Exact solutions to advection-diffusion problems using differential constraints: applications in heat exchangers design

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Abstract:*This work proposes a new analytical method for solving advection-diffusion equations via differential constraints obtained from boundary conditions or other restrictions to be prescribed. The differential constraints are employed for solving the target equation by splitting or successive reductions of order. Applications in heat exchangers design and simulation are reported.*

Keywords: *advection-diffusion equation, closed-form solutions, differential constraints, heat exchangers*

1 - Introduction

The new analytical methods conceived after the development of the symbolic packages leads to some fundamental changes in the strategies for solving partial differential equations. For many practical purposes, few particular solutions to a given differential equation are often sufficient to solve several boundary value problems of special interest. These particular solutions also satisfy other auxiliary equations which are easier to solve. These equations are known as differential constraints [1],and act as restrictions over the general solution.

If one concerns about practical applications, perhaps the most important question about differential constraintsis whether is really necessary to find the general solution of a given differential equation before applying restrictions, such as boundary conditions.In fact, such a "filtering" process could be applied before finding the solution, simplifying the target equationand improving the computational performance of the corresponding source codes. Moreover, when at least one solution of a given auxiliary equation is known, it becomes possible to replace this variety into the remaining equations in order to specify some arbitrary elements (parameters or functions). This procedure can be regarded as a direct genesis of differential equations, although the candidate solution being not simply prescribed, but produced in a systematic way. Each exact solution achieved can be employed to obtain a new variety via variation of parameters, defining an iterative scheme which stops when an invariant solution is achieved.

In certain sense, this procedure constitutes an alternative method to obtain symmetries without employing Lie groups. From the operational point of view, this procedure presents an important advantage over the methods based on Lie groups: the method dispenses solving the determinant equations, whose solution furnishes the infinitesimals which appears as variable coefficients of the corresponding generators.

For physical scenarios where reliable boundary conditions can be directly prescribed, the genesis process based on variation of parametersmust not be carried out iteratively.In this case, the differential constraints can be obtained from boundaryconditions or other restrictions arising in the problem to be solved. Thus, it becomes possible to find invariant solutions bymeans of the differentiation method. More specifically, when the expressions for certain high order derivatives definedby two or more differential equations are compared, new auxiliary equations arises. These equations furnish new functional relations between the derivatives, which reduces the order of the target equation. This process can be effected recursively, and stops when some derivatives are eliminated from the resulting expressions, yielding a simpler differential equation or even an implicit solution.

Another direct way to find differential constraints consists in finding the intersection of the space solutions of two or more differential equations which are valid in adjacent domains. This procedure is useful in cases where no reliable boundary conditions are available. Since no information about the solution is available along the internal boundaries, the arbitrary elements eventually arising in the solutions could be specified by imposing that thesefunctionsbeing smooth at the interfaces. However, when the differential constraints are obtained from the own target equations, there is no need to imposing continuity conditions over the temperature distribution and its derivatives. Hence, no extra equations appear in the mathematical model. As will be showed in the next section, the proposed formulation constitutes a simple way to ensure the regularity of the solutions obtained.

2 – Basic formulation

The former argument leads to exploit the applications of differential constraints in a very straightforward way. In order to solve practical problems, it is often sufficient to generate a particular solution and perform some changes of variables according with the

additional mechanisms and effects to be considered. In the proposed work adifferential constraintis employed to solve aadvection-diffusion problem in a shell-tube heat exchanger. The differential equations describing the heat transfer inside the ducts, in the tube wall and in the cross flow region (shell) are given, respectively, by

$$w\frac{\partial T}{\partial z} = \propto \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad (1),$$
(internal flow)

$$-\propto \frac{\partial^2 T}{\partial z^2} = \propto \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad (2)$$
(wall) and

$$v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} = \propto$$

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) \qquad (3)$$
(crossflow around cylinders)

