On Taylor Expansion Methods for Multivariate Integral Equations of the Second Kind

Boriboon Novaprateep, Khomsan Neamprem, and Hideaki Kaneko

***Abstract*—A new Taylor series method that the authors originally developed for the solution of one-dimensional integral equations is extended to solve multivariate integral equations. In this paper, the new method is applied to the solution of multivariate Fredholm equations of the second kind. A comparison is given of the new method and the traditional Taylor series method of solving integral equations. The new method is adapted to parallel computation and can therefore be highly efficient on modern computers. The method also gives highly accurate approximations for all derivatives of the solution up to the order of the Taylor series approximation. Numerical examples are given to illustrate the efficiency and accuracy of the method.**

***Keywords*—Taylor-Series Expansion Method, Multivariate Fredholm Integral Equations, Galerkin Method, Collocation Method.**

I. Introduction

INTEGRAL equations are often matched with mathematical models describing some phenomena in Physics and Engineer [3]. The approximation solution can be found by several methods [7], [6], [9]. In a recent paper [2], a Taylor-series expansion method to approximate the solution of a class of Fredholm integral equation of the second kind was considered. Subsequently, the method of [2] was improved in a recent paper [4]. The Taylor series method developed in [4] is different from the traditional Taylor series method discussed in [1], in which the kernel is expanded in the series and the resulting equation is treated within the framework of the degenerate kernel method. We will review the traditional Taylor series method in the next section so as to delineate and highlight the advantages of the current method. Improvements made in [4] over the paper [2] primarily lie in two areas. First, it was generalized so as to be applicable to a wider class of integral equations. The method discussed in the paper [2] applies only to equations with a convolution type kernel $k(|t-s|)$ in which it decays rapidly as $|t-s|$ increases to infinity. The new method in [4] on the other hand can be applied to any differentiable kernel. Secondly, the method [4] produces more accurate approximation than the one given in [2]. It is also important to note that numerical implementation of the new method can be carried out in parallel. This is a critical point to note since most of the numerical methods currently available for approximating the solution of linear as well as nonlinear equations are designed to approximate the solution globally all in one calculation. The Galerkin method and collocation method are two of the most popular methods used by practitioners. For a detailed reference to numerical solutions of integral equations, we recommend a book by Atkinson [1]. The Galerkin and collocation methods normally leads to a large scale linear system of equations which are computationally expensive to solve since matrices involved are most often dense rather than sparse. Therefore, reducing an expensive cost involved in the solution process is always an important issue in approximating the solution of integral equations. In addition, this paper provides a method to calculate a solution at a specific region of interest without calculating the solution over the whole region. The purpose of this paper is to extend the idea explored in [4] to multivariate Fredholm equations. We anticipate that the current approach will find new applications in numerical solution of boundary value problems. This will be discussed in a future paper.

B. Novaprateep is with the Department of Mathematics, Faculty of Science, Mahidol University, Bangkok, 10400, THAILAND and Centre of Excellence in Mathematics, CHE, Si Ayutthaya RD., Bangkok 10400, THAILAND, e-mail: boriboon.nov@mahidol.ac.th.

K. Neamprem is with Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, THAILAND and Centre of Excellence in Mathematics, CHE, Si Ayutthaya RD., Bangkok 10400, THAILAND, email: khomsann@kmutnb.ac.th.

H. Kaneko is with Department of Mathematics, Department of Mathematics and Statistics,Old Dominion University, Norfolk, Virginia, 23529-0077, USA, email: hkaneko@odu.edu.

We also note that recent papers [5] and [8] extend the idea developed in [4] to Volterra integral equations and to nonlinear Hammerstein equations, respectively.

The article is organized as follows: Section II, we introduce a traditional Taylor-Series Method for the Fredholm integral equation of the second kind over an one dimensional space R. In section III, we extend idea of the Taylor expansion method from [4] to the Fredholm integral equations of the second kind over a multidimensional space R^n. Section IV, we illustrate on the numerical experiments of the new Taylor expansion method and show the efficiency and accuracy of this new method.

II. A Traditional Taylor-Series Method

In this section, we review briefly the traditional Taylor-series method, see [1].

The review will serve to distinguish the new Taylor series method which was developed recently in [4] and identify its strength and advantages. Recall that one dimensional Fredholm integral equation of the second kind is given by,

$$x(t) - \int_0^1 k(t,s)x(s)ds = f(t), \qquad 0 \le t \le 1, \qquad (2.1)$$

where k and f are know functions and x is the function to be determined. This type of equation arises as an integral reformulation of the two-point boundary value problem. The traditional Taylor series approximation method is to expand the kernel k in Taylor series with respect to the variable t or s. For example, if k is expanded in s variable to the nth order, we have an finite rank approximation k_n of k which is give by

$$k_n(t,s) = \sum_{i=0}^{n} \frac{\partial^i}{\partial s^i} k(t,a)(s-a)^i. \tag{2.2}$$

Upon substituting (2.2) into (2.1), we obtain an approximation

$$x_n(t) = f(t) + \sum_{i=0}^{n} c_i k_i(t) \tag{2.3}$$

where $k_i(t) := \frac{\partial^i}{\partial s^i} k(t,a)$
and $c_i := \int_0^1 (s-a)^i x_n(s)ds$ for $i = 0, 1, \ldots n$.

