Fitted-Modified Upwind Finite Difference Method for Solving Singularly Perturbed Differential Difference Equations

Gemechis File and Y.N. Reddy

Abstract— A fitted modified upwind finite difference method is presented for solving singularly perturbed boundary value problems with delay δ and advance η parameters that are sufficiently small. The second order singularly perturbed differential difference equation is replaced by an asymptotically equivalent singularly perturbed boundary value problem. A fitting factor is introduced in a modified finite difference scheme and is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system and its stability is investigated. The method is demonstrated by implementing several model examples by taking various values for the delay parameter δ , advance parameter η and the perturbation parameter ε .

Keywords— Advance parameter, Boundary layer, Delay parameter, Differential difference equations, Fitted finite difference method, Singular perturbations.

I.INTRODUCTION

Boundary value problems involving differential-difference equations arise in studying variational problems of control theory where the problem is complicated by the effect of time delays (here, the terms ‘‘delay’’ and ‘‘advance’’). This occurs in signal transmission [1] and in depolarization in the Stein model [2], which is a continuous-time, continuous-state space Markov process whose sample paths have discontinuities of the first kind [3]. The time between nerve impulses is the time of first passage to a level at or above a threshold value; and determining the moments of this random variable involves differential-difference equations. This biological problem motivates the investigation of boundary-value problems for differential-difference equations with small shifts. Furthermore, the applications of differential-difference equations permeate all branches of contemporary sciences such as physics, engineering, economics, and biology [4] certainly merits its own volume.

As a result, many researchers tried to develop and present numerical schemes for solving such problems. For example, reference [5], [6] gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations with small shifts. Reference [7] presented ε-uniform Ritz-Galerkin finite element method for solving singularly perturbed delay differential equations with small shifts. Furthermore, reference [8] constructed an ε-uniform numerical scheme comprising of a standard upwind finite difference operator on a fitted piecewise uniform mesh for a class of singularly perturbed boundary value problems of differential-difference equations with small shifts.

The objective of this paper is, therefore, to present a fitted modified upwind finite difference method for solving singularly perturbed differential difference equations with the delay and the advance parameters (sometimes referred to as “negative shift “and “positive shift”, respectively as in [5], [6]) having the boundary layer at one end (left or right). It is based on the concept that the singularly perturbed differential difference equation is replaced by an asymptotically equivalent second order singularly perturbed two point boundary value problem. Then a fitting factor is introduced in a modified upwind finite difference scheme and is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system and the stability of the algorithm is also considered. The method is demonstrated by implementing several model examples by taking various values for the delay parameter δ , advance η and the perturbation parameter ε .

II. DESCRIPTION OF THE METHOD

A. LEFT END BOUNDARY LAYER PROBLEMS

Consider singularly perturbed differential equation with small delay as well as advance of the form:

$$\begin{aligned}&\varepsilon y''(x)+a(x)y'(x)+\alpha(x)y(x-\delta)+\omega(x)y(x)\\&+\beta(x)y(x+\eta)=f(x)\end{aligned}\tag{1}$$

$\forall x\in(0,1)$ and subject to the interval and boundary conditions

$$y(x)=\phi(x),\quad \text{on}\quad -\delta\le x\le 0 \tag{2}$$

$$y(x)=\gamma(x),\quad on\quad 1\le x\le 1+\eta \tag{3}$$

Where

$a(x),\ \alpha(x),\ \beta(x),\ \omega(x),\ f(x),\ \phi(x),\ and\ \gamma(x)$

are bounded and continuously differentiable functions on (0, 1), $0<\varepsilon<<1$ is the singular perturbation parameter; and $0<\delta=o(\varepsilon)$ and $0<\eta=o(\varepsilon)$ are the delay and the advance parameters respectively.

By using Taylor series expansion in the neighborhood of the point *x,* we have

$$y(x-\delta)\approx y(x)-\delta y'(x) \tag{4}$$

$$y(x+\eta)\approx y(x)+\eta y'(x) \tag{5}$$

Gemechis File is with the Department of Mathematics, National Institute of Technology Warangal- 506 004, India (Corresponding author, Phone: +919700325821; e-mail: gammeef@yahoo.com).

Y. N. Reddy is with the Department of Mathematics, National Institute of Technology Warangal- 506 004, India (Phone: +9197849268724; e-mail: ynreddy@nitw.ac.in).