In this model, v_r, v_θ and w are, respectively the components of the velocity vector in r, θ and z directions, and a steady fully developed flow along the cylindrical duct is assumed. For simplicity, it is also supposed that the distance between baffles is small enough to ensure that the thermal effects due to the small axial component of the velocity field in the shell side are negligible. In the tube side a parabolic velocity profile for w with a global maximum at the z axis ($r=0$) is prescribed:

$$w = w_0 + w_1 r^2 \qquad (4)$$

A -Solution for the region $0 < r < Ri$

Instead of solving each equation before connecting the corresponding solutions along the respective interfaces, it is more convenient to connect the own differential equations at these boundaries. Imposing simultaneously equations (1) and (2) over the internal surface ($r=R_i$), the first differential constraint is obtained:

$$w\frac{\partial T}{\partial z} = -\propto \frac{\partial^2 T}{\partial z^2}\text{at } R = R_i \qquad (5)$$

Here r_i is the internal radius. The former result, which arises because the right hand sides of (1) and (2) are equal, generates a boundary condition which couples the corresponding solutions at $r=R_i$. Once this differential constraint is valid for all z, it can be solved in order to obtain a local prescription for the temperature along the interface:

$$T = a + be^{\frac{-w}{\propto}z}\text{at } R = R_i \qquad (6)$$

Herea and b are arbitrary parameters. This first kind boundary condition can be readily converted into a candidate solution through variation of parameters. The resulting

variety, whose validity is extended to the entire inner region, may be defined as

$$T = a(r) + b(r)e^{c(r)z} \qquad (7)$$

Replacing (7) in (1) and regrouping terms it results

$$\propto \left[\frac{1}{r}\frac{\partial a}{\partial r} + \frac{\partial^2 a}{\partial r^2}\right]$$
$$+ \left[(w_0 + w_1 r^2)bc - \propto \left(\frac{1}{r}\frac{\partial b}{\partial r} + \frac{\partial^2 b}{\partial r^2}\right)\right]e^{c(r)z} -$$
$$\propto \left[\frac{b}{r}\frac{\partial c}{\partial r} + 2\frac{\partial b}{\partial r}\frac{\partial c}{\partial r} + b\frac{\partial^2 c}{\partial r^2}\right]ze^{c(r)z}$$
$$+\propto b\left[\frac{\partial c}{\partial r}\right]^2 = 0 \qquad (8)$$

Once a, b and c does not depend upon z, all the expressions between brackets must be null, producing a system of highly decoupled equations:

$$\frac{1}{r}\frac{\partial a}{\partial r} + \frac{\partial^2 a}{\partial r^2} = 0 \qquad (9)$$

$$(w_0 + w_1 r^2)bc - \propto \left(\frac{1}{r}\frac{\partial b}{\partial r} + \frac{\partial^2 b}{\partial r^2}\right) = 0 (10)$$

$$\frac{b}{r}\frac{\partial c}{\partial r} + 2\frac{\partial b}{\partial r}\frac{\partial c}{\partial r} + b\frac{\partial^2 c}{\partial r^2} = 0 \qquad (11)$$

$$\frac{\partial c}{\partial r} = 0 \qquad (12)$$

From equation (12) it turns out that c is a parameter ($c=c_0$), a result which also identically satisfies (11). The general solution of (9) is obtained directly by integration:

$$a = a_0 + a_1 lnr \qquad (13)$$

Therefore, (10) is the only remaining equation, which admits the following Gaussian-like solution:

$$T = a_0 e^{a_1 r^2} \qquad (14)$$

It occurs because successive differentiations generate, via chain rule, polynomial coefficients multiplying the exponential. Indeed, substituting (14) in (10) and dividing by the exponential term, an algebraic equation is produced:

$$b_0[(c_0 w_1 - 4 \propto b_1^2)r^2 + c_0 w_0 - 4 \propto b_1]$$
$$=0 \qquad (15)$$

Once the coefficients of the polynomial in r must be equal tozero,

$$b_1 = \frac{c_0 w_0}{4 \propto} \qquad (16)$$

and

$$c_0 = 4 \propto \frac{w_1}{w_0^2} \qquad (17)$$

Therefore, the temperature distribution is a particular solution of (1) which contains only five arbitrary parameters:

$$T = a_0 + a_1 \ln r + b_0 e^{\frac{w_1}{w_0} r^2 + 4 \propto \frac{w_1}{w_0^2} z} \qquad (18)$$