Multiplying (2.3) by $(s-a)^i$ and integrating over $(0,1)$, we obtain the following linear system whose solution gives an approximation x_n;

$$c_i - \sum_{j=0}^{n} \int_0^1 k_j(s)(s-a)^i ds = \int_0^1 f(s)(s-a)^i ds, \tag{2.4}$$

$$i = 0, 1, \ldots, n.$$

Note that the coefficients c_i in equation (2.3) is independent of t value and thus an approximation is obtained for $x(t)$. Any property of the derivatives of the solution is not derived from this analysis.

III. An Expansion Method for Multivariate Equation

In this section, we apply the Taylor expansion method developed in [4] to the Fredholm integral equations of the second kind proposed over a multidimensional space R^n;

$$x(\mathbf{t}) - \int_I k(\mathbf{t},\mathbf{s})x(\mathbf{s})d\mathbf{s} = f(\mathbf{t}), \tag{3.1}$$

where $\mathbf{t}, \mathbf{s} \in I$, $I \subseteq R^n$ is a compact set. We recall the Taylor series for $x(\mathbf{t})$,

$$\begin{aligned} x(\mathbf{s}) &= x(\mathbf{t}) + \textstyle\sum_{k=1}^{r} \frac{1}{k!} D^k x(\mathbf{t}) \cdot (\underbrace{\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}}_{k \text{ times}}) + R_r(\mathbf{t},\mathbf{h}) \\ &= x(\mathbf{t}) + \textstyle\sum_{k=1}^{r} \frac{1}{k!} (\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n})^k x(\mathbf{t}) \cdot (\underbrace{\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}}_{k \text{ times}}) \\ &\quad + R_r(\mathbf{t},\mathbf{h}) \end{aligned} \tag{3.2}$$

where $\mathbf{h} = \mathbf{s} - \mathbf{t}$, $\mathbf{c}$ is a point on the line segment $\mathbf{s} - \mathbf{t}$ and $R_r(\mathbf{t},\mathbf{h})/\|\mathbf{h}\|^r \to 0$ as $\mathbf{h} \to \mathbf{0}$, $\mathbf{h} \in R^n$. Here of course, $Dx(\mathbf{t})$ is the gradient of x at $\mathbf{t}$ and

$$D^2x(\mathbf{t}) = \begin{bmatrix} \frac{\partial^2 x}{\partial t_1 \partial t_1}(\mathbf{t}) & \frac{\partial^2 x}{\partial t_1 \partial t_2}(\mathbf{t}) & \cdots & \frac{\partial^2 x}{\partial t_1 \partial t_n}(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 x}{\partial t_n \partial t_1}(\mathbf{t}) & \frac{\partial^2 x}{\partial t_n \partial t_2}(\mathbf{t}) & \cdots & \frac{\partial^2 x}{\partial t_n \partial t_n}(\mathbf{t}) \end{bmatrix}.$$

Differentiating (3.1) r times, we obtain

$$\begin{aligned} Dx(\mathbf{t}) - \textstyle\int_I Dk(\mathbf{t},\mathbf{s})x(\mathbf{s})d\mathbf{s} &= Df(\mathbf{t}) \\ D^2x(\mathbf{t}) - \textstyle\int_I D^2k(\mathbf{t},\mathbf{s})x(\mathbf{s})d\mathbf{s} &= D^2f(\mathbf{t}) \\ \vdots \qquad\qquad & \quad \vdots \\ D^r x(\mathbf{t}) - \textstyle\int_I D^r k(\mathbf{t},\mathbf{s})x(\mathbf{s})d\mathbf{s} &= D^r f(\mathbf{t}). \end{aligned} \tag{3.3}$$

Now, ignoring the error term in (3.2) for a moment and substituting $\sum_{k=0}^{r} \frac{1}{k!} D^k x(\mathbf{t}) \cdot (\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h})$ for $x(\mathbf{s})$ in (3.1) and (3.3) [to simplify our notation, we write $(\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}) = (\underbrace{\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}}_{k \text{ times}})$] , we obtain an approximation $\bar{x}(\mathbf{t})$ for the solution $x(\mathbf{t})$ of (3.1) by solving the following equations:

$$\begin{aligned} \bar{x}(\mathbf{t}) - \textstyle\sum_{k=0}^{r} \frac{1}{k!} D^k \bar{x}(\mathbf{t}) \int_I k(\mathbf{t},\mathbf{s}) \cdot (\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}) d\mathbf{s} &= f(\mathbf{t}) \\ D\bar{x}(\mathbf{t}) - \textstyle\sum_{k=0}^{r} \frac{1}{k!} D^k \bar{x}(\mathbf{t}) \int_I Dk(\mathbf{t},\mathbf{s}) \cdot (\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}) d\mathbf{s} &= Df(\mathbf{t}) \\ \vdots \qquad\qquad & \quad \vdots \\ D^r\bar{x}(\mathbf{t}) - \textstyle\sum_{k=0}^{r} \frac{1}{k!} D^k \bar{x}(\mathbf{t}) \int_I D^r k(\mathbf{t},\mathbf{s}) \cdot (\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}) d\mathbf{s} &= D^r f(\mathbf{t}). \end{aligned} \tag{3.4}$$

Direct substitution of (3.2) into (3.3) and comparing it with (3.4), we obtain a set of equations which yields the following error estimate of the current method.

