Queuing with Reneging, Balking and Retention of Reneged Customers

R. Kumar, S. K. Sharma

***Abstract*—**Customer impatience has become a serious problem in the business world. Customer retention is the key issue in the organizations facing the problem of customer impatience. Keeping in mind this hot topic, we study single as well as multi-server Markovian queuing systems with reneging, balking and retention of reneged customers. We study four queuing models and present their steady-state solutions. Some important measures of performance are derived and finally, some queuing models are derived as particular cases of these models.

***Keywords*—** Customer Retention, Reneging, Balking, Markovian queuing systems, Steady-state Solution.

I. INTRODUCTION

QUEUING models have been effectively used in the design and analysis of telecommunication systems, traffic systems, service systems and many more. A number of extensions in the basic queuing models have been made and the concepts like vacations queuing, correlated queuing, retrial queuing, queuing with impatience and catastrophic queuing have come up. Of these, queuing with customer impatience has special significance for the business world as it has a very negative effect on the revenue generation of a firm. The notion of customer impatience appears in queuing theory in the work of Haight [5]. Haight [6] studies queuing with reneging. Ancker and Gafarian [1] study M/M/1/N queuing system with balking and reneging and derive its steady-state solution. Ancker and Gafarian [2] obtain results for a pure balking system (no reneging) by setting the reneging parameter equal to zero..

Abou-El-Ata and Hariri [7] consider multi-server finite capacity Markovian queue with balking and reneging. Al-seedy et al. [8] studied M/M/c queue with balking and reneging and derived its transient solution by using the probability generating function technique and the properties of Bessel function. Choudhury and Medhi [9] studied customer impatience in multi-server queues. They considered both balking and reneging as functions of system state by taking into consideration the situations where the customer was aware of its position in the system. Kapodistria [10] studied a single server Markovian queue with impatient customers and considered the situations where customers abandoned the system simultaneously. He considered two abandonment scenarios. In the first one, all present customers became impatient and performed synchronized abandonments, while in the second scenario; the customer in service was excluded from the abandonment procedure. He extended this analysis to the M/M/c queue under the second abandonment scenario also. Kumar [11] investigated a correlated queuing problem with catastrophic and restorative effects with impatient customers which have special applications in agile broadband communication networks.

This study is motivated by the fact that customer impatience leads to loss of potential customers. It has become a highly challenging problem in the current era of cut-throat competition. Customers are the backbone of any business, because without customers there will be no reason for a business to operate. Therefore, the concept of customer retention assumes a tremendous importance for the business management. Firms are employing a number of customer retention strategies to sustain their businesses. Keeping in mind the burning problem of customer impatience, the concept of retention of impatient (reneged) customers has been introduced in queuing modeling in this paper. It is envisaged that the reneged customers may be convinced to stay in the waiting line for their service by employing certain customer retention strategies (mechanisms). Thus, a reneged customer may be retained in the queue for his service with probability q (say) and may not be retained with probability $p = (1 - q)$, that is, he may not be convinced and finally decides to leave.

Recently, Kumar and Sharma [12] study a finite capacity, Markovian multi-server queuing model with reneging and retention of reneged customers. Kumar and Sharma [13] also perform economic analysis of multi-server Markovian queuing systems with of reneged customers. In this paper we study infinite capacity, Markovian single as well as multi-server queuing models and obtain some important measures of performance.

Rest of the paper is arranged as follows: Section 2 deals with steady-state solution and measures of performance of single-server Markovian queuing system with retention of reneged customers. In section 3, we study a Markovian queuing model with retention of reneged customers and

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balking. In section 4, the steady-state solution and measures of performance of a multi-server-server Markovian queuing system with retention of reneged customers are obtained. In section 5, the concept of balking is incorporated in case of multi-server Markovian queuing system with retention of reneged customers where steady-state solution, measures of performance and some particular cases of the model are obtained. The conclusions are presented in section 6.