In this equation a_1 must vanishto prevent a singularity at r=0. In order to determine the parameters in (39) it becomes convenient to start by applying an asymptotic restriction as a first kind boundary condition:

$$\lim_{z \to \infty} T = T_{io} \qquad (19)$$

Here T_{io} denotes the inlet temperature of the outer fluid. Once the exponential must tend to zero at large distances, it comes out

$$a_0 = T_{io} \qquad (20)$$

Applying now the inlet condition

$$T(0,0) = T_{ii} \qquad (21)$$

which prescribes the temperature at the inlet (z=0) of the centerline (r=0) for the inner fluid, it yields

$$a_1 = T_{ii} - T_{io} \qquad (22)$$

The remaining parameters are specified by restrictions about the flow. The first is an extension of the classical no slip condition at the wall:

$$w = w_w \ at \ r = R_i \qquad (23)$$

Here, w_w denotes the velocity near the wall. This restriction furnishes

$$w_0 = w_w - w_1 R_i^2 \qquad (24)$$

The second restriction states that the volumetric flow rate crossing each transversal section of the tube, defined by

$$Q = \int_0^R 2\pi r w \ dr \qquad (25)$$

isequal to the corresponding plug flow value, namely

$$Q = \pi R^2 W_\infty \qquad (26)$$

This restriction specifies the last free parameter in the model:

$$w_1 = \frac{w_w - 2W_\infty}{R_i^2} \qquad (27)$$

Hence, the temperature distribution in the region

$0 < r < R_i$ becomes

$$T = T_{io} +$$

$$(T_{ii} - T_{io}) e^{\frac{2(W_w - W_\infty)}{R_i^2 (W_w - 2W_\infty)} \left(-r^2 + \frac{4 \propto z}{(W_w - 2W_\infty)}\right)} \qquad (28)$$

An analytical expression for the velocity at the wall is deduced in section E.

B - Solution for the region $R_i < r < R_0$

In the region $R_i < r < R_0$ the solution is obtaining after replacing (7) in (2), which generates the same expressions for a and c. The differential equation defining b, given by

$$\frac{1}{r} \frac{\partial b}{\partial r} + \frac{\partial^2 b}{\partial r^2} + c_0^2 b = 0 \qquad (29)$$

is the Bessel equation,whose solution is

$$b = b_1 J_0(c_0 r) + b_2 Y_0(c_0 r) \qquad (30)$$

Therefore, the temperature distribution in this region is defined as

$$T = a_0 + [b_1 J_0(c_0 r) + b_2 Y_0(c_0 r)] e^{c_0 z} \qquad (31)$$

C - Solution for the region $r > R0$

For the external regionanother differential constraint must be obtained. Imposing simultaneously (2) and (3) at r = R0it yields

$$\frac{v_\theta}{R_0} \frac{\partial T}{\partial \theta} = \propto \left(\frac{1}{R_0^2} \frac{\partial^2 T}{\partial \theta^2} \right) \qquad (32)$$

In this equation, the radial component of the velocity field vanishes due to the non-penetration boundary condition. This differential constraint also imposes an exponential behavior for the temperature along the angular variable. Solving (32), it results

$$T = a + b e^{\frac{v_\theta R_0}{\propto} \theta} \qquad (33)$$

where a and b are parameters. Hence, the solution model for this region is obtained by imposing simultaneously (31) and (33) at r = R_0. In this case the simplest candidate solution for genesis is given by:

$$T = a(r) + b(r) e^{c_1 \theta + c_0 z} \qquad (34)$$

This model yet considered that the arbitrary elements in the argument of the exponential function are constants, a restriction which would naturally appear even if these elements were previously prescribed as arbitrary functions of

r.Substituting (34) in (3) it produces the same model for a, given by (9), and the following auxiliary equation for b:

$$\propto \left(\frac{1}{r}\frac{\partial b}{\partial r} + \frac{\partial^2 b}{\partial r^2} \right) +$$

$$\left(\propto c_0^2 + \frac{\propto c_1^2}{r^2} - \frac{v_\theta c_1}{r} \right) b = 0 \quad (35)$$

