$$

$$J(X:(X\cup Y)) + J(X:(X\cap Y)) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$= J(X:Y)$$

Hence Proved.

Corollary 5.2: If $X \; and \; Y \in [F_M]_{m\times n}$ then we have

$J(Y:(X\cup Y)) + J(Y:(X\cap Y)) = J(X:Y)$

Proof: Again from the part c and d of theorem (5.1) we can prove it easily.

As we know that,

$$J(Y:(X\cap Y)) = \sum_{x_{ij},y_{ij}\in S_2}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

And also,

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$$J\big(Y:(X\cup Y)\big) = \sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

And now,

$$J\big(Y:(X\cup Y)\big)+J(Y:(X\cap Y)) = \begin{bmatrix}\sum_{x_{ij},y_{ij}\in S_2}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]\\ +\sum_{x_{ij},y_{ij}\in S_1}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]\end{bmatrix}$$

$$J\big(Y:(X\cup Y)\big)+J(Y:(X\cap Y)) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

$$= J(X:Y)$$

Hence Proved.

Corollary 5.3: If $X\ and\ Y \in [F_M]_{m\times n}$ then we have

$J((X\cap Y):X) = J((X\cup Y):Y) \le J(X:Y)$

Proof: From symmetric property of divergence measure

$J(X:(X\cap Y)) = J((X\cap Y):X)$ & $J((X\cup Y):Y)$
$= J(Y:(X\cup Y))$

Now from part (d) of theorem 5.1 we have,

$J(X:(X\cap Y)) = J(Y:(X\cup Y)$

Hence,

$J((X\cap Y):X) = J((X\cup Y):Y)$

Now by using corollary (5.1) we have,

$$J\big(X:(X\cup Y)\big)+J(X:(X\cap Y)) = J(X:Y)$$
$$J((X\cap Y):X) = J(X:(X\cap Y))$$
$$= J(X:Y) - J\big(X:(X\cup Y)\big)$$

Thus,

$$J((X\cap Y):X) = J((X\cup Y):Y) \le J(X:Y)$$

Hence proved.

Corollary 5.4: If $X\ and\ Y \in [F_M]_{m\times n}$ then we have

$$J\big((X\cup Y):X\big) = J((X\cap Y):Y) \le J(X:Y)$$

Proof: By symmetric properties of divergence measure,

$J\big(X:(X\cup Y)\big) = J\big((X\cup Y):X\big)$ & $J((X\cap Y):Y)$
$= J(Y:(X\cap Y))$

Now by using part (c) of theorem (5.1) we have,

$$J\big(X:(X\cup Y)\big) = J(Y:(X\cap Y))$$

Hence,

$$J\big((X\cup Y):X\big) = J((X\cap Y):Y)$$

Now using corollary (5.2) we have,

$$J\big(Y:(X\cup Y)\big)+J(Y:(X\cap Y)) = J(X:Y)$$
$$J((X\cap Y):Y) = J(Y:(X\cap Y))$$
$$= J(X:Y) - J\big(Y:(X\cup Y)\big)$$

Therefore,

$$J\big((X\cup Y):X\big) = J((X\cap Y):Y) \le J(X:Y)$$

Hence Proved.

Theorem 5.2: Let $X\ and\ Y \in [F_M]_{m\times n}$ then the following properties are satisfied by $J(X:Y)$.

a. $J(X:Y^c) = J(X^c:Y)$
b. $J(X:X^c) = 0$ when $x_{ij} = 0\ or\ 1$ for all $i\ and\ j$.
c. $J(X:X^c) = 0$ when $x_{ij} = 0.5$ for all $i\ and\ j$.