II. M/M/1 QUEUING MODEL WITH RETENTION OF RENEGED CUSTOMERS

A. Steady-State Solution

We consider an M/M/1 queuing model with reneging. The reneging times are assumed to exponentially distributed with parameter ξ. It is envisaged that a reneged customer may be convinced by applying certain convincing mechanism to stay in the system for his service. Thus, there is a probability say, q that a reneged customer may be retained in the system and may not be retained with some complementary probability say, p (=1-q).

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad \ldots (1)$$

$$\frac{dP_n(t)}{dt} = -[\lambda + \mu + (n-1)\xi p]P_n(t) + (\mu + n\xi p)P_{n+1}(t) + \lambda P_{n-1}(t)$$

$$n \geq 1 \qquad \ldots (2)$$

Theorem 1: If the steady-state equations of M/M/1 queuing system with retention of reneged customers are

$$0 = -\lambda P_0 + \mu P_1 \qquad \ldots (3)$$

$$0 = -[\lambda + \mu + (n-1)\xi p]P_n + (\mu + n\xi p)P_{n+1} + \lambda P_{n-1}$$

$$n \geq 1 \qquad \ldots (4)$$

then, the steady-state-probabilities of system size are given by

$$P_n = \prod_{k=1}^{\infty} \frac{\lambda}{\mu + (k-1)\xi p} P_0 ; n \geq 1 \qquad \ldots (5)$$

with $P_0 = \dfrac{1}{\left(1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \dfrac{\lambda}{\mu + (k-1)\xi p}\right)}$ $\qquad \ldots (6)$

Proof: We obtain the steady-state-probabilities by using iterative method. Re-arranging (3), we get the value of P_1 as

$$P_1 = \frac{\lambda}{\mu} P_0$$

From (4), $n = 1$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$(\mu + \xi p)P_2 - (\lambda + \mu)P_1 - \lambda P_0$$

Using (3), we have

$$(\mu + \xi p)P_2 - \lambda P_1$$

$$P_2 = \prod_{k=1}^{2} \frac{\lambda}{\mu + (k-1)\xi p} P_0$$

Similarly, for $n = 2$, the above procedure gives

$$P_3 = \prod_{k=1}^{3} \frac{\lambda}{\mu + (k-1)\xi p} P_0$$

Therefore, for $n > 2$, we can easily obtain

$$P_n = \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} P_0 ; n > 3$$

Thus, equations (5) completely determine all the steady-state probabilities $P_n ; n \geq 1$.

To find P_0, we use normalization condition $\sum_{n=0}^{\infty} P_n = 1$ and the values of $P_n ; n \geq 1$.

$$\left(1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p}\right) P_0 = 1$$

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p}\right)}$$

which is (6).

The steady-state probabilities exist if $\left(1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p}\right) < \infty$.

This completes the proof.

B. Measures of Performance

The Expected System Size, L_s

$$L_s = \sum_{n=0}^{\infty} n P_n$$

$$L_s = \sum_{n=1}^{\infty} n \left(\prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} \right) P_0$$

Using little's formula $L = \lambda W$, we can obtain the related steady-state measures of performances like average number of customers in the queue (Lq), average waiting time in the system (Ws), average waiting time in the queue (Wq) etc.

The Expected Number of customers Served, $E(Customer\ Served)$**:**

The expected number of customers served is given by

$$E(customerServed) = \sum_{n=1}^{\infty} n\mu P_n$$

$$E(customerServed) = \sum_{n=1}^{\infty} n\mu \left(\prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} \right) P_0$$

Rate of Abandonment, R_{aband}

$$R_{aband} = \lambda \sum_{n=0}^{\infty} P_n - E(customerServed)$$

$$R_{aband} = \lambda - \sum_{n=1}^{\infty} n\mu \left(\prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} \right) P_0$$

Expected number of waiting customers, who actually wait, $E(CustomerWaiting)$:

$$E(CustomerWaiting) = \frac{\sum_{n=2}^{\infty} (n-1) P_n}{\sum_{n=2}^{\infty} P_n}$$

$$E(CustomerWaiting) = \frac{\sum_{n=2}^{\infty} (n-1) \left[\prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0}{\sum_{n=2}^{\infty} \left[\prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0}$$

where P_0 is derived in (6).