This solution for b also depends on the model adopted for vq(r). For simplicity, a linear dependence on r is chosen:

$$v_\theta = k_1 r \qquad\qquad (36)$$

Although the resulting equation do not admits exact solutions in Gaussian form, this model constitutes a very accurate approximate one. In fact, replacing

$$b = b_0 e^{b_1 r^2} \qquad\qquad (37)$$

in (56), the following set of algebraic equations is obtained:

$$\propto b_0 c_1^2 = 0 \qquad\qquad (38)$$

$$\propto b_0 b_1^2 = 0 \qquad\qquad (39)$$

$$c_1 k_1 - 4 \propto c_1 - \propto c_0^2 = 0 \qquad\qquad (40)$$

In practice, typical values for a are about 10-6, while the magnitude of b1 and c1 are 10-3 and b0 ranges from 10 to 100, so equations (38) and (39) may be considered yet satisfied. Indeed, the typical magnitude of the left hand side of these equations are less than 10-9so equation (40) may be solved for k1, yielding

$$k_1 = \propto \left(4 + \frac{c_0^2}{c_1} \right) \qquad\qquad (41)$$

while (38) and (39) are taken as identities. The corresponding velocity profile and temperature distribution for this region are then given, respectively, by

$$v_\theta = \propto \left(4 + \frac{c_0^2}{c_1} \right) r \qquad\qquad (42)$$

and

$$T = a_0 + a_1 \ln r + b_0 e^{b_1 r^2 + c_1 \theta + c_0 z} \qquad (43)$$

D- Estimating heat dissipation, area and mass flow rate

The arbitrary parameters in (31) and (33) are specified in terms of the original data input when the solutions are compared at the interfaces. Once (28) and (31) are identical at r = R_i, at least two parameters can be explicitly defined: a1=0 and

$$c_0 = \frac{-2\propto}{R_i^2 W_\infty} \qquad\qquad (44)$$

Hence, in equation (31) b1=0, in order to prevent imaginary terms generated by Y0 for negative arguments. The solution in the wall region then reduces to

$$T = a_0 + b_1 J_0 \left(\frac{-2\propto}{R_i^2 W_\infty} r \right) e^{\frac{-2\propto}{R_i^2 W_\infty} z} \quad (45)$$

In equation (43) the asymptotic condition yet applied to the internal region remains validsince the reference temperatures being substituted by the corresponding values of the external fluid. Therefore,

$$a_0 = T_{eo} \qquad\qquad (46)$$

and

$$a_1 = \frac{T_{eo} - T_{ei}}{n_t n_c} \qquad\qquad .(47)$$

Herent is the number of rows of the tube bank, nc is the number of times the flow pass the bundleand the firstsubscript denotes the external fluid. Hence, equation (43) becomes

$$T = T_{e0} + \frac{T_{eo} - T_{ei}}{n_t n_c} e^{b_1 r^2 + c_1 \theta + \frac{-2\propto}{R_i^2 W_\infty} z} \quad .(48)$$

Notice that equation (48) is valid around a single row of tubes, so the remaining parametersare determined numerically. Each temperature profile obtained downstream is prescribed as an inlet boundary condition for the next row. The first numerical value to be prescribed upstream comes from the own energy balance:

$$m_e c_e (T_{eo} - T_{ei}) = m_i c_i (T_{ii} - T_{io}) \quad (49)$$

Taking into account that theoutlet temperature of the internal fluid is usually prescribed, there are two unknown elements in (49): the mass flux and the outlet temperature of the external fluid. Consequently, is more convenient to prescribe the mass flux and then estimatingthe outlet temperature:

$$T_{eo} = T_{ei} + \frac{m_i c_i}{m_e c_e} (T_{ii} - T_{io}) \qquad .(50)$$

The heat transferred by each interface of a single tube, defined as

$$q = -2\pi k_i \int_0^L r \frac{\partial T}{\partial r} dz \, \text{at} \, r = R_i \qquad (51)$$