$$\begin{aligned} &(x(\mathbf{t}) - \bar{x}(\mathbf{t})) - \textstyle\sum_{k=0}^{r} \frac{1}{k!} D^k (x(\mathbf{t}) - \bar{x}(\mathbf{t})) \times \\ &\qquad \textstyle\int_I k(\mathbf{t},\mathbf{s}) \cdot (\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}) d\mathbf{s} \\ &\qquad = f(\mathbf{t}) + \textstyle\int_D k(\mathbf{t},\mathbf{s}) R_r(\mathbf{s},\mathbf{h}) d\mathbf{s} \\ &D(x(\mathbf{t}) - \bar{x}(\mathbf{t})) - \textstyle\sum_{k=0}^{r} \frac{1}{k!} D^k (x(\mathbf{t}) - \bar{x}(\mathbf{t})) \times \\ &\qquad \textstyle\int_I Dk(\mathbf{t},\mathbf{s}) \cdot (\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}) d\mathbf{s} \\ &\qquad = Df(\mathbf{t}) + \textstyle\int_I Dk(\mathbf{t},\mathbf{s}) R_r(\mathbf{s},\mathbf{h}) d\mathbf{s} \\ &\quad \vdots \qquad\qquad \vdots \\ &D^r(x(\mathbf{t}) - \bar{x}(\mathbf{t})) - \textstyle\sum_{k=0}^{r} \frac{1}{k!} D^k (x(\mathbf{t}) - \bar{x}(\mathbf{t})) \times \\ &\qquad \textstyle\int_I D^r k(\mathbf{t},\mathbf{s}) \cdot (\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}) d\mathbf{s} \\ &\qquad = D^r f(\mathbf{t}) + \textstyle\int_I D^r k(\mathbf{t},\mathbf{s}) R_r(\mathbf{s},\mathbf{h}) d\mathbf{s}. \end{aligned} \tag{3.5}$$

For example, with $n = 2$ and $r = 2$, equations (3.5) represent a system of 6 equations in 6 unknowns $x(\mathbf{t})$, $\frac{\partial x}{\partial t_1}(\mathbf{t})$, $\frac{\partial x}{\partial t_2}(\mathbf{t})$, $\frac{\partial^2 x}{\partial^2 t_1}(\mathbf{t})$, $\frac{\partial^2 x}{\partial t_1 \partial t_2}(\mathbf{t})$, $\frac{\partial^2 x}{\partial^2 t_2}(\mathbf{t})$. As a simple illustration, with $n = 2$ and $r = 1$,

$$\begin{bmatrix} 1 - \int_I k(\mathbf{t},\mathbf{s})d\mathbf{s} & -\int_I k(\mathbf{t},\mathbf{s})(\mathbf{s}-\mathbf{t})d\mathbf{s} & -\int_I k(\mathbf{t},\mathbf{s})(\mathbf{s}-\mathbf{t})d\mathbf{s} \\ -\int_I k_{t_1}(\mathbf{t},\mathbf{s})d\mathbf{s} & 1 - \int_I k_{t_1}(\mathbf{t},\mathbf{s})(\mathbf{s}-\mathbf{t})d\mathbf{s} & -\int_I k_{t_1}(\mathbf{t},\mathbf{s})(\mathbf{s}-\mathbf{t})d\mathbf{s} \\ -\int_I k_{t_2}(\mathbf{t},\mathbf{s})d\mathbf{s} & -\int_I k_{t_2}(\mathbf{t},\mathbf{s})(\mathbf{s}-\mathbf{t})d\mathbf{s} & 1 - \int_I k_{t_2}(\mathbf{t},\mathbf{s})(\mathbf{s}-\mathbf{t})d\mathbf{s} \end{bmatrix}$$

$$\times \begin{bmatrix} x(\mathbf{t}) - \bar{x}(\mathbf{t}) \\ \frac{\partial x}{\partial t_1}(\mathbf{t}) - \frac{\partial \bar{x}}{\partial t_1}(\mathbf{t}) \\ \frac{\partial x}{\partial t_2}(\mathbf{t}) - \frac{\partial \bar{x}}{\partial t_2}(\mathbf{t}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2!} \int_I k(\mathbf{t},\mathbf{s})(D^2x)(\mathbf{c})(\mathbf{s}-\mathbf{t})^2 d\mathbf{s} \\ \frac{1}{2!} \int_I k_{t_1}(\mathbf{t},\mathbf{s})(D^2x)(\mathbf{c})(\mathbf{s}-\mathbf{t})^2 d\mathbf{s} \\ \frac{1}{2!} \int_I k_{t_2}(\mathbf{t},\mathbf{s})(D^2x)(\mathbf{c})(\mathbf{s}-\mathbf{t})^2 d\mathbf{s} \end{bmatrix}$$

where $\mathbf{c}$ is a point between $\mathbf{s}$ and $\mathbf{t}$.

In general, we have the following theorem regarding the accuracy of the current expansion method.