Using equations (4) and (5) in (1) we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{6}$$

$$y(0) = \phi(0) = \phi_0 \tag{7}$$

$$y(1) = \gamma(1) = \gamma_1 \tag{8}$$

where $p(x) = a(x) + \beta(x)\eta - \alpha(x)\delta$ (9)

and $q(x) = \alpha(x) + \beta(x) + \omega(x)$ (10)

The transition from (1) to (6) is admitted, because of the condition that $0 < \delta << 1$ and $0 < \eta << 1$ are sufficiently small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in [9]. Thus, the solution of (6) will provide a good approximation to the solution of (1). Further, we assume that $q(x) = \alpha(x) + \beta(x) + \omega(x) \le 0$, $p(x) = a(x) + \beta(x)\eta - \alpha(x)\delta \ge M > 0$ throughout the interval [0, 1], where M is some constant. Under these assumptions, (5) has a unique solution $y(x)$ which in general, exhibits a boundary layer of width O(ε) on the left side ($x = 0$) of the underlying interval.

From the theory of singular perturbations in [10] it is known that the solution of (5) - (6) is of the form:

$$y(x) = y_0(x) + \frac{p(0)}{p(x)}(\phi_0 - y_0(0))\, e^{-\int_0^x \left(\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)}\right)dx} + O(\varepsilon) \tag{11}$$

where $y_0(x)$ is the solution of the reduced problem:

$p(x)y_0'(x) + q(x)y_0(x) = f(x)$, $y_0(0) = \gamma_1$. (12)

By taking the Taylor's series expansion for $p(x)$ and $q(x)$ about the point '0' and restricting to their first terms, (11) becomes:

$$y(x) = y_0(x) + (\phi_0 - y_0(0))\, e^{-\left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right)x} + O(\varepsilon) \tag{13}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0,\ x_1,\ x_2,\ ...,\ x_N = 1$ be the mesh points. Then we have $x_i = ih,\ \ i = 0, 1,\ 2,\ ...,\ N$. From (13) we have:

$$\lim_{h\to 0} y(ih) = y_0(0) + (\phi_0 - y_0(0))\, e^{-\left(\frac{p^2(0) - \varepsilon q(0)}{p(0)}\right)i\rho} + O(\varepsilon) \tag{14}$$

where $\rho = \dfrac{h}{\varepsilon}$.

Furthermore, by Taylor's series expansion:

$$y_i' = \frac{y_{i+1} - y_i}{h} - \frac{h}{2}y_i'' + O(h^2) \tag{15}$$

Thus, the modified upwind scheme corresponding to (6)-(8) is:

$$\left(\varepsilon - \frac{h}{2}p_i\right)\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + p_i\left[\frac{y_{i+1} - y_i}{h}\right] + q_i y_i = f_i + O(h^2) \tag{16}$$

$$y_0 = \phi_0, \quad y_N = \gamma_1 \tag{17}$$

Where

$p(x_i) = p_i,\ \ q(x_i) = q_i,\ \ f(x_i) = f_i,\ \ y(x_i) = y_i$.

Introducing a fitting factor $\sigma(\rho)$ into (16) we get

$$\sigma(\rho)\left(\varepsilon - \frac{h}{2}p_i\right)\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + p_i\left[\frac{y_{i+1} - y_i}{h}\right] + q_i y_i = f_i + O(h^2) \tag{18}$$

With $y_0 = \phi_0,\ \ y_N = \gamma_1$ which is to be determined in such a way that the solution of (18) with the boundary conditions converges uniformly to the solution of (6) - (8) which is in turn a good approximation to the solution of (1) - (3). Multiplying (18) by h and taking the limit as $h \to 0$; we get

$$\sigma(\rho)\lim_{h\to 0}\left(\varepsilon - \frac{p_i}{2}\right)\left[(y_{i+1} - 2y_i + y_{i-1}) + p_i(y_{i+1} - y_i)\right] = 0 \tag{19}$$

Provided $f(x_i) - q(x_i)y_i$ is bounded. By substituting (14) in (19) and simplifying, we get the constant fitting factor

$$\sigma = \left[\frac{2\rho p(0)}{2 - \rho p(0)}\right]\left[\frac{1}{\exp\left[\left(\frac{p^2(0) - \varepsilon q(0)}{p(0)}\right)\rho\right] - 1}\right] \tag{20}$$