can be obtained from equation (28),which contains only known parameters::

$$q = 2\pi k_i R_i^2 \frac{W_\infty}{\propto} (T_{ii} - T_{io}) \left[e^{-\frac{2\propto z}{R_i^2 W_\infty}} - 1 \right]$$

$$(52)$$

In these equations, L is the tube length and ki is the thermal conductivity of the internal fluid. The total heat flux transferred by the tube bank, given by

$$Q = 2\pi N k_i R_i^2 \frac{W_\infty}{\propto}(T_{ii} - T_{io})\left[e^{-\frac{2\propto z}{R_i^2 W\infty}} - 1\right]$$

(53)

where N is the number of tubes, must be consistent with the right hand side of (49):

$$Q = m_i c_i (T_{ii} - T_{io})$$ (54)

Hence, N is defined by

$$N = \frac{\propto m_i c_i (T_{ii} - T_{io})}{2\pi k_i R_i^2 W_\infty (T_{ii} - T_{io})\left[e^{-\frac{2\propto z}{R_i^2 W\infty}} - 1\right]}$$

(55)

Equations (28),(49), (52) and (55) can be employed to define a direct method to perform heat exchangers design in three steps:

- In order to determine the length of the heat exchanger, it becomes necessary to solve a system containing equation (52) and any constraint which accounts for costs minimization. If there is no extra constraint to impose, the outlet temperature of the inner fluid must be prescribed over equation (28), which is solved for z.

- The mass flow rate or the outlet temperature of the water can be determined using equation (49). A typical prescription establishes the maximum flow rate, whose numerical value depends essentially upon the upper bound for the pressure drop [3,4].

- Finally, the number of tubes (and so the total area) is obtained from (55).

This scheme can be simplified to a single step calculation if the free stream velocity in the tube side is written as

$$W_\infty = \frac{m}{\rho \pi R_i^2 N}$$ (56)

in equation (53). In this case, the heat transferred is expressed as a function of the number of tubes in the bank and the respective tube length.

Once defined the desired parameters it is possible to check the temperature distribution in all the regions either to establish operational conditions or to verify the need for extra passes.

Although the analytical character of the solutions obtained allows formulating flexiblecodesapplicable to a wide class of problems involving design and simulation, the accuracy of the solutions must be improved in order to account for turbulence, which affects the velocity near the wall and the thermal diffusion coefficient.

E - Estimating the velocity near the wall

In order to define Ww an extra boundary condition is applied over the velocity profile:

$$\tau = \mu \frac{\partial w}{\partial r} \qquad (57)$$

For the parabolic profile defined by (4) this restriction furnishes a definition for Ww as a function of the shear stress at the wall:

$$W_w = W_\infty - \frac{R_i \tau}{4\mu} \qquad (58)$$

The shear stress is estimated using the following empirical relation [2]:

$$\tau = 0.0296 \, Re_z^{0.2} \, \rho \, W_\infty^2 \qquad (59)$$

In this equation, the Reynolds number based on the axial coordinate is given by

$$Re_z = \frac{\rho W_\infty i z}{\mu} \quad (60)$$

F -Correcting the thermal diffusivity

The proposed model do not considers the effects of turbulence over the thermal diffusivity. Taking into account that the diffusion coefficient is defined in microscopic scale as [3]

$$D = \frac{l^2}{2\tau} \qquad (61)$$

where l is the mean free path and τ the average time delay between two successive collisions, it becomes possible to deduce a dimensionless amplification factor in the form

$$f = \frac{D_{turbulent}}{D_{Brownian\ motion}} \qquad (62)$$

in order to obtain a definition for the turbulent heat diffusivity:

$$\propto = f \, D_{Brownian\ motion} \qquad (63)$$

The amplification factor can be obtained bydefining the mean free path and the mean velocity for two different scales. For the macroscopic scale, the mean free path is about R_i and the corresponding velocity is approximately W_∞.In molecular scale these values need not to be defined, because the momentum diffusion coefficient for the Brownian motion is the own kinematic viscosity. Hence, the turbulent diffusion is defined as

$$D = \frac{l^2}{2\tau} = \frac{l}{2}\frac{l}{\tau} = \frac{R_i}{2}W_\infty \qquad (64)$$

Therefore, the dimensionless factor is the Reynolds number based on a mean free path about a half of the internal tube radius.