Theorem 3.1: Let $x(\mathbf{t})$ be the solution of equation (3.1) and $\bar{x}(\mathbf{t})$ be the solution of equation (3.4). Suppose that the kernel k is such that $\int_I D_t^{i+1} k(\mathbf{t},\mathbf{s})d\mathbf{s} < \infty$ for all $i \leq r+1$. Then

$$\|D_{\mathbf{t}}^i (x(\mathbf{t}) - \bar{x}(\mathbf{t}))\|_\infty = \frac{M}{(r+1)!}\mathbf{h}^{r+1},$$

where $\mathbf{h} = \mathbf{s} - \mathbf{t}$,
and $M = \max_{0\leq i,j\leq r+1} \int_I D^j_{\mathbf{t}} k(\mathbf{t}, \mathbf{s})(D^i x)(\mathbf{c}) d\mathbf{s}$.

It is helpful to compare the current expansion method with more traditional numerical methods, such as the Galerkin method and the collocation method, to identify its strength. Suppose that $\text{meas}(I) = 1$ and we apply the current expansion method with $r = 4$. That is, we apply the Taylor series of order 4 in (3.2). Then the resulting system (3.4) represents 14 linear equations in 14 unknowns, with its error term $\mathcal{O}(\frac{1}{5!})$. This means that we expect the accuracy of approximation $\bar{x}(\mathbf{t})$ of $x(\mathbf{t})$ as well as all of its derivatives of order less than 5 to be approximately ± 0.008. Now, in order to attain the same order of accuracy using the Galerkin or collocation method with, say, the linear spline basis, we must establish a 440×440 system, since its order of accuracy of linear spline is $\mathcal{O}(\bar{d}^2)$, where $I = \cup_{\alpha\in\Gamma} I_\alpha$ and $\bar{I} = \max_{\alpha\in\Gamma} I_\alpha$ and $\bar{d} = \text{meas}(\bar{I})$, and it requires a basis function with 4 degree of freedom over each I_α. Thus the comparison shows that the current Taylor expansion method, when an approximation is desired for the solution as well as for its derivatives over a specific region of interest, provides a much more efficient way of calculating them than the traditional Galerkin and the collocation methods.

IV. Numerical Results

In this section, we present five numerical examples. The computer programs are run on a personal computer with 2.8GHz CPU and 8GB memory.

Example 4.1: Consider the equation

$$x(s_1, s_2) - \int_0^1 \int_0^1 k(s_1, s_2, t_1, t_2) x(t_1, t_2) dt_1 dt_2 = f(s_1, s_2),$$
$$[s_1, s_2] \in [0, 1] \times [0, 1]$$

where $k(s_1, s_2, t_1, t_2) = s_1 + s_2 - t_1 - t_2$
and $f(s_1, s_2)$ is chosen so that the isolated solution $x(s_1, s_2) = s_1 + 2s_2$.
The numerical results can see in Table I.

In Example 4.1, the solution is taken to be a polynomial of degree 2. Thus, the current method with $r = 2$ delivers an approximation which is exact to the machine accuracy.

In the next example, the exponential kernel is used, $\exp(s_1 + s_2 - t_1 - t_2)$, and the solution is taken to be a polynomial of degree 4. Thus, the current method with $r = 4$ delivers an approximation with the high accuracy.

TABLE I
COMPUTATIONAL RESULTS WITH $r = 2$ FOR EXAMPLE 4.1.

s_1	s_2	Exact	Approx.	Abs. Error
0.0	0.0	0.0000	0.0000	1.6653e-16
0.0	0.2	0.4000	0.4000	5.5511e-17
0.0	0.4	0.8000	0.8000	1.1102e-16
0.0	0.6	1.2000	1.2000	0.0000e+00
0.0	0.8	1.6000	1.6000	2.2204e-16
0.0	1.0	2.0000	2.0000	2.2204e-16
0.2	0.0	0.2000	0.2000	5.5511e-17
0.2	0.2	0.6000	0.6000	2.2204e-16
0.2	0.4	1.0000	1.0000	1.1102e-16
0.2	0.6	1.4000	1.4000	0.0000e+00
0.2	0.8	1.8000	1.8000	2.2204e-16
0.2	1.0	2.2000	2.2000	0.0000e+00
0.4	0.0	0.4000	0.4000	5.5511e-17
0.4	0.2	0.8000	0.8000	1.1102e-16
0.4	0.4	1.2000	1.2000	2.2204e-16
0.4	0.6	1.6000	1.6000	0.0000e+00
0.4	0.8	2.0000	2.0000	0.0000e+00
0.4	1.0	2.4000	2.4000	0.0000e+00
0.6	0.0	0.6000	0.6000	1.1102e-16
0.6	0.2	1.0000	1.0000	2.2204e-16
0.6	0.4	1.4000	1.4000	0.0000e+00
0.6	0.6	1.8000	1.8000	2.2204e-16
0.6	0.8	2.2000	2.2000	0.0000e+00
0.6	1.0	2.6000	2.6000	4.4409e-16
0.8	0.0	0.8000	0.8000	0.0000e+00
0.8	0.2	1.2000	1.2000	2.2204e-16
0.8	0.4	1.6000	1.6000	4.4409e-16
0.8	0.6	2.0000	2.0000	0.0000e+00
0.8	0.8	2.4000	2.4000	0.0000e+00
0.8	1.0	2.8000	2.8000	0.0000e+00
1.0	0.0	1.0000	1.0000	1.1102e-16
1.0	0.2	1.4000	1.4000	2.2204e-16
1.0	0.4	1.8000	1.8000	4.4409e-16
1.0	0.6	2.2000	2.2000	0.0000e+00
1.0	0.8	2.6000	2.6000	0.0000e+00
1.0	1.0	3.0000	3.0000	0.0000e+00

Example 4.2: Consider the equation

$$x(s_1, s_2) - \int_0^1 \int_0^1 k(s_1, s_2, t_1, t_2) x(t_1, t_2) dt_1 dt_2 = f(s_1, s_2),$$
$$[s_1, s_2] \in [0, 1] \times [0, 1]$$

where $k(s_1, s_2, t_1, t_2) = \exp(s_1 + s_2 - t_1 - t_2)$ and $f(s_1, s_2)$ is chosen so that the isolated solution $x(s_1, s_2) = s_1^4 + s_2^4$.
The numerical results can see in TABLE II.