Now, from (18) we have:

$$\frac{\sigma}{h^2}\left(\varepsilon - \frac{h}{2}p(x_i)\right)y_{i-1} - \left(\frac{2\sigma}{h^2}\left(\varepsilon - \frac{h}{2}p(x_i)\right) + \frac{p(x_i)}{h} - q(x_i)\right)y_i + \left(\frac{\sigma}{h^2}\left(\varepsilon - \frac{h}{2}p(x_i)\right) + \frac{p(x_i)}{h}\right)y_{i+1} = f(x_i) \tag{21}$$

$i = 1,\ 2,\ ...,\ N-1$ where the fitting factor σ is given by (20). Equation (21) can be written as the three term recurrence relation of form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \ ;\ i = 1,\ 2,\ ...,\ N-1 \tag{22}$$

where

$$E_i = \frac{\sigma}{h^2}\left(\varepsilon - \frac{h}{2}p(x_i)\right) \tag{23}$$

$$F_i = \frac{2\sigma}{h^2}\left(\varepsilon - \frac{h}{2}p(x_i)\right) + \frac{p(x_i)}{h} - q(x_i) \tag{24}$$

$$G_i = \frac{\sigma}{h^2}\left(\varepsilon - \frac{h}{2}p(x_i)\right) + \frac{p(x_i)}{h} \tag{25}$$

$$H_i = f(x_i) \tag{26}$$

This gives us the tridiagonal system which can be solved easily by Thomas Algorithm described in the next section.

B. THOMAS ALGORITHM

A brief description for solving the tri-diagonal system using Thomas algorithm is presented as follows: Consider the scheme:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \ ; \ i = 1,\ 2,\ ...,\ N-1 \tag{27}$$

subject to the boundary conditions

$$y_0 = y(0) = \phi_0; \tag{28}$$

$$y_N = y(1) = \gamma_1. \tag{29}$$

We set $y_i = W_i y_{i+1} + T_i \ , i = N-1,\ N-2,\ ...,\ 2,\ 1$ (30)

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined. From (30), we have:

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{31}$$

By substituting (31) in (27) and comparing with (30) we get the recurrence relations:

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) \tag{32}$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right). \tag{33}$$

To solve these recurrence relations for $i = 1,\ 2,\ ...,\ N-1$, we need the initial conditions for W_0 and T_0. For this we take $y_0 = \phi_0 = W_0 y_1 + T_0$. We choose $W_0 = 0$ so that the value of $T_0 = \phi_0$. With these initial values, we compute W_i and T_i for $i = 1,\ 2,\ ...,\ N-1$ from (32) and (33) in forward process, and then obtain y_i in the backward process from (29) and (30).

C. STABILITY ANALYSIS

We will now show that the algorithm is computationally stable. By stability, we mean that the effect of an error made in one stage of the calculation is not propagated into larger errors at later stages of the calculations. Let us now examine the recurrence relation given by (32). Suppose that a small error e_{i-1} has been made in the calculation of W_{i-1}; then, we have $\overline{W}_{i-1} = W_{i-1} + e_{i-1}$ a, where $\overline{W}_{i-1}$ the exact value at $(i-1)$ step and calculating

$$\overline{W}_i = \left(\frac{G_i}{F_i - E_i \overline{W}_{i-1}}\right). \tag{34}$$

From (32) and (34), we have

$$e_i = \left(\frac{G_i}{F_i - E_i(W_{i-1} + e_{i-1})}\right) - \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) = \left(\frac{W_i^2 E_i}{G_i}\right) e_{i-1} \tag{35}$$

under the assumption that the error is small initially.

From the assumptions made earlier that $p(x) = a(x) + \beta(x)\eta - \alpha(x)\delta > 0$ and $q(x) = \alpha(x) + \beta(x) + \omega(x) \le 0$, we have

$$F_i \ge E_i + G_i; \ i = 1,\ 2,\ ...,\ N-1 .$$

Form (32), we have $W_1 = \frac{G_1}{F_1} < 1$, since $F_1 > G_1$

$$W_2 = \frac{G_2}{F_2 - E_2 W_1} < \frac{G_2}{F_2 - E_2}, \text{ since } W_1 < 1,$$

$$< \frac{G_2}{E_2 + G_2 - E_2} = 1, \text{ since } F_2 \ge E_2 + G_2$$

Successively, it follows that $|e_i| = |W_i|^2 \left|\frac{E_i}{G_i}\right| |e_{i-1}|$

$< |e_{i-1}|$, since $|E_i| \le |G_i|$.