C. Special Cases

In this section, we derive some important Markovian queuing models from the M/M/1model with retention of reneged customers.

(i) In the absence of retention of reneged customer (i.e. for q=0), our model reduces to an M/M/1 queuing model with reneging as studied by Haight [6].

(ii) In the absence of reneging (i.e. for ξ=0), our model reduces to a simple M/M/1 queuing model as studied by Gross and Harris [4].

(iii) When the capacity of the system is taken as finite say, N, the resulting model is an M/M/1/N queuing model with retention of reneged customers as studied by Kumar and Sharma [14] with

$$P_n = \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} P_0 ; 1 \le n \le N$$

and

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{N} \prod_{k=1}^{n} \frac{\lambda}{\mu + (k-1)\xi p} \right)}$$

III. M/M/1 QUEUING MODEL WITH RETENTION OF RENEGED CUSTOMERS AND BALKING

A. Steady-State Solution

We consider an M/M/1 queuing model with reneging. The reneging times are assumed to exponentially distributed with parameter ξ. It is envisaged that a reneged customer may be convinced by applying certain convincing mechanism to stay in the system for his service. Thus, there is a probability say, q that a reneged customer may be retained in the system and may not be retained with some complementary probability say, p (=1-q). An arriving customer who finds the server busy on arrival, may balk with probability q_1 or may join the system with probability p_1 such that $p_1 + q_1 = 1$.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad \dots (7)$$

$$\frac{dP_1(t)}{dt} = -[\lambda p_1 + \mu] P_1(t) + (\mu + \xi p) P_2(t) + \lambda P_0(t) \qquad \dots (8)$$

$$\frac{dP_n(t)}{dt} = -[\lambda p_1 + \mu + (n-1)\xi p] P_n(t) + (\mu + n\xi p) P_{n+1}(t) + \lambda p_1 P_{n-1}(t)$$

$$n \ge 2 \qquad \dots (9)$$

Theorem 2: If the steady-state equations of M/M/1 queuing system with retention of reneged customers and balking are

$$0 = -\lambda P_0 + \mu P_1 \qquad \dots (10)$$

$$0 = -[\lambda p_1 + \mu] P_1 + (\mu + \xi p) P_2 + \lambda P_0 \qquad \dots (11)$$

$$0 = -[\lambda p_1 + \mu + (n-1)\xi p] P_n + (\mu + n\xi p) P_{n+1} + \lambda p_1 P_{n-1}$$

$$n \ge 2 \qquad \dots (12)$$

then, the steady-state-probabilities of system size are given by

$$P_n = \frac{\lambda}{\mu} \left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1} \mu + k\xi p} \right] P_0 ; n \ge 1 \qquad \dots (13)$$

with $$P_0 = \frac{1}{\left(1 + \frac{\lambda}{\mu} + \sum_{n=1}^{\infty} \frac{\lambda}{\mu} \left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1} \mu + k \xi p} \right] \right)} \qquad \dots (14)$$

Proof: We obtain the steady-state-probabilities by using iterative method. Re-arranging (10), we get the value of P_1 as

$$P_1 = \frac{\lambda}{\mu} P_0$$

From (11), $n = 1$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$(\mu + \xi p) P_2 = (\lambda p_1 + \mu) P_1 - \lambda P_0$$

Using (10), we have

$$(\mu+\xi p)P_2=\lambda P_1$$

$$P_2=\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{2-1}}{\prod_{k=1}^{2-1}\mu+k\xi p}\right]P_0$$

Similarly, for $n=2$, the above procedure gives

$$P_3=\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{3-1}}{\prod_{k=1}^{3-1}\mu+k\xi p}\right]P_0$$

Therefore, for $n>2$, we can easily obtain

$$P_n=\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]P_0$$

Thus, equations (13) completely determine all the steady-state probabilities $P_n; n\geq 1$.