TABLE II
COMPUTATIONAL RESULTS WITH $r = 4$ FOR EXAMPLE 4.2.

s_1	s_2	Exact	Approx.	Abs. Error
0.00	0.00	0.0000	0.0000	1.7971e-12
0.00	0.20	0.0016	0.0016	3.2242e-12
0.00	0.40	0.0256	0.0256	6.8141e-12
0.00	0.60	0.1296	0.1296	3.7350e-11
0.00	0.80	0.4096	0.4096	1.0869e-10
0.00	1.00	1.0000	1.0000	1.4238e-10
0.20	0.00	0.0016	0.0016	3.2471e-12
0.20	0.20	0.0032	0.0032	5.9512e-12
0.20	0.40	0.0272	0.0272	1.3397e-11
0.20	0.60	0.1312	0.1312	1.2587e-10
0.20	0.80	0.4112	0.4112	1.0854e-10
0.20	1.00	1.0016	1.0016	2.4390e-11
0.40	0.00	0.0256	0.0256	6.8142e-12
0.40	0.20	0.0272	0.0272	1.3422e-11
0.40	0.40	0.0512	0.0512	4.0969e-11
0.40	0.60	0.1552	0.1552	1.4185e-10
0.40	0.80	0.4352	0.4352	2.0112e-11
0.40	1.00	1.0256	1.0256	7.8340e-12
0.60	0.00	0.1296	0.1296	3.7296e-11
0.60	0.20	0.1312	0.1312	1.2618e-10
0.60	0.40	0.1552	0.1552	1.4183e-10
0.60	0.60	0.2592	0.2592	5.0445e-11
0.60	0.80	0.5392	0.5392	1.4321e-11
0.60	1.00	1.1296	1.1296	6.2204e-12
0.80	0.00	0.4096	0.4096	1.0928e-10
0.80	0.20	0.4112	0.4112	1.0784e-10
0.80	0.40	0.4352	0.4352	2.0171e-11
0.80	0.60	0.5392	0.5392	1.4190e-11
0.80	0.80	0.8192	0.8192	4.4130e-12
0.80	1.00	1.4096	1.4096	2.2133e-12
1.00	0.00	1.0000	1.0000	1.4171e-10
1.00	0.20	1.0016	1.0016	2.4625e-11
1.00	0.40	1.0256	1.0256	7.9217e-12
1.00	0.60	1.1296	1.1296	6.1835e-12
1.00	0.80	1.4096	1.4096	2.1945e-12
1.00	1.00	2.0000	2.0000	1.0716e-12

Next, we show the comparison of numerical experimentations with several orders of Taylor expansion method.

Example 4.3: Consider the equation

$$x(s_1,s_2)-\int_0^1\int_0^1 k(s_1,s_2,t_1,t_2)x(t_1,t_2)dt_1dt_2 = f(s_1,s_2),$$
$$[s_1,s_2]\in[0,1]\times[0,1]$$

where $k(s_1,s_2,t_1,t_2)=\exp(-s_1-s_2-t_1-t_2)$
and $f(s_1,s_2)$ is chosen so that the isolated solution $x(s_1,s_2)=\sin(s_1+s_2)$.
The numerical results can see in TABLE III and TABLE VI.

From Table III and VI, we found that the numerical errors deceased when order of Taylor expansion increased for each point. Moreover, we may see it from from Fig. 1 to Fig. 3.

TABLE III
COMPUTATIONAL RESULTS WITH $r = 5$ FOR EXAMPLE 4.3.

s_1	s_2	Exact	$r=5$	
			Approx.	Abs. Error
0.00	0.00	0.0000	0.0003	3.4599e-4
0.00	0.25	0.2474	0.2476	1.7759e-4
0.00	0.50	0.4794	0.4795	5.5449e-5
0.00	0.75	0.6816	0.6817	1.2704e-5
0.00	1.00	0.8415	0.8415	8.6911e-6
0.25	0.00	0.2474	0.2476	1.7759e-4
0.25	0.25	0.4794	0.4795	5.5449e-5
0.25	0.50	0.6816	0.6817	1.2704e-5
0.25	0.75	0.8415	0.8415	8.6911e-6
0.25	1.00	0.9490	0.9490	3.5614e-5
0.50	0.00	0.4794	0.4795	5.5449e-5
0.50	0.25	0.6816	0.6817	1.2704e-5
0.50	0.50	0.8415	0.8415	8.6911e-6
0.50	0.75	0.9490	0.9490	3.5614e-5
0.50	1.00	0.9975	0.9976	1.2216e-4
0.75	0.00	0.6816	0.6817	1.2704e-5
0.75	0.25	0.8415	0.8415	8.6911e-6
0.75	0.50	0.9490	0.9490	3.5614e-5
0.75	0.75	0.9975	0.9976	1.2216e-4
0.75	1.00	0.9840	0.9843	3.1838e-4
1.00	0.00	0.8415	0.8415	8.6911e-6
1.00	0.25	0.9490	0.9490	3.5614e-5
1.00	0.50	0.9975	0.9976	1.2216e-4
1.00	0.75	0.9840	0.9843	3.1838e-4
1.00	1.00	0.9093	0.9100	6.6880e-4