Therefore, the recurrence relation (32) is stable. Similarly we can prove that the recurrence relation (33) is also stable. Finally the convergence of the Thomas Algorithm is ensured by the condition $|W_i| < 1$, $i = 1,\ 2,\ ...,\ N-1$.

D. NUMERICAL EXAMPLES WITH LEFT END BOUNDARY LAYER

To demonstrate the applicability of the method we have applied it to three boundary value problems of the type given by equations (1)-(3) with left-end boundary layer. The approximate solution is compared with exact solution.

The exact solution of such boundary value problems having constant coefficients (i.e. $a(x) = a$, $\alpha(x) = \alpha$, $\beta(x) = \beta$, $\omega(x) = \omega$, $f(x) = f$, $\phi(x) = \phi$ and $\gamma(x) = \gamma$ are constants) is given by:

$$y(x) = c_1 \exp(m_1 x) + c_2 \exp(m_2 x) + \frac{f}{c}, \tag{36}$$

where

$$c_1 = \frac{-f + \gamma c + \exp(m_2)(f - \phi c)}{(\exp(m_1) - \exp(m_2))c}$$

$$c_2 = \frac{f - \gamma c + \exp(m_1)(-f + \phi c)}{(\exp(m_1) - \exp(m_2))c}$$

$$m_1 = \frac{-(a - \alpha\delta + \beta\eta) + \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4\varepsilon c}}{2\varepsilon}$$

$$m_2 = \frac{-(a - \alpha\delta + \beta\eta) - \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4\varepsilon c}}{2\varepsilon}$$

$$c = \alpha + \beta + \omega \tag{37}$$

Example 1: Consider the model boundary value problem given by equations (1)-(3) with $a(x) = 1,\ \alpha(x) = 2,\ \beta(x) = 0,\ \omega(x) = -3,$ $f(x) = 0,\ \phi(x) = 1,\ \gamma(x) = 1.$

The exact solution of the problem is given by (36)-(37).

The numerical results are given in tables 1, 2 for ε=0.01 and 0.005 respectively.

Example 2. Consider the model boundary value problem given by equations (1)-(3) with $a(x) = 1,\ \alpha(x) = 0,\ \beta(x) = 2,\ \omega(x) = -3,$ $f(x) = 0,\ \phi(x) = 1,\ \gamma(x) = 1.$

The exact solution of the problem is given by (36)-(37).

The numerical results are given in tables 3, 4 for ε=0.01 and 0.005 respectively.

Example 3. Consider the model boundary value problem given by equations (1)-(3) with $a(x) = 1,\ \alpha(x) = -2,\ \beta(x) = 1,\ \omega(x) = -5,$ $f(x) = 0,\ \phi(x) = 1,\ \gamma(x) = 1.$

The exact solution of the problem is given by (36)-(37).

The numerical results are given in tables 5, 6 for ε=0.01 and 0.005 respectively.

E. RIGHT-END BOUNDARY LAYER PROBLEMS

We now consider (6)-(8) and assume that $p(x) = a(x) + \beta(x)\eta - \alpha(x)\delta \le M < 0$ throughout the interval [0, 1], where M is constant. This assumption merely implies that the boundary layer will be in the neighborhood of x = 1.

Thus, from the theory of singular perturbations the solution of (6) - (8) is of the form:

$$y(x) = y_0(x) + \frac{p(1)}{p(x)}(\gamma_1 - y_0(1))\, e^{\int_x^1 \left(\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)}\right)dx} + O(\varepsilon) \tag{38}$$

where $y_0(x)$ is the solution of the reduced problem

$$p(x)y_0'(x) + q(x)y_0(x) = f(x), \quad y_0(0) = \phi_0. \tag{39}$$

By taking the Taylor's series expansion for $p(x)$ and $q(x)$ about the point '1' and restricting to their first terms, (38) becomes

$$y(x) = y_0(x) + (\gamma_1 - y_0(1))\, e^{\left(\frac{p(1)}{\varepsilon} - \frac{q(1)}{p(1)}\right)(1-x)} + O(\varepsilon) \tag{40}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0,\ x_1,\ x_2,\ \ldots,\ x_N = 1$ be the mesh points. Then we have $x_i = ih,\ \ i = 0, 1,\ 2,\ \ldots,\ N$. From (40) we have

$$\lim_{h\to 0} y(ih) = y_0(0) + (\gamma_1 - y_0(1))\, e^{\left(\frac{p^2(1) - \varepsilon q(1)}{a(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)} + O(\varepsilon) \tag{41}$$

where $\rho = \dfrac{h}{\varepsilon}$. Applying the same procedure as in section 2.1 and using (41) we can get the tri-diagonal system (22)-(26) with a fitting factor as

$$\sigma = \left[\frac{2\rho p(0)}{2 - \rho p(0)}\right]\left[\frac{1}{\exp\left[\left(\frac{p^2(1) - \varepsilon q(1)}{p(1)}\right)\rho\right] - 1}\right]$$

which can be solved by Thomas Algorithm described in section (B).