To find P_0, we use normalization condition $\sum_{n=0}^{\infty}P_n=1$ and the values of $P_n; n\geq 1$.

$$\left(1+\frac{\lambda}{\mu}+\sum_{n=1}^{\infty}\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]\right)P_0=1$$

$$P_0=\frac{1}{\left(1+\frac{\lambda}{\mu}+\sum_{n=1}^{\infty}\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]\right)}$$

which is (6).

The steady-state probabilities exist if

$$\left(1+\frac{\lambda}{\mu}+\sum_{n=1}^{\infty}\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]\right)<\infty.$$

This completes the proof.

B. *Measures of Performance*

The Expected System Size, L_s

$$L_s=\sum_{n=0}^{\infty}nP_n$$

$$L_s=\sum_{n=1}^{\infty}n\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]P_0$$

Using little's formula $L=\lambda W$, we can obtain the related steady-state measures of performances like average number of customers in the queue (Lq), average waiting time in the system (Ws), average waiting time in the queue (Wq) etc.

The Expected Number of customers Served, $E(Customer\ Served)$**:**

The expected number of customers served is given by

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$$E(customerServed)=\sum_{n=1}^{\infty}n\mu\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]P_0$$

Rate of Abandonment, R_{aband}

$$R_{aband}=\lambda\sum_{n=0}^{\infty}P_n-E(customerServed)$$

$$R_{aband}=\lambda-\sum_{n=1}^{\infty}n\mu\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]P_0$$

Expected number of waiting customers, who actually wait, $E(CustomerWaiting)$:

$$E(CustomerWaiting)=\frac{\sum_{n=2}^{\infty}(n-1)P_n}{\sum_{n=2}^{\infty}P_n}$$

$$E(CustomerWaiting)=\frac{\sum_{n=2}^{\infty}(n-1)\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]P_0}{\sum_{n=2}^{\infty}\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu+k\xi p}\right]P_0}$$

where P_0 is derived in (14).

C. *Special Cases*

In this section, we derive some important Markovian queuing models from the M/M/1 model with retention of reneged customers.

(i) In the absence of retention of reneged customer (i.e. for q=0), our model reduces to an M/M/1 queuing model with reneging and balking as studied by Haight [].

(ii) In the absence of reneging (i.e. for ξ=0), our model reduces to a simple M/M/1 queuing with balking model.

(iii) When the capacity of the system is taken as finite say, N, the resulting model is an M/M/1/N queuing model with retention of reneged customers and balking with

$$P_n = \frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu + k\xi p}\right]P_0 ; 1 \le n \le N$$

and

$$P_0 = \frac{1}{\left(1 + \frac{\lambda}{\mu} + \sum_{n=1}^{N}\frac{\lambda}{\mu}\left[\frac{(\lambda p_1)^{n-1}}{\prod_{k=1}^{n-1}\mu + k\xi p}\right]\right)}$$

(iv) In the absence of balking, when $p_1 = 1$, the model reduces to M/M/1 queuing model with retention of reneged customers as studied in section 2.

IV. M/M/c Queuing Model with Retention of Reneged Customers

A. Steady-State Solution

In this section, we consider a multi-server Markovian queuing model with reneging. The arrival process is Poisson with average arrival rate λ. Assume that there are c servers and the service times at each server are independently, identically and exponentially distributed with parameter μ. The mean service rate is given by $\mu_n = \{n\mu; 0 \le n \le c-1 \,\&\, c\mu; n \ge c\}$. A reneged customer may be convinced by applying certain convincing mechanism to stay in the system for his service. A queue gets developed when the number of customers exceeds the number of servers, that is, when $n > c$. Each customer upon joining the queue will wait a certain length of time for his service to begin. If it has not begun by then, he will get reneged and may leave the queue without getting service with probability p and may remain in the queue for his service with probability $q(=1-p)$. The reneging times follow exponential distribution with parameter ξ.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad \ldots (15)$$