TABLE IV
COMPUTATIONAL RESULTS WITH 6 FOR EXAMPLE 4.3.

s_1	s_2	Exact	$r=6$	
			Approx.	Abs. Error
0.00	0.00	0.0000	0.0003	1.4657e-4
0.00	0.25	0.2474	0.2475	7.2008e-5
0.00	0.50	0.4794	0.4794	1.1629e-5
0.00	0.75	0.6816	0.6816	1.2607e-6
0.00	1.00	0.8415	0.8415	5.4338e-7
0.25	0.00	0.2474	0.2475	7.2008e-5
0.25	0.25	0.4794	0.4794	1.1629e-5
0.25	0.50	0.6816	0.6816	1.2607e-6
0.25	0.75	0.8415	0.8415	5.4338e-7
0.25	1.00	0.9490	0.9490	2.4488e-6
0.50	0.00	0.4794	0.4794	1.1629e-5
0.50	0.25	0.6816	0.6816	1.2607e-6
0.50	0.50	0.8415	0.8415	5.4338e-7
0.50	0.75	0.9490	0.9490	2.4488e-6
0.50	1.00	0.9975	0.9975	4.8117e-6
0.75	0.00	0.6816	0.6816	1.2607e-6
0.75	0.25	0.8415	0.8415	5.4338e-7
0.75	0.50	0.9490	0.9490	2.4488e-6
0.75	0.75	0.9975	0.9975	4.8117e-6
0.75	1.00	0.9840	0.9840	5.3417e-5
1.00	0.00	0.8415	0.8415	5.4338e-7
1.00	0.25	0.9490	0.9490	2.4488e-6
1.00	0.50	0.9975	0.9975	4.8117e-6
1.00	0.75	0.9840	0.9840	5.3417e-5
1.00	1.00	0.9093	0.9093	3.5252e-5

Example 4.4: Consider the equation

$$x(s_1,s_2)-\int_0^1\int_0^1 k(s_1,s_2,t_1,t_2)x(t_1,t_2)dt_1dt_2 = f(s_1,s_2),$$
$$[s_1,s_2]\in[0,1]\times[0,1]$$

where $k(s_1,s_2,t_1,t_2)=\exp(-s_1-s_2-t_1-t_2)$ and $f(s_1,s_2)$ is chosen so that the isolated solution $x(s_1,s_2)=\exp(-2s_1-s_2^2)$.
The numerical results can see in TABLE VII - TABLE XII.

In Example 4.4, the numerical approximation for $x(s_1,s_2)$ and its first and second derivatives with $r = 8$ are shown

TABLE V
COMPUTATIONAL RESULTS WITH $r = 7$ FOR EXAMPLE 4.3.

s_1	s_2	Exact	$r = 7$ Approx.	$r = 7$ Abs. Error
0.00	0.00	0.0000	0.0000	6.1874e-5
0.00	0.25	0.2474	0.2474	5.2460e-6
0.00	0.50	0.4794	0.4794	1.2549e-6
0.00	0.75	0.6816	0.6816	1.9802e-7
0.00	1.00	0.8415	0.8415	1.0014e-7
0.25	0.00	0.2474	0.2474	5.2460e-6
0.25	0.25	0.4794	0.4794	1.2549e-6
0.25	0.50	0.6816	0.6816	1.9802e-7
0.25	0.75	0.8415	0.8415	1.0014e-7
0.25	1.00	0.9490	0.9490	6.5633e-7
0.50	0.00	0.4794	0.4794	1.2549e-6
0.50	0.25	0.6816	0.6816	1.9802e-7
0.50	0.50	0.8415	0.8415	1.0014e-7
0.50	0.75	0.9490	0.9490	6.5633e-7
0.50	1.00	0.9975	0.9975	3.2645e-6
0.75	0.00	0.6816	0.6816	1.9802e-7
0.75	0.25	0.8415	0.8415	1.0014e-7
0.75	0.50	0.9490	0.9490	6.5633e-7
0.75	0.75	0.9975	0.9975	3.2645e-6
0.75	1.00	0.9840	0.9840	1.1619e-5
1.00	0.00	0.8415	0.8415	1.0014e-7
1.00	0.25	0.9490	0.9490	6.5633e-7
1.00	0.50	0.9975	0.9975	3.2645e-6
1.00	0.75	0.9840	0.9840	1.1619e-5
1.00	1.00	0.9093	0.9093	3.1942e-5

TABLE VI
COMPUTATIONAL RESULTS WITH 8 FOR EXAMPLE 4.3.