Notice that the dimensionless factor should be defined locally, once the viscosity varies with temperature. Besides, the own definition of the Reynolds number could be extended in order to account for microscopic effects [4].

These improvements are still out of the scope of this work, but the model yet produces reasonableresults, as will be showed in the next section.

3- RESULTS AND DISCUSSION

A practical application of the proposed method is now presented [2].Water is heatedfrom 38°C to 54°C in the ducts of a shell-tube heat exchanger. The external fluid is also water, which enters the shell at 93°C. The mass flow rates for the tube and shell sides are, respectively, 3.8 kg/s and 1.9 kg/s and. the numerical values for the physical properties are showed in table 1.

Table 1 – Physical properties of water

K (W/m^{o}C)	ρ (Kg/m^3)	ν (m^2/s)	C (J/Kg^{o}C)
0,5	1000	10^{-6}	4200

These values correspond to a thermal diffusivity about 10-6 m2/s, which is corrected by the dimensionless amplification factor presented in section 2.6. The internal tube radius is 0,0095m, and the thermal conductivity of the interface (considering made of steel) is 16W/m^{o}C.Table 2 shows the heat transferred per second from the shell side as a function of number of tubes and the corresponding length.

Table 2 – Power transferred (W)

L(m)	N = 10	N = 20	N = 30	N = 40	N = 50
2.0	58019	111474	160025	203715	242737
2.2	63661	121921	174417	221238	262648
2.4	69276	132261	188568	238348	281950
2.6	74867	142495	202487	255062	300672
2.8	80434	152627	216180	271396	318838
3.0	85976	162660	229655	287364	336475

The experimental data available for 36 tubes with length about 2.9 m is q = 264 kW, which is consistent with the results obtained.

The calculations were executed in a low performance microcomputer (AMD Sempron 1.8GHz, with 512Mb RAM), using MapleV. The total processing time demanded to run the mws code (including symbolical manipulations) is less than5s.It is important to stress that the proposed formulation dispenses the previous information required by LMTD schemes [5-7] to carry out an iterative process, such as the global heat transfer coefficient and the outlet temperatures of the fluids.

Notice that the global heat transfer coefficient cannot be considered asa data input in this problem, since convective coefficients are unknown. Besides, these coefficients vary along the interfaces, because it depends upon the temperature distribution at the boundaries. Therefore, the proposed approach is more realistic than the schemes usually adopted to estimate the thermal energy dissipated by the shell. Once the temperature distribution is not known along internal interfaces, it makes no sense to prescribe boundary conditions or even define convection coefficients at these regions.

The connectivity conditions are applied at boundaries not only to avoid iterative schemes, but also to prevent an error which often appears in undergraduate texts: employing weak solutions whose first derivatives are discontinuous at the interfaces. This prescription, which is more commonly employed in one dimensional diffusion models for inhomogeneous media, clearly violates the target equation, at the interfaces, provided that the normal second order derivative is a Dirac delta function.

4 - CONCLUSIONS

The most important information about themethodcan be summarized as follows.

I. Advection-diffusion equations can be solved by genesis, using differential constraints to obtain a model for the solution instead of performing a trial and error process based on arbitrary prescriptions. The differential constraints can be constructed from boundary conditions or other restrictions to be prescribed, as well as from the own differential equations.

II. These constraints often constitute first order PDEs which allows reducing the order of the target equation or even eliminate some derivatives respect to spatial variables. This simplification provides a considerable reduction in the corresponding computational code.

III. When the differential constraints are obtained from the owntarget equations, it becomes possible to ensure the regularity of the corresponding solutions without imposing connectivity conditions at the interfaces. Therefore, the computational code becomes even simpler. In addition, differential constraints produced from differential equations are more reliable than those originated from boundary conditions.

