Identification of nonlinear systems with hard nonlinearity

A. Brouri, S. Ziani

***Abstract*—** The problem of identifying Wiener-Hammerstein systems is proposed in the presence of hard nonlinearity. Wiener-Hammerstein systems consist of a series connection including a nonlinear element sandwiched with two linear subsystems. Presently, the two linear subsystems are allowed to be nonparametric and structure entirely unknown, but are supposed asymptotically stable. Furthermore, the system nonlinearity is of hard type and no knowledge is priori required. Interestingly, the system nonlinearity is separately identified first. In turn, the linear subsystems are identified in the second stage using a frequency approach.

***Keywords*—**Hard nonlinearity, frequency system identification, Wiener models, Hammerstein models, Wiener-Hammerstein systems.

I. Introduction

The problem of system identification based on different variants of the Wiener-Hammerstein model has been given a great deal of interest, especially on the last decade, and several solutions are now available. Wiener-Hammerstein models consist of a series connection including a nonlinear static element sandwiched with two linear subsystems (Fig. 1). Accordingly, this structure of models can be viewed as a generalization of Hammerstein and Wiener models and so it is expected to feature a superior modeling capability. This has been confirmed by several practical applications e.g. paralyzed skeletal muscle dynamics [1]. As a matter of fact, Hammerstein-Wiener systems are more difficult to identify than the simpler Wiener and Hammerstein systems [2].

Note that, the internal signals: $v(t)$, $w(t)$, $x(t)$ and $\xi(t)$ are not accessible to measurements. The only measurable signals are the system input $u(t)$ and output $y(t)$.

In view of these difficulties, it is not surprising that few solutions are available that deal with Wiener-Hammerstein system identification.

The available methods have been developed following three main approaches i.e. iterative nonlinear optimization procedures [3]; stochastic methods e.g. [4], [5]; frequency methods e.g. [6], [7].

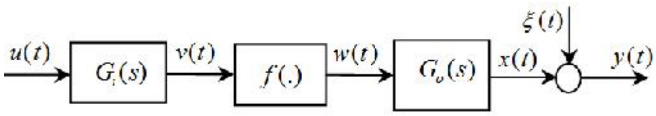

Fig. 1 Wiener-Hammerstein Model structure

A. Brouri is with the ENSAM-Meknes, Moulay Ismail University, L2MC, AEEE, Morocco (e-mail: a.brouri@ensam-umi.ac.ma).

In this paper, a frequency-domain identification scheme is designed for Wiener-Hammerstein systems involving two linear subsystems (asymptotically stable) of entirely unknown structure, unlike many previous works. Furthermore, the static nonlinearity is of hard type (Fig. 2), it's also of unknown structure and is not required to be invertible. The system nonlinearity can have several effects [7]. Given the system nonparametric nature, the identification problem is presently dealt with by developing a two-stage frequency identification method, involving periodic inputs.

First, the identification of system nonlinearity can be achieved by using a set of constant points. Then, the linear subsystems can be dealt by developing a frequency identification method, using simple sine inputs. An accurate estimates of complex gain (gain modulus and phase) can be determined, for any frequency $\omega_k \in \{\omega_1; \ldots; \omega_m\}$.

The outline of the remaining part of this paper consists of 4 sections. The identification problem is formally described in Section II. The identification of the nonlinearity is coped with in Section III. the identification of the linear subsystems is dealt with in Section IV.

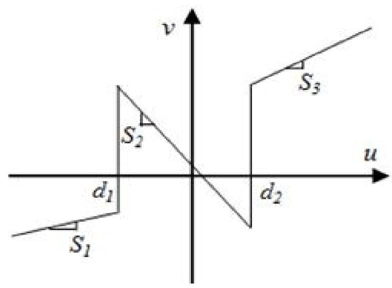

Fig. 2 Example of hard nonlinearity

II. Identification Problem Statement

We are interested in systems that can be described by the Wiener-Hammerstein structure (Fig. 1) with hard nonlinearity (Fig. 2). The above model is analytically described by the following equations:

$$v(t) = g_i(t) * u(t) \quad (1a)$$

$$w(t) = f\left(v(t)\right) = f\left(g_i(t) * u(t)\right) \tag{1b}$$

$$y(t) = g_o(t) * w(t) + \xi(t) = g_o(t) * f\left(v(t)\right) + \xi(t) \tag{1c}$$

where $g_i(t) = L^{-1}\left(G_i(s)\right)$ and $g_o(t) = L^{-1}\left(G_o(s)\right)$ are the inverse Laplace transform of $G_i(j\omega)$ and $G_o(j\omega)$ (respectively);
* refers to the convolution operation. The linear subsystems are supposed to be asymptotically stable with non-null static gain. For a problem of identifiability, at least one of these segments has nonzero slope. The external noise $\xi(t)$ is supposed to be a zero-mean stationary sequence of independent random variables and ergodic.