s_1	s_2	Exact	$r = 8$ Approx.	$r = 8$ Abs. Error
0.00	0.00	0.0000	-0.0000	1.1875e-5
0.00	0.25	0.2474	0.2474	1.9100e-6
0.00	0.50	0.4794	0.4794	2.2970e-7
0.00	0.75	0.6816	0.6816	1.8252e-8
0.00	1.00	0.8415	0.8415	5.0046e-9
0.25	0.00	0.2474	0.2474	1.9100e-6
0.25	0.25	0.4794	0.4794	2.2970e-7
0.25	0.50	0.6816	0.6816	1.8252e-8
0.25	0.75	0.8415	0.8415	5.0046e-9
0.25	1.00	0.9490	0.9490	3.4603e-8
0.50	0.00	0.4794	0.4794	2.2970e-7
0.50	0.25	0.6816	0.6816	1.8252e-8
0.50	0.50	0.8415	0.8415	5.0046e-9
0.50	0.75	0.9490	0.9490	3.4603e-8
0.50	1.00	0.9975	0.9975	9.1357e-8
0.75	0.00	0.6816	0.6816	1.8252e-8
0.75	0.25	0.8415	0.8415	5.0046e-9
0.75	0.50	0.9490	0.9490	3.4603e-8
0.75	0.75	0.9975	0.9975	9.1357e-8
0.75	1.00	0.9840	0.9840	5.8804e-8
1.00	0.00	0.8415	0.8415	5.0046e-9
1.00	0.25	0.9490	0.9490	3.4603e-8
1.00	0.50	0.9975	0.9975	9.1357e-8
1.00	0.75	0.9840	0.9840	5.8804e-8
1.00	1.00	0.9093	0.9093	1.5824e-6

in Table VII to XII. We point out that in the example the numerical errors of $x(s_1, s_2)$ and its first and second derivatives are the same at the each point.

Finally, we extend our expansion technique with order $r = 3$ and $r = 4$ to three dimensional Fredholm equations shown below.

Example 4.5: Consider the equation

$$x(\vec{s}) - \int_0^1 \int_0^1 \int_0^1 k(\vec{s}, \vec{t}\,) x(\vec{t}\,) dt_1 dt_2 dt_3 = f(\vec{s}),$$

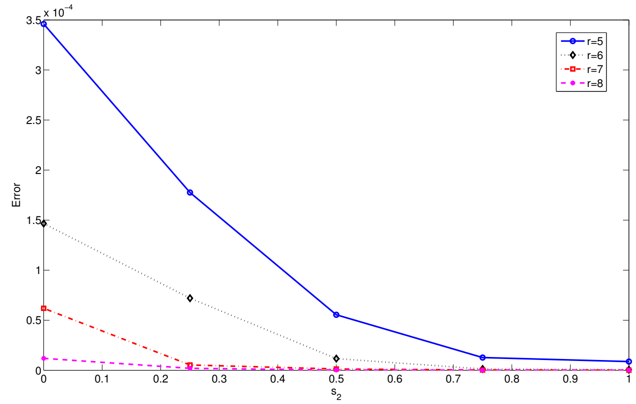

Fig. 1. The comparison of the numerical errors with fixed $s_1 = 0.00$.

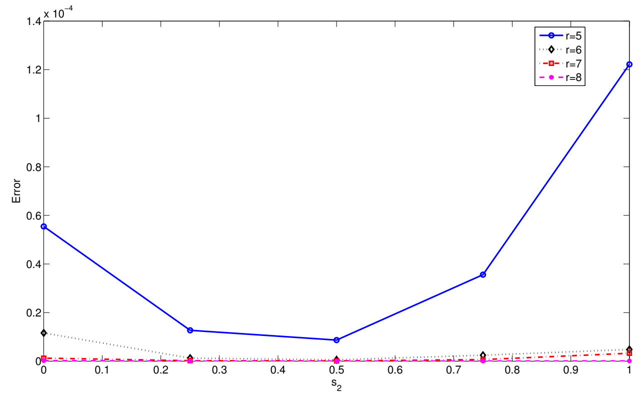

Fig. 2. The comparison of the numerical errors with fixed $s_1 = 0.50$.

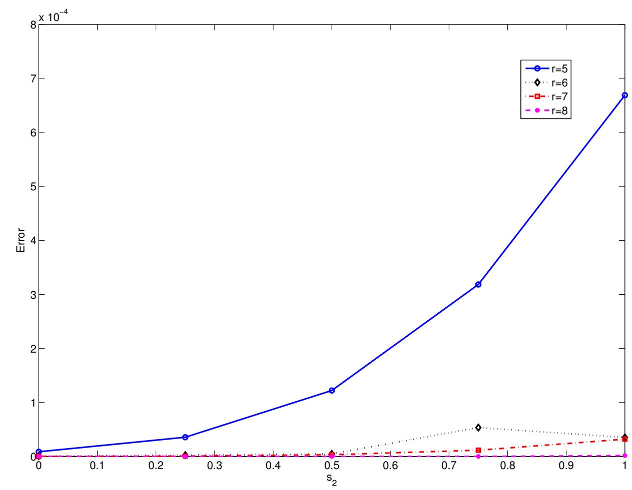

Fig. 3. The comparison of the numerical errors with fixed $s_1 = 1.00$.

$$\vec{t} \equiv [t_1, t_2, t_3], \vec{s} \equiv [s_1, s_2, s_3] \in [0,1] \times [0,1] \times [0,1]$$

where $k(\vec{s}, \vec{t}\,) = \exp(-s_1 - s_2 - s_3 - t_1 - t_2 - t_3)$
and $f(\vec{s})$ is chosen so that the isolated solution
$x(\vec{s}) = \exp(-s_1 - s_2 - s_3)$.
The numerical results can see in TABLE XIII and TABLE XIV.