Proof: We have

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-y_{ij})log\frac{x_{ij}}{y_{ij}}\right\}+\left\{(y_{ij}-x_{ij})log\frac{1-x_{ij}}{1-y_{ij}}\right\}\right]$$

(a) First taking left hand side

$$J(X:Y^c) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\big(x_{ij}-(1-y_{ij})\big)log\frac{x_{ij}}{1-y_{ij}}+\big((1-y_{ij})-x_{ij}\big)\,log\frac{1-x_{ij}}{1-(1-y_{ij})}\right]$$

$$J(X{:}Y^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(x_{ij}+y_{ij}-1\right)log\frac{x_{ij}}{1-y_{ij}}+\left(1-y_{ij}-x_{ij}\right)log\frac{1-x_{ij}}{y_{ij}}\right]$$

Now, taking right hand side

$$J(X^c{:}Y)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(\left(1-x_{ij}\right)-y_{ij}\right)log\frac{1-x_{ij}}{y_{ij}}+\left(y_{ij}-(1-x_{ij})\right)log\frac{1-(1-x_{ij})}{(1-y_{ij})}\right]$$

$$J(X^c{:}Y)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(\left(1-x_{ij}-y_{ij}\right)\right)log\frac{1-x_{ij}}{y_{ij}}+\left(x_{ij}+y_{ij}-1\right)log\frac{x_{ij}}{(1-y_{ij})}\right]$$

L.H.S. = R.H.S.

(b) We have

$$J(X{:}X^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(x_{ij}-(1-x_{ij})\right)log\frac{x_{ij}}{(1-x_{ij})}+\left(\left(1-x_{ij}\right)-x_{ij}\right)log\frac{(1-x_{ij})}{1-(1-x_{ij})}\right]$$

$$J(X{:}X^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(-1)log\frac{x_{ij}}{\left(1-x_{ij}\right)}+(1-2x_{ij})log\frac{1-x_{ij}}{x_{ij}}\right]$$

When $x_{ij}=0\ or\ 1\ or\ 0.5$ for all $i\ and\ j$.

$$J(X{:}X^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}[0]=0$$

Hence Proved.

(c) As we know that,

$$J(X{:}X^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(-1)log\frac{x_{ij}}{\left(1-x_{ij}\right)}+(1-2x_{ij})log\frac{1-x_{ij}}{x_{ij}}\right]$$

When $x_{ij}=0.5$ for all $i\ and\ j$, then we have

$$J(X{:}X^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(-1)log\frac{0.5}{(0.5)}+(-1)log\frac{0.5}{(0.5)}\right]$$

$$J(X{:}X^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}[-2\log(1)]$$

$$J(X{:}X^c)=\sum_{i=1}^{m}\sum_{j=1}^{n}[0]=0$$

Hence proved.

Theorem 5.3: Let $X,\ Y\ and\ Z\in[F_M]_{m\times n}$ then

a) $J(X{:}Z)+J(Y{:}Z)-J\left((X\cup Y){:}Z\right)=J((X\cap Y){:}Z)$
b) $J(X{:}Z)+J(Y{:}Z)-J((X\cap Y){:}Z)=J((X\cup Y){:}Z)$

Proof: To prove this theorem we define fuzzy matrices into two sets as given below:

$$\left.\begin{array}{l}S_1=\left\{x_{ij}\ or\ y_{ij}\ ;\ x_{ij}\in X\ or\ y_{ij}\in Y;\ x_{ij}\ \geq\ y_{ij}\right\}\\ S_2=\left\{x_{ij}\ or\ y_{ij}\ ;\ x_{ij}\in X\ or\ y_{ij}\in Y;\ x_{ij}\ <\ y_{ij}\right\}\end{array}\right\}$$