IV. The formulation based on genesis furnishes exact solutions and closed form approximations, demanding a time processing about few seconds to be obtained in symbolical packages.

V. From these solutions, a straightforward method for heat exchangers design is proposed.This method dispenses a priori estimation for the global heat transfer coefficient or even knowledge about inlet and outlet temperatures, such as in the LMTD schemes.

VI. The most serious limitation of the proposed method relies on the fact that the variations of viscosity with temperature were neglected, and thus eventually may produce unrealistic results [8].

In order to overcome this limitation, our work is now focused in obtaining an explicit solution which considers the variation of the kinematic viscosity with temperature. In this case there are at least two ways to avoid implicit solutions. The first is based on two-level recursive definitions where the argument of an empirical expression for the viscosity as a function of temperature is employed. In this model, the

argument of this function is the own temperature distribution, so the viscosity becomes a known function of the own coordinates. The second is a perturbation scheme based on low order Taylor expansions ofhigh level recurrence relations, which will be applied in cases where the first formulation fails.

There is also a more rigorous model to formulate, in order to take into account the variable viscosity. This model describes the heat transfer phenomena in micro scale. This heat transfer model is a Boltzmann-type equation whose integral scattering is replaced by a nonlinear term. Although conceptually complex, this approach can be converted in a viable analytical method by applying differential constraints and Bäcklund transformations. The foundations of the proposed formulation, which are yet being developed, are briefly described as follows.

A –A possible model in micro scale

Although the formulation based on differential constraints being a straightforward strategy for solving partial differential equations, the ideas behind this simple method are by no means trivial. Some of these ideas may give raise to new unexpected applications in fluid mechanics and quantum field theory. This feature can be readily understood when some connections between differential constraints and other methods employed for solving PDEs are established.

The proposed method is closely related to other interesting subjects in the field of differential equations, such as Bäcklund transformations and Lie group analysis. Indeed, this formulation can be considered as an attempt to avoid the factorizationrequired to obtain Bäcklund transformations, a task which is not completely systematic. This situation is analogous to find the Lax pair in the inverse scattering methods: there are no algorithmfor finding two differential operators whose commutator is the own operator which applied over the unknown function defines the target equation.In this case, the differential constraints obtained from third kind boundary conditions allow reducing the order of the target equation, dispensing the factorization process.

On the other hand, the method can be regarded as a way to increase the dimension of the Lie group of a given differential equation. Notice that the number of symmetries admitted by the target equation increases whenapplying differential constraints, provided that the space of solutions is reduced to a subset which obeys the corresponding boundary conditions. Therefore, applying differential constraints is equivalent to add new generators to the symmetry group admitted by the target equation.

The former conclusion has a very interesting consequence. In most practical cases, where there is only one reliable boundary condition available to apply over the variety which defines the solution obtained, this restriction is often not sufficient to specify its arbitrary elements. However, once the target equation changes after using the differential constraint, a new symmetry group arises after solving the corresponding determinant equations. After finding the new generators, it becomes possible to obtain other elements of the Lie group by means of commutators or Jacobi's identity. In this case, new natural boundary conditions emerge, and so

new differential constraints can be also obtained via Noether´s theorem. This recursive process stops when the new group results closed respect to the commutator and Jacobi's identity, or when the differential constraints remain invariant.

The former conclusionsinduce to step forward in a specific direction. If one considers that Bäcklund transformations perform a mapping between two apparently uncorrelated models, such as purely diffusive transport and nonlinear advection-diffusion (Burgers-type) equations, another fundamental conclusion arises: it is possible to reduce nonlinear partial differential equations to purely diffusive models using mappings and differential constraints? In order to answer this question, it becomes necessary to consider the nature of the most common nonlinearities which arises in quantum field models and advection-diffusion equations, such as Navier-Stokes and Helmholtz.

When one search for a specific Lagrangian to describe a given physical phenomena, such as scattering or chemical reactions, the resulting Euler-Lagrange equation contains polynomial nonlinearities which defines the interaction between particles.