The problem complexity also lies in the fact that the internal signals are not uniquely defined from an input-output viewpoint. In effect, if $\left(G_i(s), f(v), G_o(s)\right)$ is representative of the system then, any model of the form $\left(G_i(s)/k_1\ ,\ k_2 f(k_1 v)\ ,\ G_o(s)/k_2\right)$ is also representative whatever the real numbers $k_1 \neq 0$ and $k_2 \neq 0$. In the present study, we assume that: $G_i(0) \neq 0$ and $G_o(0) \neq 0$. To get benefit from models plurality, it is judicious to take:

$$k_1 = G_i(0) \quad \text{and} \quad k_2 = G_o(0) \tag{2}$$

Accordingly, without loss of generality, the model to be identified is described by the following equations:

$$\bar{G}_i(s) = \frac{G_i(s)}{G_i(0)} \quad \text{and} \quad \bar{G}_o(s) = \frac{G_o(s)}{G_o(0)} \tag{3a}$$

$$\bar{f}(x) = G_o(0) f\left(G_i(0) x\right) \tag{3b}$$

The considered model is characterized by unit static gains, i.e. the system (3a-b) satisfy the properties: $\bar{G}_i(0) = \bar{G}_o(0) = 1$.

On the other hand, to avoid the use of several notations, the system to be identified will be noted $\left(G_i(s), f(v), G_o(s)\right)$. It is interesting to note that, the hard nature of nonlinearity makes its orthogonal polynomials series approximation not too suitable. To capture discontinuities, one needs too large series troncature degree (Fig. 3). Besides, the suitable degree is not easy to select because the shape of the N.L is a priori unknown. Let approximate an example of hard nonlinearity $f(v)$ with series expansion for various degrees m: $f(v) \approx \sum_{i=1}^{m} \mu_i v^i$ (Fig. 3). Then, a large values of m are necessary to conveniently capture $f(v)$. Accordingly, the suitable values of m depend on $f(.)$ which is unknown.

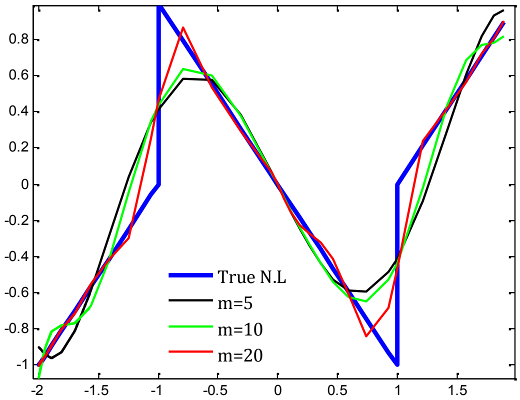

Fig. 3 Hard element compared with series expansion of degree *m*

III. IDENTIFICATION OF SYSTEM NONLINEARITY

In this section, we are interest to develop the nonlinear system identification. The Wiener-Hammerstein system is excited with piecewise constant signal. When the system is successively excited by a set of constant inputs $u(t) \in \left\{U_1; \ldots ; U_N\right\}$, then the steady state of the internal signal $v(t)$ takes N constant values. It follows (1a) and (3a), $v(t)$ can be expressed as follows:

$$v(t) \in \left\{U_1; \ldots ; U_N\right\} \tag{4}$$

where the number N is arbitrarily chosen by the user. Then, it is readily obtained from (1b), (4) and (3a-b) that, the steady-state internal signal $w(t)$ can simply be expressed as follows:

$$w(t) \in \left\{f\left(U_1\right); \ldots ; f\left(U_N\right)\right\} \tag{5}$$