F. NUMERICAL EXAMPLES WITH RIGHT END BOUNDARY LAYER

Here we considered four boundary value problems of the type given by equations (1)-(3) with right-end boundary layer. The approximate solution is compared with the exact solution. The exact solution of such boundary value problems having constant coefficients (i.e. $a(x) = a,\ \alpha(x) = \alpha,\ \beta(x) = \beta,\ \omega(x) = \omega,\ f(x) = f,\ \phi(x) = \phi$ and $\gamma(x) = \gamma$ are constants) is given by equation (36)-(37).

Example 4: Consider the model boundary value problem given by equations (1)-(3) with $a(x) = -1,\ \alpha(x) = -2,\ \beta(x) = 0,\ \omega(x) = 1,$ $f(x) = 0,\ \phi(x) = 1,\ \gamma(x) = -1.$

The exact solution of the problem is given by (36)-(37). The numerical results are given in tables 7, 8 for ε=0.01 and 0.005 respectively.

Example 5: Consider the model boundary value problem given by equations (1)-(3) with $a(x) = -1,\ \alpha(x) = 0,\ \beta(x) = -2,\ \omega(x) = 1,$
$f(x) = 0,\ \phi(x) = 1,\ \gamma(x) = -1.$

The exact solution of the problem is given by (36)-(37). The numerical results are given in tables 9, 10 for ε=0.01 and 0.005 respectively.

Example 6: Consider the model boundary value problem given by equations (1)-(3) with $a(x) = -1,\ \alpha(x) = -2,\ \beta(x) = -2,\ \omega(x) = 1,$
$f(x) = 0,\ \phi(x) = 1,\ \gamma(x) = -1.$

The exact solution of the problem is given by (36)-(37). The numerical results are given in tables 11, 12 for ε=0.01 and 0.005 respectively

Example 7: Consider the model boundary value problem given by equations (1)-(3) with $a(x) = -(1 + \exp(x^2)),\ \alpha(x) = -x,$
$\beta(x) = -(1 - \exp(-x)),\ \omega(x) = x^2,\ f(x) = 1,$
$\phi(x) = 1,\ \gamma(x) = -1.$

The exact solution of the problem is not known. The numerical results are given in tables 13, 14 for ε=0.01 and 0.005 respectively

III. DISCUSSIONS AND CONCLUSIONS

We have presented a fitted-modified upwind finite difference method to solve singularly perturbed differential difference equations with the delay and advance parameters. To demonstrate the efficiency of the method, we considered three examples with left end boundary layer and four with right end boundary layer for different values of δ, η and ε . The approximate solution is compared with the exact solution. From the results presented in tables; we observed that the present method approximates the exact solution very well.

Table 1: Numerical Results **of Example 1** for ε=0.01, N=100

x	δ=0.001, η=0.005		δ=0.005=η		δ=0.009, η= 0.005	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.4606003	0.4620298	0.4593767	0.4608593	0.4581702	0.4597025
0.04	0.3967817	0.3969578	0.3942694	0.3944706	0.3917536	0.3919752
0.06	0.3951908	0.3950116	0.3923633	0.3921947	0.3895214	0.3893581
0.08	0.4018598	0.4016091	0.3989754	0.3987311	0.3960733	0.3958297
0.10	0.4097450	0.4094820	0.4068559	0.4065984	0.4039479	0.4036901
0.20	0.4524234	0.4521629	0.4495846	0.4493290	0.4467248	0.4464683
0.40	0.5516438	0.5514057	0.5490459	0.5488116	0.5464243	0.5461891
0.60	0.6726243	0.6724306	0.6705107	0.6703200	0.6683748	0.6681828
0.80	0.8201368	0.8200186	0.8188472	0.8187307	0.8175418	0.8174245
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 2: Numerical Results of **Example 1** for ε=0.005, η=0.0005, N=100