(a) We have

$$J(X:Z)+J(Y:Z)-J\big((X\cup Y):Z\big)$$

$$=\left\{\begin{array}{l}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(x_{ij}-z_{ij})log\frac{x_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-x_{ij})log\frac{1-x_{ij}}{1-z_{ij}}\right\}\right]\\+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left\{(y_{ij}-z_{ij})log\frac{y_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-y_{ij})log\frac{1-y_{ij}}{1-z_{ij}}\right\}\right]\\-\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}(\max(x_{ij},y_{ij})-z_{ij})log\frac{\max(x_{ij},y_{ij})}{z_{ij}}\\+(z_{ij}-\max(x_{ij},y_{ij}))log\frac{1-\max(x_{ij},y_{ij})}{1-z_{ij}}\end{array}\right]\end{array}\right\}$$

$$=\left\{\begin{array}{l}\sum_{x_{ij},y_{ij},z_{ij}\in S_1}\left(\begin{array}{l}\left[\left\{(x_{ij}-z_{ij})log\frac{x_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-x_{ij})log\frac{1-x_{ij}}{1-z_{ij}}\right\}\right]\\+\left[\left\{(y_{ij}-z_{ij})log\frac{y_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-y_{ij})log\frac{1-y_{ij}}{1-z_{ij}}\right\}\right]\\-\left[\left\{(x_{ij}-z_{ij})log\frac{x_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-x_{ij})log\frac{1-x_{ij}}{1-z_{ij}}\right\}\right]\end{array}\right)\\+\sum_{x_{ij},y_{ij},z_{ij}\in S_2}\left(\begin{array}{l}\left[\left\{(x_{ij}-z_{ij})log\frac{x_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-x_{ij})log\frac{1-x_{ij}}{1-z_{ij}}\right\}\right]\\+\left[\left\{(y_{ij}-z_{ij})log\frac{y_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-y_{ij})log\frac{1-y_{ij}}{1-z_{ij}}\right\}\right]\\-\left[\left\{(y_{ij}-z_{ij})log\frac{y_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-y_{ij})log\frac{1-y_{ij}}{1-z_{ij}}\right\}\right]\end{array}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{l}\sum_{x_{ij},y_{ij},z_{ij}\in S_1}\left[\left\{(y_{ij}-z_{ij})log\frac{y_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-y_{ij})log\frac{1-y_{ij}}{1-z_{ij}}\right\}\right]\\+\sum_{x_{ij},y_{ij},z_{ij}\in S_2}\left[\left\{(x_{ij}-z_{ij})log\frac{x_{ij}}{z_{ij}}\right\}+\left\{(z_{ij}-x_{ij})log\frac{1-x_{ij}}{1-z_{ij}}\right\}\right]\end{array}\right\}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}(\min(x_{ij},y_{ij})-z_{ij})log\frac{\min(x_{ij},y_{ij})}{z_{ij}}\\+(z_{ij}-\min(x_{ij},y_{ij}))log\frac{1-\min(x_{ij},y_{ij})}{1-z_{ij}}\end{array}\right]$$

$=J((X\cap Y):Z)$

Hence Proved.

(b) Using the result of part (a) of this theorem we know that,

$$J(X:Z)+J(Y:Z)-J\big((X\cup Y):Z\big)=J((X\cap Y):Z)$$

Thus we have

a. $J\big((X\cup Y):Z\big)\le J(X:Z)+J(Y:Z)$

$$J(X:Z)+J(Y:Z)-J((X\cap Y):Z)=J\big((X\cup Y):Z\big)$$

Hence Proved.

Corollary 5.5: If $X,\ Y\ and\ Z\in[F_M]_{m\times n}$ then

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b. $J((X \cap Y): Z) \leq J(X: Z) + J(Y: Z)$

Proof:

(a) Using part 'b' of theorem (5.3) we know that,

$$J\left((X \cup Y): Z\right) = J(X: Z) + J(Y: Z) - J((X \cap Y): Z)$$

Since $J((X \cap Y): Z) \geq 0$,

Hence,

$$J\left((X \cup Y): Z\right) \leq J(X: Z) + J(Y: Z)$$

Hence Proved.