Then, as the linear subsystem $G_o(s)$ is asymptotically stable with unit static gain, it is readily obtained from (1c), (3a-b) and (5) that, the steady state undisturbed output $x(t)$ takes N constant values, that can be expressed as follows:

$$x(t) \in \left\{f\left(U_1\right); \ldots ; f\left(U_N\right)\right\} \tag{6}$$

On the other hand, using the fact $y(t) = x(t) + \xi(t)$ and (6), the steady state the system output $y(t)$ can expressed as:

$$Y_j = f(U_j) + \xi(t) \quad \text{where } j \in \left\{1; \ldots ; N\right\} \tag{7}$$

Then, as the system is asymptotically stable, its step undisturbed response settles down (i.e. gets very close to final value) after a transient period.

Finally, notice that the steady-state undisturbed output $X_j = f(U_j)$ $(j = 1 \ldots N)$ can simply be estimated using the fact that $y(t)=x(t)+\xi(t)$ and $\xi(t)$ is zero-mean. Specifically, X_j $(j=1\ldots N)$ can be recovered by averaging $y(t)$ on a sufficiently large interval.

$$\hat{X}_j(M)=\frac{1}{NT_r}\int_{(j-1)MT_r}^{jMT_r} y(t)dt \quad \text{for} \quad j=1,\ \ldots\ ,N \qquad (8)$$

where T_r is theoretically any positive real number. Practically, it is convenient to let T_r be comparable to the system rise time i.e. the time that is necessary for a system step response to reach 95% of its final value.

Hence, a number of points of the nonlinear function $f(.)$ can thus be accurately estimated. Then, the set of couples $\left(U_j\ ,\hat{X}_j(L)\right)=\left(U_j\ ,\hat{f}_L(U_j)\right)$ $(j=1\ldots N)$, where $\hat{f}_L(U_j)$ designates the estimate of $f(U_j)$, are estimates of N points all belonging to the nonlinear function $f(.)$. This yields the following statement:

Proposition 1. The couple of points $\left(U_j\ ,\hat{X}_j(L)\right)$, for $j=1\ldots N$, determined in the Nonlinearity Estimator, converge (in probability) to the trajectory of $f(.)$.

Proof. First, recall that, as the linear subsystems are asymptotically stable, the unit response of system settles down i.e. gets very close to its final value after a transient regime.

Using the rescaling model (3a-b) of Wiener-Hammerstein systems, the steady state undisturbed output depends only on the system nonlinearity $f(.)$.

The Nonlinearity Estimator was shown to be consistent in [8] i.e.:

$$\lim_{L\to\infty}\hat{X}_j(L)=X_j \qquad \text{for all} \quad j=1\ldots N \qquad (9)$$

Then, it is readily obtained from (6) and (9), the set of couples $\left(U_j\ ,\hat{X}_j(L)\right)$ $(j=1\ldots N)$ converge (in probability) to $\left(U_j\ ,f\left(U_j\right)\right)$, for $j=1\ldots N$. This establishes and completes the proof of Proposition 1.

Remark 1. If the Wiener-Hammerstein system is excited with constant inputs, the resulting system in the steady state boils down to the linearity part $f(.)$ (see Proposition 1). As a matter of fact, a set of points of the nonlinearity is identified (Fig. 4). Likewise, knowing that the nonlinearity $f(.)$ is of hard type, then it can be accurately estimated (Fig. 4). If necessary, the system can be excited by other points.

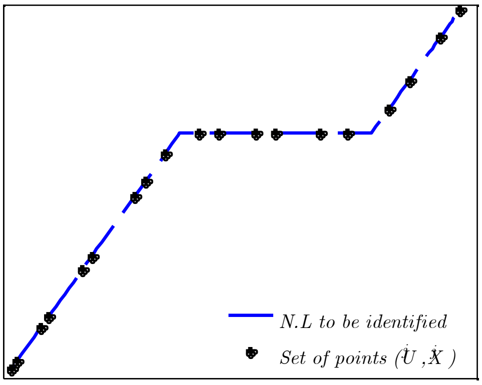

Fig. 4 Example of obtained points $\left(U_j\ ,f(U_j)\right)$

IV. Linear Subsystems Identification

In this section, a frequency identification method is proposed to obtain estimates of the complex gain corresponding to the two linear subsystems $G_i(j\omega)$ and $G_o(j\omega)$ for a set of frequencies $\left\{\omega_1;\ldots;\omega_m\right\}$. For simplicity, we presently suppose that the nonlinearity estimator, established in section III, have been exactly determined.