	δ=0.0001, η=0.0005		δ=0.0005=η		δ=0.0009, η= 0.0005	
x	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3887189	0.3883907	0.3884719	0.3881406	0.3882209	0.3878904
0.04	0.3854013	0.3848477	0.3851155	0.3845578	0.3848257	0.3842677
0.06	0.3929417	0.3923809	0.3926550	0.3920900	0.3923642	0.3917988
0.08	0.4008259	0.4002658	0.4005396	0.3999753	0.4002493	0.3996845
0.10	0.4088717	0.4083128	0.4085861	0.4080229	0.4082963	0.4077327
0.20	0.4515897	0.4510409	0.4513093	0.4507563	0.4510248	0.4504713
0.40	0.5508814	0.5503792	0.5506247	0.5501186	0.5503643	0.5498578
0.60	0.6720043	0.6715958	0.6717955	0.6713839	0.6715837	0.6711716
0.80	0.8197587	0.8195094	0.8196315	0.8193802	0.8195021	0.8192506
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 3: Numerical Results of **Example 2** for ε=0.01, N=100

	δ=0.005, η=0.001		δ=0.005=η		δ=0.005, η= 0.009	
x	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.4612147	0.4626200	0.4624456	0.4638100	0.4636980	0.4650123
0.04	0.3980323	0.3981982	0.4005193	0.4006724	0.4030082	0.4031376
0.06	0.3965950	0.3964127	0.3993813	0.3992000	0.4021595	0.4019676
0.08	0.4032912	0.4030393	0.4061293	0.4058825	0.4089564	0.4087026
0.10	0.4111782	0.4109147	0.4140196	0.4137619	0.4168488	0.4165850
0.20	0.4538308	0.4535700	0.4566194	0.4563647	0.4593939	0.4591334
0.40	0.5529305	0.5526921	0.5554766	0.5552441	0.5580059	0.5577688
0.60	0.6736697	0.6734761	0.6757361	0.6755477	0.6777859	0.6775938
0.80	0.8207739	0.8206559	0.8220317	0.8219170	0.8232776	0.8231609
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 4: Numerical Results **of Example 2** for ε=0.005, N=100

x	**δ=0.0005, η=0.0001**		**δ=0.0005=η**		**δ=0.0005, η= 0.0009**	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3888459	0.3885157	0.3890998	0.3887657	0.3893480	0.3890156
0.04	0.3855476	0.3849926	0.3858402	0.3852821	0.3861268	0.3855714
0.06	0.3930885	0.3925262	0.3933820	0.3928168	0.3936695	0.3931071
0.08	0.4009725	0.4004109	0.4012656	0.4007010	0.4015525	0.4009909
0.10	0.4090181	0.4084576	0.4093105	0.4087471	0.4095968	0.4090364
0.20	0.4517334	0.4511832	0.4520206	0.4514674	0.4523015	0.4517514
0.40	0.5510128	0.5505093	0.5512754	0.5507694	0.5515324	0.5510292
0.60	0.6721113	0.6717017	0.6723247	0.6719133	0.6725337	0.6721245
0.80	0.8198239	0.8195741	0.8199540	0.8197031	0.8200815	0.8198320
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 5: Numerical Results of **Example 3** for ε=0.01, N=100

x	**δ=0.001, η=0.005**		**δ=0.005=η**		**δ=0.009, η= 0.005**	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.1094809	0.1227591	0.1081350	0.1211090	0.1068117	0.1194889
0.04	0.0156566	0.0186041	0.0155188	0.0183522	0.0153938	0.0181171
0.06	0.0062106	0.0066617	0.0063555	0.0067770	0.0065071	0.0069001
0.08	0.0057535	0.0057689	0.0059596	0.0059658	0.0061710	0.0061684
0.10	0.0063126	0.0062567	0.0065448	0.0064836	0.0067823	0.0067158
0.20	0.0110616	0.0109575	0.0114247	0.0113141	0.0117945	0.0116773
0.40	0.0341085	0.0338675	0.0349449	0.0346909	0.0357898	0.0355229
0.60	0.1051741	0.1046780	0.1068863	0.1063678	0.1086025	0.1080617
0.80	0.3243055	0.3235398	0.3269347	0.3261408	0.3295490	0.3287274
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 6: Numerical Results of **Example 3** for ε=0.005, N=100