An analogous situation occurs in macroscopic scale, when one defines material derivatives in order to account for advection terms in transport equations. In this case, when the chain rule is employed to define the velocity field, the parameterization of a path followed by each molecule of the corresponding fluid is implicitly assumed.

In both cases the medium is ultimately considered as composed by particles which are supposed to preserve their own identities along time.

This assumption must not be taken so seriously for scattering processes and chemical reactions. Roughly speaking, in scattering processes there are no indicia that the particles emerging from a given scattering medium are really the ones whichwere in the incident beam. The scattering amplitudes and angles induce to conclude that particles behave much more as interference patterns between fields than as independent massive objects.

In the second case, when a given chemical reaction occurs, the conclusion seems to be essentially the same. The changes in the electronic cloud in a given molecule due to the field produced by its neighbors(inductive effect) are associated with Eigen functions related to a new energy spectrum, such as occurs in the Zeeman effect. Naturally, the production of a fine structure, which characterizes the new spectrum, is a direct consequence of the changes in the interaction potential, which depends upon the total wave function. But this is exactly the effect expected when new atoms are added to a given molecule, which produces a new bound state. Thus, it becomes necessary totake into account the radiation interacting with the molecules in order to compose a realistic total wave function(otherwise it isnot possible to investigate the consequences of the inductive effect over the reaction mechanism).

Once the potential depends on the total wave function and its derivatives, the corresponding interaction term may be

considered as an advectionnonlinear one. Once this vector field can also be expressed as a material derivative, any particle arising from the field would be more adequately described by a N-soliton model.

Consequently, the concepts of fermion and boson seem to be much more an arbitrary way to distinguish particles and fields than a consistent form of thinking about statistical physics. Moreover, this point of view often induces to choose some specific variables as "natural candidates" for unknown functions to a given problem.

This choice seems to be the origin of the nonlinearities appearing in macroscopic and microscopic models. These nonlinear terms would not necessarily appear in some alternative formulations where the unknown functions were chosen in order to suppress the material derivatives. For instance, the Helmholtz equation can be viewed as an advection diffusion model where the solid interfaces acts as "sources of vorticity" distributed along the flow. If the kinetic energy were chosen as the unknown variable instead of the vorticity function, the interfaces would be considered as sinks, so the physical interpretation of the corresponding scenario would be essentially equivalent. Nevertheless, from the operational point of view, the last interpretation is advantageous, because advection terms are not expected to arise in a hydrodynamic model based on kinetic energy. Consequently, the resulting equation should be easily converted into a linear model whose solutions are mapped into ones of the original problem by applying nonlinear operators.

Aside from considerations about symmetries and conservation laws, the only practical limitation of this approach is that it ever produces only particular solutions of the original problem. However, this is not a serious limitation, once the subspace of solutions can be easily generalized using symmetries admitted by the own target equation.

In future workswe step forward by showing that the Bäcklund-type transformations are more than a mapping procedure. Behind these transformations arises a systematic way to obtain new dependent variables which furnishes a useful point of view for simplifying the way of reasoning about modeling and solving nonlinear problems in Physics and Engineering.

In this new point of view the differential constraints play a crucial role. These auxiliary differential equations are directly related to certain nontrivial conservation laws. These conservation laws naturally arise when the concept of point particles whose interactions are mediated by bosons is replaced by interacting complex fields.

In this picture, the fermionicandbosoniccharacteris, in some sense, a projection of a more comprehensive behavior of a local field, and becomes merely a consequence of a mapping induced by operators.

This point of view allows treating heat transfer problems in a completely different way: by solving an auxiliary problem involving only radiation scattering, without using the Boltzmann equation in its original form.

Once the ultimate heat transfer mechanism is in fact photon emission and absorption, the only technical difficulty in solving Boltzmann-type equations, e. g., solving the SN equations produced by discretization of the scattering kernel, can be finally surmounted. The scattering integral can be eliminated from the Boltzmann equation by applying a differential operator whose fixed point is the scattering cross section. Once the cross section depends on the scattering potential, and consequently on the wave function, the Boltzmann equation can be mapped into a nonlinear PDE term which defines a new local model for heat conduction problems.

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