The frequency identification approach that is proposed in this section relies on the investigation of the system response to sine excitations. The identification problem under study is dealt using a method based on the frequency approach. For a set of a priori chosen frequencies $(k=1\ldots m)$. The Wiener-Hammerstein system is excited with a given sine input:

$$u(t)=u_o+U\ \sin(\omega_k t) \qquad (10)$$

where the frequency $\omega_k>0$ is kept constant, u_o is the offset and U the amplitude of sine signal. The choice of couple of parameters $\left(u_o,\ U\right)$ will be made after.

On the other hand, it is supposed first that the input signal $u(t)$ in (4) is chosen such that $v(t)$ spans only one segment of $f(.)$ having a nonzero slope. Then, as the linear subsystem $G_i(s)$ is asymptotically stable, it follows from (3a)-(10) that the internal signal $v(t)$, in steady state, is of the form:

$$v(t)=u_o+U\ \left|G_i(j\omega_k)\right|\ \sin(\omega_k t+\varphi_i(\omega_k)) \qquad (11)$$

where $\varphi_i(\omega)=\arg\left(G_i(j\omega)\right)$. If $v(t)$ spans only the chosen segment, then the internal signals $v(t)$ and $w(t)$ are related by a linear relationship:

$$w(t) = S^* v(t) + P^* \tag{12}$$

where S^* is the slope of segment l and P^* is the value of $w(t)$ when $v(t) = 0$. Thereafter, from (11)-(12), the internal signal $w(t)$ is written in the following form:

$$w(t) = US^* |G_i(j\omega_k)| \sin(\omega_k t + \varphi_i(\omega_k)) + S^* u_o + P^* \tag{13}$$

As the linear subsystem $G_o(s)$ is asymptotically stable, it follows from (13) that, the steady state undisturbed output $x(t)$ can be expressed as follows:

$$x(t) = US^* |G_i(j\omega_k)||G_o(j\omega_k)| \sin\left(\omega_k t + \varphi_i(\omega_k) + \varphi_o(\omega_k)\right) + S^* u_o + P^* \tag{14}$$

where $\varphi_o(\omega) = \arg\left(G_o(j\omega)\right)$. Finally, as $y(t) = x(t) + \xi(t)$, one immediately gets from (14):

$$y(t) = US^* |G_i(j\omega_k)||G_o(j\omega_k)| \sin\left(\omega_k t + \varphi_i(\omega_k) + \varphi_o(\omega_k)\right) + S^* u_o + P^* + \xi(t) \tag{15}$$

On the other hand, let define the variables for any ω:

$$\varphi(\omega) = \varphi_i(\omega) + \varphi_o(\omega) \tag{16a}$$

$$|G(j\omega)| = |G_i(j\omega)||G_o(j\omega)| \tag{16b}$$

$$y_0 = S^* u_o + P^* \tag{16c}$$

Then, from (15)-(16c), it is readily seen that the output system $y(t)$ can be rewritten in the following form:

$$y(t) = US^* |G(j\omega_k)| \sin\left(\omega_k t + \varphi(\omega_k)\right) + y_0 + \xi(t) \tag{17}$$

Notice that that, the steady state undisturbed output $x(t)$ is a sin signal. Furthermore, the system output $y(t)$ becomes sine or constant signal (up to noise) after a transient period.

On the other hand, a judicious choice of the couple $(u_o,\ U)$ in (10) can be performed using the experimental data of nonlinearity estimator, established in section III. Based upon the curve of the nonlinearity $f(.)$, let choose any segment l of $f(.)$ with nonzero slope (Fig. 4). Then, the identified system is submitted to the sine input (10) within the selected segment. The component u_o can take any value close to the center.

Regarding the choice of the amplitude U in (10), if $v(t)$ spans only one segment, the system output $y(t)$ becomes sine signal (up to noise) after a transient period. The amplitude $U > 0$ is initialized to a small value. Then, the steady state system is sine signal. The amplitude may be increased to obtain a sufficient output signal, but it must keep the sine form. The amplitude U can be reduced if necessary.