(b) Using part 'a' of theorem (5.3) we know that,

$$J((X \cap Y): Z) = J(X: Z) + J(Y: Z) - J\left((X \cup Y): Z\right)$$

Since $J((X \cup Y): Z) \geq 0$

Hence,

$$J((X \cap Y): Z) \leq J(X: Z) + J(Y: Z)$$

Hence Proved.

VI. Application in Decision Making and Data Reduction

The proposed measure has application in various decision making problems as well as in data reduction. Here a real life example is considered:

The Department of Statistics, Maharshi Dayanand University, Rohtak has a problem which software should be introduced for the students. We have collected data about this problem from teachers and also from outside about the software in the form of fuzzy set $\{\dot{S}_1, \dot{S}_2, \dots\dots, \dot{S}_6\}$ having parameters $\{\dot{P}_1, \dot{P}_2, \dots\dots, \dot{P}_6\}$. The data is collected by interview method. There are six software SPSS, C++, R, MATLAB, C language, TORA from which only one is included in the syllabus for students on the basis of selected parameters. We have taken 8 parameters/attributes based on the software chosen. The parameters are

1. Job efficient
2. Latest
3. Availability of tutor
4. Useful in Statistics
5. Cheap
6. Easy to learn
7. Basic
8. Curriculum related

Now, we have chosen only six parameters from all, some biased parameters are removed. They are:

1. Job efficiency
2. Latest
3. Useful in Statistics
4. Cheap
5. Easy to learn
6. Curriculum related

Now for evaluating six softwares, the following six fuzzy sets will be constructed as:

$$\dot{S}_1 = \{(\dot{p}_1, 0.9), (\dot{p}_2, 0.8), (\dot{p}_3, 0.9), (\dot{p}_4, 0.1), (\dot{p}_5, 0.8), (\dot{p}_6, 1.0)\}$$

$$\dot{S}_2 = \{(\dot{p}_1, 0.2), (\dot{p}_2, 0.3), (\dot{p}_3, 0.4), (\dot{p}_4, 1.0), (\dot{p}_5, 0.2), (\dot{p}_6, 0.1)\}$$

$$\dot{S}_3 = \{(\dot{p}_1, 0.9), (\dot{p}_2, 0.8), (\dot{p}_3, 0.8), (\dot{p}_4, 0.3), (\dot{p}_5, 0.9), (\dot{p}_6, 0.9)\}$$

$$\dot{S}_4 = \{(\dot{p}_1, 0.6), (\dot{p}_2, 0.6), (\dot{p}_3, 0.6), (\dot{p}_4, 0.2), (\dot{p}_5, 0.5), (\dot{p}_6, 0.7)\}$$

$$\dot{S}_5 = \{(\dot{p}_1, 0.3), (\dot{p}_2, 0.2), (\dot{p}_3, 0.3), (\dot{p}_4, 1.0), (\dot{p}_5, 0.7), (\dot{p}_6, 0.2)\}$$

$$\dot{S}_6 = \{(\dot{p}_1, 0.4), (\dot{p}_2, 0.7), (\dot{p}_3, 0.7), (\dot{p}_4, 0.7), (\dot{p}_5, 0.7), (\dot{p}_6, 0.8)\}$$

Here, a table is presented, these six sets which is further converted in a fuzzy matrix $'\dot{S}'$ having order (6×6) where rows represent software and column represents parameters as follows:

Software/parameter	$\dot{p}_1$	$\dot{p}_2$	$\dot{p}_3$	$\dot{p}_4$	$\dot{p}_5$	$\dot{p}_6$
$\dot{S}_1$	0.9	0.8	0.9	0.1	0.8	0.8
$\dot{S}_2$	0.2	0.3	0.4	0.9	0.2	0.2
$\dot{S}_3$	0.9	0.8	0.8	0.5	0.9	0.9
$\dot{S}_4$	0.6	0.6	0.6	0.2	0.5	0.5
$\dot{S}_5$	0.3	0.3	0.3	0.9	0.7	0.7
$\dot{S}_6$	0.4	0.7	0.7	0.7	0.7	0.5