$$\frac{dP_n(t)}{dt} = -(\lambda + n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) + \lambda P_{n-1}(t); 1 \le n \le c-1 \quad \ldots (16)$$

$$\frac{dP_n(t)}{dt} = -[\lambda + c\mu + (n-c)\xi p]P_n(t) + [c\mu + \{(n+1)-c\}\xi p]P_{n+1}(t) + \lambda P_{n-1}(t); n \ge c \quad \ldots (17)$$

Theorem 3: If the steady-state equations of M/M/c queuing system with retention of reneged customers are:

$$0 = -\lambda P_0 + \mu P_1 \quad \ldots (18)$$

$$0 = -(\lambda + n\mu)P_n + (n+1)\mu P_{n+1} + \lambda P_{n-1}; 1 \le n \le c-1 \quad \ldots (19)$$

$$0 = -[\lambda + c\mu + (n-c)\xi p]P_n + [c\mu + \{(n+1)-c\}\xi p]P_{n+1} + \lambda P_{n-1}; n \ge c \quad \ldots (20)$$

then the steady-state-probabilities of system size are given by

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 ; 1 \le n \le c \quad \ldots (21)$$

$$P_n = \prod_{k=c+1}^{n}\frac{\lambda}{c\mu + (k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0 ; n > c \quad \ldots (22)$$

With

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{c}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c+1}^{\infty}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu + (k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)} \quad \ldots (23)$$

Proof: We prove the theorem by using iterative method. Rearranging (18), we get the value of P_1 as

$$P_1 = \frac{\lambda}{\mu}P_0$$

For $n = 1$, (19) yields

$$2\mu P_2 = (\lambda + \mu)P_1 - \lambda P_0$$

$$2\mu P_2 = \lambda P_1$$

$$P_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

For $n = 2$, (11) yields

$$3\mu P_3 = (\lambda + 2\mu)P_2 - \lambda P_1$$

Since $2\mu P_2 = \lambda P_1$, we have

$$3\mu P_3 = \lambda P_2$$

$$P_3 = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 P_0$$

Similarly, for $n = c-1$, the value of P_c is

$$c\mu P_c = [\lambda + (c-1)\mu]P_{c-1} - \lambda P_{c-2}$$

Since $n\mu P_n = \lambda P_{n-1}$ for $1 \le n \le c-1$

Therefore $(c-1)\mu P_{c-1} = \lambda P_{c-2}$

Hence $P_c = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0$

Thus, all the probabilities $P_n; 1 \le n \le c$ satisfy (21).

For $n > c$, put $n = c$ in (20).

$$(c\mu + \xi p)P_{c+1} = (\lambda + c\mu)P_c - \lambda P_{c-1}$$

Since $c\mu P_c = \lambda P_{c-1}$

$$(c\mu+\xi p)P_{c+1}=\lambda P_c$$

$$P_{c+1}=\frac{\lambda}{(c\mu+\xi p)}P_c$$

$$P_{c+1}=\prod_{k=c+1}^{c+1}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0$$

Similarly, for $n>c$, the steady-state probabilities $P_n; n>c+1$ are obtained as

$$P_n=\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0; n>c+1$$

All these probabilities $P_n; n>c$ satisfy (22)
To find P_0, we use normalization condition $\sum_{n=0}^{\infty}P_n=1$ and the values of $P_n; n\geq 1$.

$$\left(1+\sum_{n=1}^{c}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\sum_{n=c+1}^{\infty}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)P_0=1$$

$$P_0=\frac{1}{\left(1+\sum_{n=1}^{c}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\sum_{n=c+1}^{\infty}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)}$$

Which is (23).
The steady-state probabilities exist if

$$\left(1+\sum_{n=1}^{c}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\sum_{n=c+1}^{\infty}\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)<\infty$$

Thus, equations (21)-(23) completely determine all the steady-state probabilities of the system size $P_n; n\geq 0$ which completes the proof of the theorem.