On the other hand, let $s(\omega_k)$ the amplitude of the sinusoidal part of $x(t)$ and $\psi(\omega_k)$ its phase. Getting benefit from (14)-(17), one key idea is to determine the gain modulus $|G(j\omega_k)| = |G_i(j\omega_k)||G_o(j\omega_k)|$ and the phase $\varphi(\omega_k) = \varphi_i(\omega_k) + \varphi_o(\omega_k)$. Then, using (14)-(17), it is readily seen that:

$$|G(j\omega_k)| = \left|\frac{s(\omega_k)}{S^* U}\right| \tag{18}$$

Knowing the sign of S^*, the phase $\varphi(\omega_k)$ can be recovered modulo 2π. Let us consider the parameter γ defined as follows:

$$\gamma = 0 \quad \text{if} \quad \mathrm{sign}(S^*) = \mathrm{sign}\left(s(\omega_k)\right) \quad \text{else} \quad \gamma = 1 \tag{19}$$

Then, $\psi(\omega_k)$ and $\varphi(\omega_k)$ are linked by the following relationship:

$$\varphi(\omega_k) = \psi(\omega_k) + \gamma\pi \quad (\text{modulo } 2\pi) \tag{20}$$

Equations (18)-(19) shown that the gain modulus $|G(j\omega_k)|$ and the phase $\varphi(\omega_k)$ can be accurately determined. The difficulty here resides in the fact that (18)-(20) are currently established based on the signal $x(t)$ which is not accessible to measurement. This is presently coped with making full use of the information at hand, namely the periodicity (with period $2\pi / \omega_k$) of both $u(t)$ and $x(t)$ and the ergodicity of the noise $\xi(t)$. Bearing these in mind, the relation $y(t) = x(t) + \xi(t)$ suggests the following estimator of $x(t)$:

$$\hat{x}(t, M) = \frac{1}{M} \sum_{j=1}^{M} y(t + jT_k); \quad t \in [0, T_k) \tag{21a}$$

$$\hat{x}(t + jT_k, M) = \hat{x}(t, M) \quad \text{for any integer } j > 0 \tag{21b}$$

where $T_k = 2\pi / \omega_k$ and M is a sufficiently large integer. Specifically, for a fixed time instant t, the quantity $\hat{x}(t, M)$ turns out to be the mean value of the (measured) sequence $\{y(t + jT_k);\ j = 0\ \ 1\ ...\}$.

On the other hand, Let $\hat{s}_M(\omega_k)$ denotes the estimate of $s(\omega_k)$. Then, an estimate $\left(\hat{\varphi}_M(\omega_k), |\hat{G}_M(j\omega_k)|\right)$ of

$\left(\varphi(\omega_k), \left|G(j\omega_k)\right|\right)$ can be determined, one has thus, for any frequency ω_k :

$$\hat{\varphi}_M(\omega_k) = \hat{\varphi}_i(\omega_k, M) + \hat{\varphi}_o(\omega_k, M) = \delta^* + \gamma\pi \quad (\text{modulo } 2\pi) \quad (22a)$$

$$\left|\hat{G}_M(j\omega_k)\right| = \left|\hat{G}_i(j\omega_k, M)\right|\left|\hat{G}_o(j\omega_k, M)\right| = \left|\frac{\hat{s}_M(\omega_k)}{S^*U}\right| \quad (22b)$$

Then, these remarks lead to the following proposition:

Proposition 2. Consider the identification problem statement. Then, one has:

1) The undisturbed output estimate $\hat{x}(t, M)$ Converges in probability to $x(t)$ as $M \to \infty$.

2) The frequency complex gain $\hat{G}_M(j\omega_k)$ Converges in probability to $G(j\omega_k)$ as $M \to \infty$ (gain modulus and phase estimates converge to their true value).

Proof. As $y(t) = x(t) + \xi(t)$ and using the fact that $x(t)$ is periodic with period T_k, it follows that for all $t \in [0, T_k)$ and all integers *j*: $y(t + jT_k) = x(t) + \xi(t + jT_k)$, which in turn implies that for all $t \in [0, T_k)$:

$$\hat{x}(t, M) = \frac{1}{M}\sum_{j=1}^{M} y(t + jT_k) = x(t) + \frac{1}{M}\sum_{j=1}^{M}\xi(t + jT_k) \quad (23)$$

Since $\xi(t)$ is zero mean and ergodic, the last term on the right side vanishes *w.p.1* as $M \to \infty$. This proves Part 1 of the proposition.