$$\dot{S} = \begin{bmatrix} 0.9 & 0.8 & 0.9 & 0.1 & 0.8 & 0.8 \\ 0.2 & 0.3 & 0.4 & 0.9 & 0.2 & 0.2 \\ 0.9 & 0.8 & 0.8 & 0.5 & 0.9 & 0.9 \\ 0.6 & 0.6 & 0.6 & 0.2 & 0.5 & 0.5 \\ 0.3 & 0.3 & 0.3 & 0.9 & 0.7 & 0.7 \\ 0.4 & 0.7 & 0.7 & 0.7 & 0.7 & 0.5 \end{bmatrix}$$

Let $T = \{(\dot{P}_1, 0.8), (\dot{P}_2, 0.6), (\dot{P}_3, 0.9), (\dot{P}_4, 0.8), (\dot{P}_5, 0.5), (\dot{P}_6, 0.9)\}$ be the standard of various parameters for a particular software provided by the head of department.

Then as a fuzzy row matrix T can be represented as:

$$T = [\,0.8 \quad 0.6 \quad 0.9 \quad 0.8 \quad 0.5 \quad 0.9]$$

Now divide the fuzzy matrix $\dot{S}$ into six row matrices as $\{\dot{S}_1, \dot{S}_2, \dots\dots, \dot{S}_6\}$Now we will find the divergence between each row matrix and matrix T by using proposed divergence measure

$$J(X:Y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left\{(x_{ij} - y_{ij})log\frac{x_{ij}}{y_{ij}}\right\} + \left\{(y_{ij} - x_{ij})log\frac{1 - x_{ij}}{1 - y_{ij}}\right\}$$

$$J(\dot{S}_1:T) = \left\{\begin{matrix} \left[(0.9 - 0.8)log\frac{0.9}{0.8} + (0.8 - 0.9)log\frac{1-0.9}{1-0.8}\right] \\ \left[(0.2 - 0.6)log\frac{0.2}{0.6} + (0.2 - 0.6)log\frac{1-0.2}{1-0.6}\right] \\ \left[(0.9 - 0.9)log\frac{0.9}{0.9} + (0.9 - 0.9)log\frac{1-0.9}{1-0.9}\right] \\ \left[(0.6 - 0.8)log\frac{0.6}{0.8} + (0.6 - 0.8)log\frac{1-0.6}{1-0.8}\right] \\ \left[(0.3 - 0.5)log\frac{0.3}{0.5} + (0.3 - 0.5)log\frac{1-0.3}{1-0.5}\right] \\ \left[(0.4 - 0.9)log\frac{0.4}{0.9} + (0.9 - 0.4)log\frac{1-0.4}{1-0.9}\right] \end{matrix}\right\} = 1.42566$$

We need to find divergence of T with other matrices $(\dot{S}_2, \dot{S}_3, \dots\dots.., \dot{S}_6)$ which are given below:

$J(\dot{S}_2:T) = 2.75611$,
$J(\dot{S}_3:T) = \mathbf{0.7179453}$,
$J(\dot{S}_4:T) = 1.42281$
$J(\dot{S}_5:T) = 1.66764$, $J(\dot{S}_6:T) = 0.9264$.

Third row matrix ($\dot{S}_3$) has minimum divergence from the other row matrices. The optimum solution of T is **0.7179453** which is Software $\dot{S}_3$. Thus software $\dot{S}_3$ will be best in preference of $\dot{S}_6, \dot{S}_4, \dot{S}_1, \dot{S}_5, \dot{S}_2$.

The choice of preferred software for students according to all teachers of the department has been shown in the following figure (1).The figure clearly shows that software ($\dot{S}_3$) has minimum divergence than other softwares. Thus R software is the best for the students according to teachers.