x	δ=0.0001, η=0.0005		δ=0.0005=η		δ=0.0009, η= 0.0005	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.0178655	0.0195213	0.0178369	0.0194850	0.0178084	0.0194490
0.04	0.0041111	0.0039867	0.0041262	0.0040009	0.0041414	0.0040150
0.06	0.0043845	0.0041877	0.0044024	0.0042050	0.0044205	0.0042223
0.08	0.0049182	0.0047005	0.0049379	0.0047195	0.0049578	0.0047387
0.10	0.0055205	0.0052813	0.0055422	0.0053023	0.0055640	0.0053233
0.20	0.0098372	0.0094574	0.0098715	0.0094907	0.0099061	0.0095241
0.40	0.0312358	0.0303269	0.0313175	0.0304071	0.0313998	0.0304873
0.60	0.0991824	0.0972491	0.0993555	0.0974203	0.0995293	0.0975916
0.80	0.3149324	0.3118478	0.3152070	0.3121222	0.3154826	0.3123965
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 7: Numerical Results of **Example 4** for ε=0.01, N=100

x	δ=0.001, η=0.005		δ=0.005=η		δ=0.009, η= 0.005	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8201371	0.8200187	0.8188482	0.8187308	0.8175434	0.8174245
0.40	0.6726251	0.6724306	0.6705123	0.6703200	0.6683772	0.6681829
0.60	0.5516449	0.5514057	0.5490478	0.5488117	0.5464274	0.5461891
0.80	0.4524246	0.4521630	0.4495868	0.4493290	0.4467281	0.4464684
0.90	0.4096702	0.4093998	0.4067760	0.4065081	0.4038625	0.4035918
0.92	0.4012797	0.4009919	0.3983587	0.3980641	0.3954176	0.3951149
0.94	0.3907402	0.3903787	0.3876992	0.3872665	0.3846339	0.3841594
0.96	0.3626873	0.3621779	0.3590897	0.3580557	0.3554546	0.3541659
0.98	0.1994671	0.2009362	0.1941174	0.1917881	0.1887210	0.1847184
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000

Table 8: Numerical Results of **Example 4** for ε=0.005, N=100

x	δ=0.0001, η=0.0005		δ=0.0005=η		δ=0.0009, η= 0.0005	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8197590	0.8195095	0.8196320	0.8193802	0.8195019	0.8192506
0.40	0.6720049	0.6715958	0.6717966	0.6713839	0.6715834	0.6711717
0.60	0.5508822	0.5503792	0.5506259	0.5501186	0.5503638	0.5498578
0.80	0.4515907	0.4510410	0.4513105	0.4507563	0.4510241	0.4504714
0.90	0.4088726	0.4083128	0.4085873	0.4080229	0.4082956	0.4077327
0.92	0.4008266	0.4002657	0.4005406	0.3999752	0.4002483	0.3996845
0.94	0.3929316	0.3923772	0.3926452	0.3920862	0.3923523	0.3917950
0.96	0.3847817	0.3846442	0.3844923	0.3843529	0.3841966	0.3840614
0.98	0.3534923	0.3770636	0.3531339	0.3767721	0.3527690	0.3764803
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000

Table 9: Numerical Results of **Example 5** for ε=0.01, N=100

x	δ=0.005, η=0.001		δ=0.005=η		δ=0.005, η= 0.009	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8207732	0.8206559	0.8220346	0.8219171	0.8232757	0.8231609
0.40	0.6736687	0.6734761	0.6757410	0.6755477	0.6777827	0.6775939
0.60	0.5529291	0.5526921	0.5554826	0.5552442	0.5580020	0.5577688
0.80	0.4538293	0.4535700	0.4566260	0.4563647	0.4593894	0.4591334
0.90	0.4111035	0.4108346	0.4139587	0.4136878	0.4167818	0.4165221
0.92	0.4027260	0.4024331	0.4056069	0.4053128	0.4084550	0.4082114
0.94	0.3922449	0.3918253	0.3952396	0.3948204	0.3981973	0.3981311
0.96	0.3644656	0.3634869	0.3679978	0.3670035	0.3714807	0.3731723
0.98	0.2021188	0.1999686	0.2073903	0.2049766	0.2126045	0.2309708
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000