To prove Part 2, using Part 1, one immediately gets:

$$\left|\hat{s}_M(\omega_k)\right| \underset{M\to\infty}{\to} \left|s(\omega_k)\right| \quad w.p.1 \quad (24)$$

Likewise, it follows (8) and Part 1, that:

$$US^*\left|G(j\omega_k)\right|\sin\left(\omega_k t + \varphi(\omega_k)\right) = s(\omega_k)\sin\left(\omega_k t + \psi(\omega_k)\right) \quad (25)$$

Finally, from (14), (18), (19) and (25), it is readily seen that:

$$\left|\hat{G}_M(j\omega_k)\right| \underset{M\to\infty}{\to} \left|G(j\omega_k)\right| \quad w.p.1 \quad (26a)$$

$$\hat{\varphi}_M(\omega_k) \underset{M\to\infty}{\to} \varphi(\omega_k) \quad w.p.1 \quad (26b)$$

This completes the proof of Proposition 2.

V. CONCLUSION

We have developed a new frequency identification method to deal with Wiener-Hammerstein systems; the identification problem is addressed in presence of hard nonlinearity and two linear subsystems of structure entirely unknown.

Firstly, the hard nonlinearity is determined using the algorithm established in section III. Then, a frequency-domain identification method is developed to estimate the product of modulus gains $\left|G_i(j\omega_k)\right|$ and $\left|G_o(j\omega_k)\right|$, as well as sum of the phases $\varphi_i(\omega_k)$ and $\varphi_o(\omega_k)$ (at a number of frequencies).

The present study constitutes a significant progress in frequency-domain identification of block-oriented nonlinear system identification. The originality of the present study lies in the fact that the system is not necessarily parametric and of structure totally unknown. Another feature of the method is the fact that the exciting signals are easily generated and the estimation algorithms can be simply implemented.

It is interesting to point that the estimation of linear subsystem and nonlinearities are performed separately.

REFERENCES

[1] E.W. Bai, Z. Cai, S. Dudley-Javorosk, and R.K. Shields, “Identification of a modified Wiener-Hammerstein system and its application in electrically stimulated paralyzed skeletal muscle modeling,” *Automat. Contr.*, vol. 45, no. 3, pp. 736-743, 2009.

[2] E.W. Bai and F. Giri, “*Block-Oriented Nonlinear System Identification*,” U.K: Springer, 1 edition, 2010.

[3] A. Marconato, J. Sjoberg, and J. Schoukens, “Initialization of nonlinear state-space models applied to the Wiener-Hammerstein benchmark,” *Control Engineering Practice*, vol. 20, pp. 1126-1132, 2012.

[4] N.J. Bershad, P. Celka, and S. McLaughlin, “Analysis of stochastic gradient identification of Wiener-Hammerstein systems for nonlinearities with Hermite polynomial expansions,” *IEEE Trans. Signal Process.*, vol. 49, pp. 1060-1071, 2001.

[5] G. Pillonetto, A. Chiuso, and G.D. Nicolao, “Prediction error identification of linear systems: a nonparametric Gaussian regression approach,” *Automatica*, vol. 47, no. 2, pp. 291-305, 2011.

[6] F. Giri, Y. Rochdi, A. Radouane, A. Brouri, F.Z. Chaoui, “Frequency Identification of nonparametric Wiener systems containing backlash nonlinearities,” *Automatica*, vol. 49, pp. 124-137, 2013.

[7] A. Brouri, F. Giri, F.Z. Chaoui, A. Amdouri, “Frequency Identification of Hammerstein-Wiener Systems with Piecewise Affine Input Nonlinearity,” in *Proc. 19th IFAC World Congress*, Cape Town, South Africa, August 24-29, 2014, pp. 10030-10035.

[8] F. Giri, A. Radouane, A. Brouri, F.Z. Chaoui, “Combined frequency-prediction error identification approach for Wiener systems with backlash and backlash-inverse operators,” Automatica, vol. 50, pp. 768-783, 2014.

[9] E.W. Weisstein, “Lissajous Curve, MathWorld, a Wolfram Web Resource,” http://mathworld.wolfram.com/LissajousCurve.html