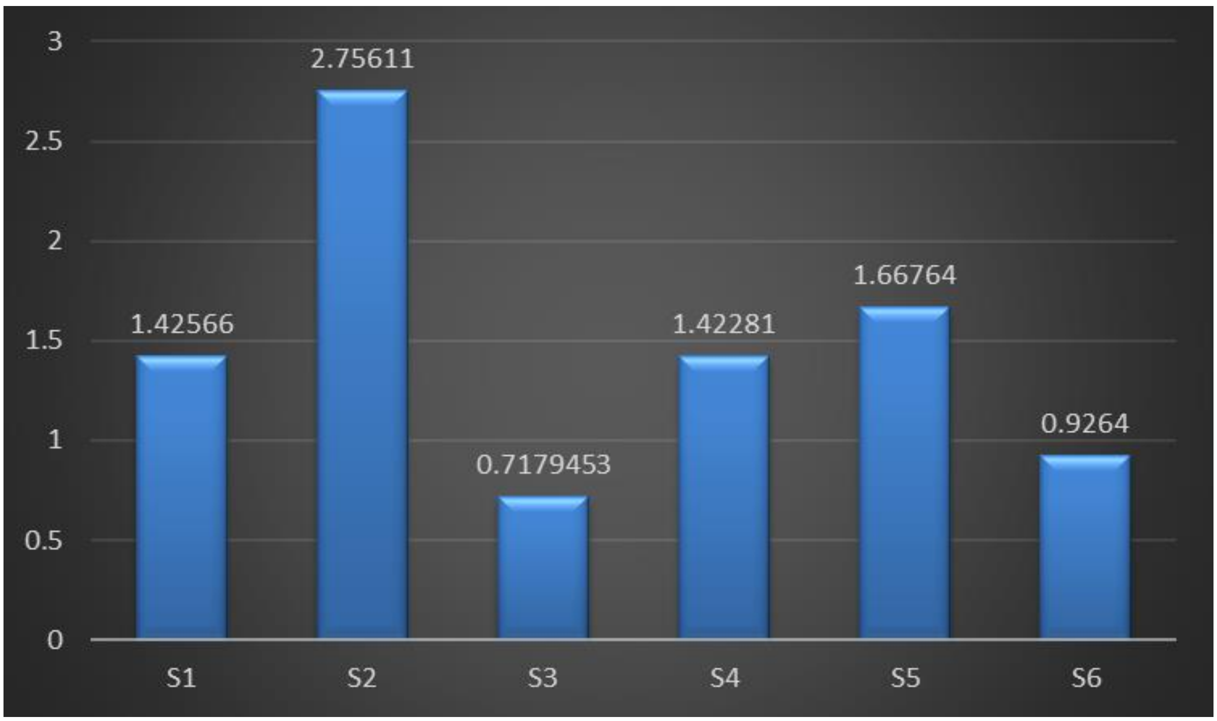

Figure 1: The comparison chart of different software showing divergence

In matrix $\dot{S}$ the columns of matrix are representation the parameters of software, thus for parameter reduction firstly we find the order of significance of these parameters.

Let $Y = \{(X_1, 0.8), (X_2, 0.2), (X_3, 0.8), (X_4, 0.4), (X_5, 0.3), (X_6, 0.5)\}$ be the preferences of head of the department for various software for a particular parameter.

Then as a fuzzy column matrix Y can be represented as:

$F = [\ 0.8 \quad 0.2 \quad 0.8 \quad 0.4 \quad 0.3 \quad 0.5]^T$

Now divide the fuzzy matrix $\dot{S}$ into column matrices as $\{F_1, F_2, \dots \dots, F_6\}$. Now we have to find the divergence between these column matrices and matrix F by using proposed divergence measure

$$J(X:Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\left\{ \left(x_{ij} - y_{ij}\right) log \frac{x_{ij}}{y_{ij}} \right\} + \left\{ \left(y_{ij} - x_{ij}\right) log \frac{1 - x_{ij}}{1 - y_{ij}} \right\} \right]$$