B. *Measures of Performances*

The Expected System Size, Ls

$$L_s=\sum_{n=0}^{\infty}nP_n$$

$$L_s=\left[\sum_{n=1}^{c}n\left(\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right)+\sum_{n=c+1}^{\infty}n\left(\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)\right]P_0$$

Using little's formula $L=\lambda W$, we can obtain the related steady-state measures of performances like average number of customers in the queue (Lq), average waiting time in the system (Ws), average waiting time in the queue (Wq) etc.

The Expected Number of customers Served, $E(Customer\ Served)$**:**
The expected number of customers served is given by

$$E(CustomerServed)=\sum_{n=1}^{c}n\mu P_n+\sum_{n=c+1}^{\infty}c\mu P_n$$

$$=\left[\begin{array}{l}\sum_{n=1}^{c}n\mu\left(\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right)\\ +\sum_{n=c+1}^{\infty}c\mu\left(\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)\end{array}\right]P_0$$

Rate of Abandonment, R_{aband}

$$R_{aband}=\lambda\sum_{n=0}^{\infty}P_n-E(customer\ Served)$$

$$R_{aband}=\lambda-\left[\begin{array}{l}\sum_{n=1}^{c}n\mu\left(\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right)\\ +\sum_{n=c+1}^{\infty}c\mu\left(\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)\end{array}\right]P_0$$

Expected number of waiting customers, who actually wait, $E(CustomerWaiting)$:

$$E(CustomerWaiting)=\frac{\sum_{n=c+1}^{\infty}(n-c)P_n}{\sum_{n=c+1}^{\infty}P_n}$$

$$=\frac{\sum_{n=c+1}^{\infty}(n-c)\left[\sum_{n=c+1}^{\infty}n\left(\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)\right]P_0}{\sum_{n=c+1}^{\infty}\left[\sum_{n=c+1}^{\infty}n\left(\prod_{k=c+1}^{n}\frac{\lambda}{c\mu+(k-c)\xi p}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)\right]P_0}$$

C. *Special Cases*

In this sub-section, we obtain some important queuing model as particular cases of our model.

(i) In the absence of retention of reneged customer (i.e. for $q=0$), one can see that the model results resemble with Montazer-Hagighi et al.[3] if we remove balking from the model studied by them.

(ii) In the absence of reneging (i.e. for ξ=0), our model reduces to a simple M/M/c queuing model.

(iii) When the capacity of the system is taken as finite say, N, the resulting model is an M/M/c/N queuing model with retention of reneged customers as studied by Kumar and Sharma [12] with

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0; 1 \le n \le c$$

$$P_n = \prod_{k=c+1}^{n} \frac{\lambda}{c\mu + (k-c)\xi p} \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0; c+1 \le n \le N$$

And

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{c} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c+1}^{N} \prod_{k=c+1}^{n} \frac{\lambda}{c\mu + (k-c)\xi p} \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right)}$$

V. M/M/c Queuing Model with Retention of Reneged Customers and Balking

A. Steady-State Solution

In this section, we consider a multi-server Markovian queuing model with reneging. The arrival process is Poisson with average arrival rate λ. Assume that there are c servers and the service times at each server are independently, identically and exponentially distributed with parameter μ. The mean service rate is given by $\mu_n = \{n\mu; 0 \le n \le c-1 \,\&\, c\mu; n \ge c\}$. A reneged customer may be convinced by applying certain convincing mechanism to stay in the system for his service. A queue gets developed when the number of customers exceeds the number of servers, that is, when $n > c$. Each customer upon joining the queue will wait a certain length of time for his service to begin. If it has not begun by then, he will get reneged and may leave the queue without getting service with probability p and may remain in the queue for his service with probability $q (= 1 - p)$. The reneging times follow exponential distribution with parameter ξ. An arriving customer who finds all the server busy, on arrival, may balk with probability q_1 or may join the system with probability p_1 such that $p_1 + q_1 = 1$.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad \dots (24)$$