Table 10: Numerical Results of **Example 5** for ε=0.005, N=100

x	δ=0.0005, η=0.0001		**δ=0.0005=η**		**δ=0.0005, η= 0.0009**	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8198244	0.8195741	0.8199533	0.8197032	0.8207421	0.8206241
0.40	0.6721120	0.6717017	0.6723235	0.6719133	0.6736173	0.6734240
0.60	0.5510136	0.5505093	0.5512739	0.5507694	0.5528658	0.5526280
0.80	0.4517343	0.4511832	0.4520189	0.4514674	0.4537601	0.4534999
0.90	0.4090191	0.4084576	0.4093089	0.4087471	0.4110329	0.4107640
0.92	0.4009733	0.4004109	0.4012638	0.4007010	0.4026548	0.4023689
0.94	0.3930786	0.3925226	0.3933697	0.3928132	0.3921708	0.3918114
0.96	0.3849301	0.3847897	0.3852240	0.3850807	0.3643779	0.3638624
0.98	0.3536752	0.3772092	0.3540379	0.3775004	0.2019874	0.2033520
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000

Table 11: Numerical Results of **Example 6** for ε=0.01, N=100

x	**δ=0.005, η=0.001**		δ=0.005=η		δ=0.005, η= 0.009	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.5562779	0.5557454	0.5587557	0.5582183	0.5612090	0.5606658
0.40	0.3094450	0.3088530	0.3122078	0.3116077	0.3149554	0.3143461
0.60	0.1721374	0.1716436	0.1744479	0.1739451	0.1767556	0.1762431
0.80	0.0957562	0.0953902	0.0974737	0.0970993	0.0991968	0.0988135
0.90	0.0713902	0.0710767	0.0728351	0.0724996	0.0742875	0.0739417
0.92	0.0671167	0.0667839	0.0685213	0.0680689	0.0699333	0.0694517
0.94	0.0615982	0.0611406	0.0630215	0.0616699	0.0644498	0.0628982
0.96	0.0442006	0.0435159	0.0459409	0.0382505	0.0476774	0.0382895
0.98	-0.0720914	-0.0679831	-0.0690684	-0.1199952	-0.0660603	-0.1315051
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000

Table 12: Numerical Results of **Example 6** for ε=0.005, N=100

x	δ=0.0005, η=0.0001		δ=0.0005=η		δ=0.0005, η= 0.0009	
	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.	Numerical Sol.	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.5548313	0.5533746	0.5550852	0.5536292	0.5553393	0.5538834
0.40	0.3078378	0.3062235	0.3081196	0.3065052	0.3084018	0.3067868
0.60	0.1707981	0.1694563	0.1710326	0.1696902	0.1712677	0.1699241
0.80	0.0947641	0.0937728	0.0949377	0.0939454	0.0951117	0.0941181
0.90	0.0705869	0.0697568	0.0707324	0.0699013	0.0708783	0.0700458
0.92	0.0665488	0.0657489	0.0666889	0.0658881	0.0668295	0.0660274
0.94	0.0627372	0.0619713	0.0628723	0.0621053	0.0630077	0.0622395
0.96	0.0588723	0.0584107	0.0590040	0.0585397	0.0591361	0.0586689
0.98	0.0385971	0.0550547	0.0387736	0.0551789	0.0389502	0.0553031
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000

Table 13: Numerical Results of **Example 7** for ε=0.01, N=100

x	δ=0.00, η=0.00	δ=0.005, η=0.001	δ=0.005=η	δ=0.005, η= 0.009
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8832572	0.8832309	0.8832549	0.8832780
0.40	0.7518808	0.7517785	0.7518653	0.7519513
0.60	0.6265452	0.6263362	0.6265016	0.6266667
0.80	0.5204743	0.5201598	0.5203944	0.5206292
0.90	0.4766997	0.4763406	0.4766020	0.4768638
0.92	0.4687217	0.4683546	0.4686207	0.4688871
0.94	0.4610007	0.4606259	0.4608964	0.4611673
0.96	0.4533263	0.4529480	0.4532202	0.4534931
0.98	0.4268595	0.4266440	0.4268158	0.4269973
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000

Table 14: Numerical Results of **Example 6** for ε=0.005,   N=100

x	δ=0.00, η=0.00	δ=0.0005, η=0.0001	δ=0.0005=η	δ=0.0005, η= 0.0009
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8834323	0.8834295	0.8834320	0.8834342
0.40	0.7520655	0.7520551	0.7520639	0.7520724
0.60	0.6266463	0.6266254	0.6266420	0.6266583
0.80	0.5204805	0.5204489	0.5204723	0.5204955
0.90	0.4766710	0.4766349	0.4766611	0.4766871
0.92	0.4686872	0.4686504	0.4686770	0.4687034
0.94	0.4609482	0.4609106	0.4609376	0.4609645
0.96	0.4527357	0.4526978	0.4527250	0.4527518
0.98	0.4115886	0.4115608	0.4115804	0.4115995
1.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000

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