$$J(F_1:F)=\left\{\begin{array}{l}\left[(0.9-0.8)log\frac{0.9}{0.8}+(0.8-0.9)log\frac{1-0.9}{1-0.8}\right]\\ \left[(0.2-0.2)log\frac{0.2}{0.2}+(0.2-0.2)log\frac{1-0.2}{1-0.2}\right]\\ \left[(0.9-0.8)log\frac{0.9}{0.8}+(0.8-0.9)log\frac{1-0.9}{1-0.8}\right]\\ \left[(0.6-0.4)log\frac{0.6}{0.4}+(0.6-0.4)log\frac{1-0.6}{1-0.4}\right]\\ \left[(0.3-0.3)log\frac{0.3}{0.5}+(0.3-0.3)log\frac{1-0.3}{1-0.3}\right]\\ \left[(0.4-0.5)log\frac{0.4}{0.5}+(0.5-0.4)log\frac{1-0.4}{1-0.5}\right]\end{array}\right\}=\mathbf{0.158482}$$

Similarly, we may find divergence of F with other matrices $(F_2, F_3, \dots\dots.., F_6)$ which are given below:

$J(F_2:F) = \mathbf{0.16744}$,
$J(F_3:F) = \mathbf{0.264444}$,
$J(F_4:F) = \mathbf{3.311562}$
$J(F_5:F) = \mathbf{0.347209}$,
$J(F_6:F) = \mathbf{0.544886}$.

The minimum divergence from Y is **0.158482** which is defined with column matrix F_1 as compare to other column matrices.

We find the perfect solution of F is **0. 158482** which is option F_1. Thus parameter P_1 is more significant in comparison to F_2, F_3, F_5, F_6, F_4.

Now we have to take away parameters which do not change the order of preference of software or maintain the optimality of decision. Here F_4 is least significant so we first abolish F_4 and check the optimality of the result.

$J(B_1:B) = \mathbf{0.33625}$,
$J(B_2:F) = 2.72089$,
$J(B_3:B) = 0.53733$
$J(B_4:B) = 0.70034$,
$J(B_5:B) = 1.63242$,
$J(B_6:B) = 0.90299$.

The decision is not optimal after eviction of F_4 so we cannot remove F_4.

When we remove F_5 then

$J(B_1:B) = 1.42504$,
$J(B_2:B) = 2.57549$,
$J(B_3:B) = \mathbf{0.33625}$
$J(B_4:B) = 1.24281$,
$J(B_5:B) = 1.59404$,
$J(B_6:B) = 0.85281$.

We can remove F_5 because optimality maintain after removal of it.

When we remove F_2 then

$J(B_1:B) = 1.34047$,
$J(B_2:B) = 2.59289$,
$J(B_3:B) = \mathbf{0.63275}$
$J(B_4:B) = 1.42281$,
$J(B_5:B) = 1.50442$,
$J(B_6:B) = 0.90721$.

We can remove F_2 because optimality maintain after removal of it.

Similarly when we remove F_6, F_1, F_4 individualy then optimality of decision is not maintained so we can't remove these parameters individually.

When we remove F_2 & F_5 both then

$J(B_1:B) = 1.15985$,
$J(B_2:B) = 2.41227$,
$J(B_3:B) = \mathbf{0.25105}$
$J(B_4:B) = 1.42281$,
$J(B_5:B) = 1.43082$,
$J(B_6:B) = 0.83362$.

Here we can see from the result, solution of the problem has not changed, decision is maintained so we can remove these parameters.

Since solution is optimal so we can remove any subset from these given parameters set. Thus we can eliminate the following parameters subset$\{\{F_2, F_5\}, \{F_2\}, \{F_5\}\}$.