$$\frac{dP_n(t)}{dt} = -(\lambda + n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) + \lambda P_{n-1}(t); 1 \le n \le c-1 \qquad \dots (25)$$

$$\frac{dP_c(t)}{dt} = -[\lambda p_1 + c\mu]P_c(t) + [c\mu + \xi p]P_{c+1}(t) + \lambda P_{c-1}(t) \qquad \dots (26)$$

$$\frac{dP_n(t)}{dt} = -[\lambda p_1 + c\mu + (n-c)\xi p]P_n(t) + [c\mu + \{(n+1)-c\}\xi p]P_{n+1}(t) + \lambda p_1 P_{n-1}(t); n > c \qquad \dots (27)$$

Theorem 4: If the steady-state equations of M/M/c queuing system with retention of reneged customers and balking are:

$$0 = -\lambda P_0 + \mu P_1 \qquad \dots (28)$$

$$0 = -(\lambda + n\mu)P_n + (n+1)\mu P_{n+1} + \lambda P_{n-1}; 1 \le n \le c-1 \qquad \dots (29)$$

$$0 = -[\lambda p_1 + c\mu]P_c + c\mu\xi p P_{c+1} + \lambda P_{c-1} \qquad \dots (30)$$

$$0 = -[\lambda p_1 + c\mu + (n-c)\xi p]P_n + [c\mu + \{(n+1)-c\}\xi p]P_{n+1} + \lambda p_1 P_{n-1}; n > c \qquad \dots (31)$$

then the steady-state-probabilities of system size are given by

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0; 1 \le n \le c \qquad \dots (32)$$

$$P_n = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right] P_0; n > c \qquad \dots (33)$$

With

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{c} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c+1}^{\infty} \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right]\right)} \qquad \dots (34)$$

Proof: We prove the theorem by using iterative method. Rearranging (28), we get the value of P_1 as

$$P_1 = \frac{\lambda}{\mu} P_0$$

For $n = 1$, (29) yields

$$2\mu P_2 = (\lambda + \mu)P_1 - \lambda P_0$$

$$2\mu P_2 = \lambda P_1$$

$$P_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

For $n = 2$, (29) yields

$$3\mu P_3 = (\lambda + 2\mu)P_2 - \lambda P_1$$

Since $2\mu P_2 = \lambda P_1$, we have

$$3\mu P_3 = \lambda P_2$$

$$P_3 = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 P_0$$

Similarly, for $n = c-1$, the value of P_c is

$$c\mu P_c = [\lambda + (c-1)\mu]P_{c-1} - \lambda P_{c-2}$$

Since $n\mu P_n = \lambda P_{n-1}$ for $1 \le n \le c-1$

Therefore $(c-1)\mu P_{c-1} = \lambda P_{c-2}$

Hence $P_c = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0$

Thus, all the probabilities $P_n; 1 \le n \le c$ satisfy (32).

From equation (), the value of P_{c+1} is

$$(c\mu + \xi p)P_{c+1} = (\lambda p_1 + c\mu)P_c - \lambda P_{c-1}$$

Since $c\mu P_c = \lambda P_{c-1}$

$$(c\mu + \xi p)P_{c+1} = \lambda P_c$$

$$P_{c+1} = \frac{\lambda p_1}{(c\mu + \xi p)} P_c$$

$$P_{c+1} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{(c+1)-c}}{\prod_{k=1}^{(c+1)-c} \mu + k\xi p}\right] P_0$$

Similarly, for $n > c$, the steady-state probabilities $P_n; n > c+1$ are obtained as

$$P_n = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right] P_0; n > c+1$$

All these probabilities $P_n; n > c$ satisfy (33).