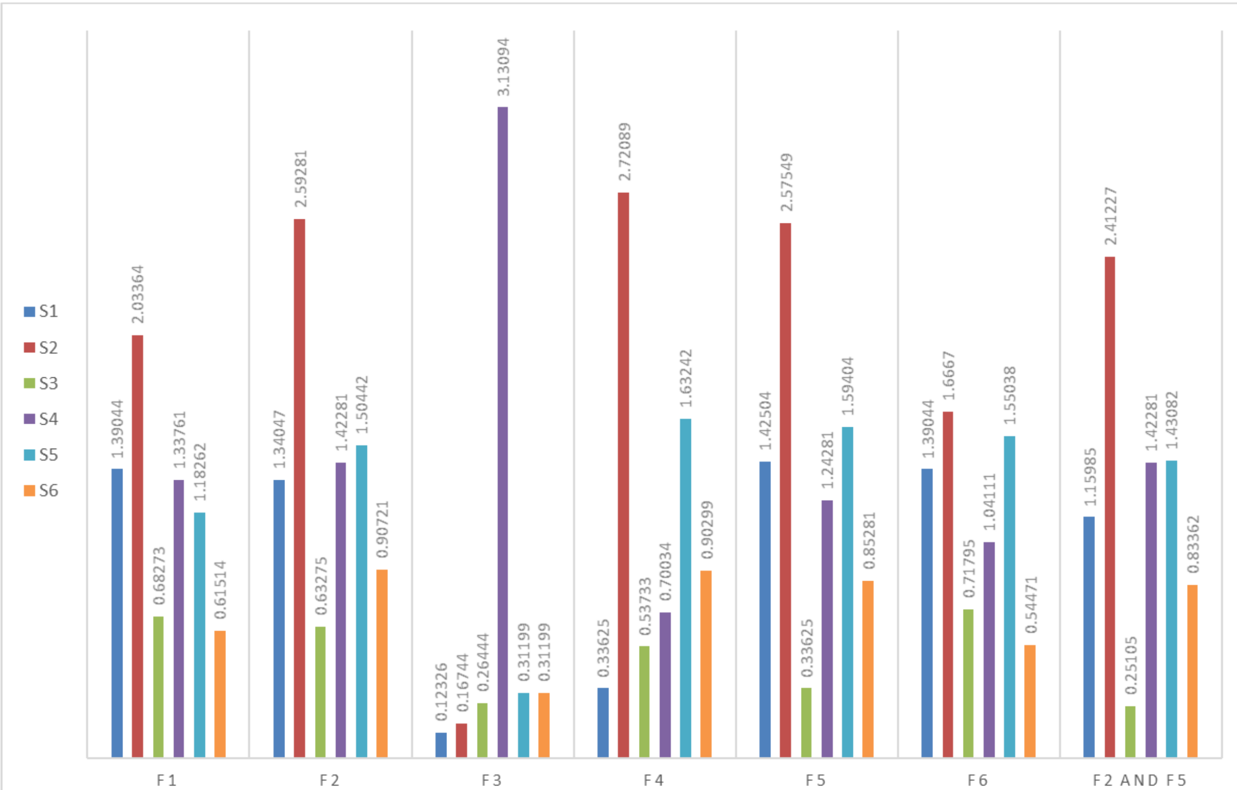

Figure 2: The comparison chart of Feature selection showing decision maintenance

To reduce any feature we check divergence between features and then reduce one by one feature to find out which feature removal does not affect the result. The figure (2) shows only removal of F_2, F_5 and both $F_2$and F_5 maintain the optimality of the result.

VII. CONCLUSION

The chapter introduced logarithmic divergence measure for fuzzy matrix. Proposed measure is a valid measure as it satisfies all the axioms. Properties in the form of theorems are also proved of the measure. An application of the measure is presented in decision making problem and parameter reduction problem. We may eclectic the best alternative in preference to other available alternatives in decision making problem. We remove those features which have inconsequential to make decision and after eviction of these features decision is maintained in feature selection problem. Finally, the proposed divergence measure is applied on a case-study to check its real application.

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