To find P_0, we use normalization condition $\sum_{n=0}^{\infty} P_n = 1$ and the values of $P_n; n \ge 1$.

$$\left(1 + \sum_{n=1}^{c} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c+1}^{\infty} \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right]\right) P_0 = 1$$

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^{c} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c+1}^{\infty} \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right]\right)}$$

Which is (34).

The steady-state probabilities exist if

$$\left(1 + \sum_{n=1}^{c} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c+1}^{\infty} \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right]\right) < \infty$$

Thus, equations (32)-(34) completely determine all the steady-state probabilities of the system size $P_n; n \ge 0$ which completes the proof of the theorem.

B. *Measures of Performances*

The Expected System Size, Ls

$$L_s = \sum_{n=0}^{\infty} nP_n$$

$$L_s = \left\{\sum_{n=1}^{c} n\left(\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right) + \sum_{n=c+1}^{\infty} n\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right]\right\} P_0$$

Using little's formula $L = \lambda W$, we can obtain the related steady-state measures of performances like average number of customers in the queue (Lq), average waiting time in the system (Ws), average waiting time in the queue (Wq) etc.

The Expected Number of customers Served, $E(Customer\ Served)$**:**

The expected number of customers served is given by

$$E(CustomerServed) = \sum_{n=1}^{c} n\mu P_n + \sum_{n=c+1}^{\infty} c\mu P_n$$

$$= \left[\begin{array}{l} \sum_{n=1}^{c} n\mu \left(\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right) \\ + \sum_{n=c+1}^{\infty} c\mu \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right] \end{array}\right] P_0$$

Rate of Abandonment, R_{aband}

$$R_{aband} = \lambda \sum_{n=0}^{\infty} P_n - E(customer\ Served)$$

$$R_{aband} = \lambda - \left[\begin{array}{l} \sum_{n=1}^{c} n\mu \left(\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right) \\ + \sum_{n=c+1}^{\infty} c\mu \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c} \mu + k\xi p}\right] \end{array}\right] P_0$$

Expected number of waiting customers, who actually wait, $E(Customer Waiting)$:

$$E(Customer Waiting) = \frac{\sum_{n=c+1}^{\infty}(n-c)P_n}{\sum_{n=c+1}^{\infty}P_n}$$

$$= \frac{\sum_{n=c+1}^{\infty}(n-c)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c}\mu + k\xi p}\right]}{\sum_{n=c+1}^{\infty}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c}\mu + k\xi p}\right]}$$

C. Special Cases

In this sub-section, we obtain some important queuing model as particular cases of our model.

(i) In the absence of retention of reneged customer (i.e. for $q = 0$), one can see that the model results resemble with Montazer-Hagighi et al. [3].

(ii) In the absence of reneging (i.e. for ξ=0), our model reduces to a simple M/M/c queuing model and balking.

(iii)When the capacity of the system is taken as finite say, N, the resulting model is an M/M/c/N queuing model with retention of reneged customers and balking with

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0; 1 \le n \le c$$

$$P_n = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c}\mu + k\xi p}\right]P_0; c+1 \le n \le N$$

And

$$P_0 = \frac{1}{\left(1+\sum_{n=1}^{c}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c+1}^{N}\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\left[\frac{(\lambda p_1)^{n-c}}{\prod_{k=1}^{n-c}\mu + k\xi p}\right]\right)}$$

(iv) In the absence of balking i.e. when $p_1 = 1$, our model reduces to an M/M/c queuing model with retention of reneged customers as studied in section 4.

VI. Conclusions

In this paper, we study single as well as multi-server Markovian queuing systems with retention of reneged customers. The steady-state solution is obtained for both the models iteratively. Some important queuing models are obtained as particular cases of the model.

The balking is also considered in single as well as multi-server case. The steady-state analysis, measures of performances and special cases are also obtained.

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