TABLE VII
COMPUTATIONAL RESULTS WITH $r = 8$ FOR EXAMPLE 4.4.

s_1	s_2	$x(s_1, s_2)$		
		Exact	Approx.	Abs. Error
0.00	0.00	1.0000	1.0001	1.3174e-4
0.00	0.25	0.9394	0.9394	1.3041e-5
0.00	0.50	0.7788	0.7788	6.1249e-6
0.00	0.75	0.5698	0.5698	1.2551e-6
0.00	1.00	0.3679	0.3678	4.8815e-5
0.25	0.00	0.6065	0.6066	6.6066e-5
0.25	0.25	0.5698	0.5698	5.1271e-7
0.25	0.50	0.4724	0.4724	1.8856e-7
0.25	0.75	0.3456	0.3456	1.0243e-6
0.25	1.00	0.2231	0.2232	3.7883e-5
0.50	0.00	0.3679	0.3679	1.6583e-5
0.50	0.25	0.3456	0.3456	5.7666e-7
0.50	0.50	0.2865	0.2865	4.4944e-8
0.50	0.75	0.2096	0.2096	2.7770e-6
0.50	1.00	0.1353	0.1354	3.9355e-5
0.75	0.00	0.2231	0.2231	2.2927e-5
0.75	0.25	0.2096	0.2096	5.3034e-7
0.75	0.50	0.1738	0.1738	1.9752e-7
0.75	0.75	0.1271	0.1271	7.5757e-7
0.75	1.00	0.0821	0.0821	4.1819e-7
1.00	0.00	0.1353	0.1353	2.6724e-6
1.00	0.25	0.1271	0.1271	1.6488e-6
1.00	0.50	0.1054	0.1054	1.1408e-6
1.00	0.75	0.0771	0.0771	3.1583e-6
1.00	1.00	0.0498	0.0498	2.3462e-5

TABLE VIII
COMPUTATIONAL RESULTS WITH $r = 8$ FOR EXAMPLE 4.4. (CONT.)

s_1	s_2	$\partial x(s_1, s_2)/\partial s_1$		
		Exact	Approx.	Abs. Error
0.00	0.00	-2.0000	-2.0001	1.3174e-4
0.00	0.25	-1.8788	-1.8788	1.3041e-5
0.00	0.50	-1.5576	-1.5576	6.1249e-6
0.00	0.75	-1.1396	-1.1396	1.2551e-6
0.00	1.00	-0.7358	-0.7357	4.8815e-5
0.25	0.00	-1.2131	-1.2131	6.6066e-5
0.25	0.25	-1.1396	-1.1396	5.1271e-7
0.25	0.50	-0.9447	-0.9447	1.8856e-7
0.25	0.75	-0.6912	-0.6912	1.0243e-6
0.25	1.00	-0.4463	-0.4463	3.7883e-5
0.50	0.00	-0.7358	-0.7357	1.6583e-5
0.50	0.25	-0.6912	-0.6912	5.7666e-7
0.50	0.50	-0.5730	-0.5730	4.4944e-8
0.50	0.75	-0.4192	-0.4192	2.7770e-6
0.50	1.00	-0.2707	-0.2707	3.9355e-5
0.75	0.00	-0.4463	-0.4462	2.2927e-5
0.75	0.25	-0.4192	-0.4192	5.3034e-7
0.75	0.50	-0.3475	-0.3475	1.9752e-7
0.75	0.75	-0.2543	-0.2543	7.5757e-7
0.75	1.00	-0.1642	-0.1642	4.1819e-7
1.00	0.00	-0.2707	-0.2707	2.6724e-6
1.00	0.25	-0.2543	-0.2543	1.6488e-6
1.00	0.50	-0.2108	-0.2108	1.1408e-6
1.00	0.75	-0.1542	-0.1542	3.1583e-6
1.00	1.00	-0.0996	-0.0996	2.3462e-5

TABLE IX
COMPUTATIONAL RESULTS WITH $r = 8$ FOR EXAMPLE 4.4. (CONT.)

s_1	s_2	$\partial x(s_1, s_2)/\partial s_2$		
		Exact	Approx.	Abs. Error
0.00	0.00	-0.0000	-0.0001	1.3174e-4
0.00	0.25	-0.4697	-0.4697	1.3041e-5
0.00	0.50	-0.7788	-0.7788	6.1249e-6
0.00	0.75	-0.8547	-0.8547	1.2551e-6
0.00	1.00	-0.7358	-0.7357	4.8815e-5
0.25	0.00	-0.0000	-0.0001	6.6066e-5
0.25	0.25	-0.2849	-0.2849	5.1271e-7
0.25	0.50	-0.4724	-0.4724	1.8856e-7
0.25	0.75	-0.5184	-0.5184	1.0243e-6
0.25	1.00	-0.4463	-0.4463	3.7883e-5
0.50	0.00	-0.0000	0.0000	1.6583e-5
0.50	0.25	-0.1728	-0.1728	5.7666e-7
0.50	0.50	-0.2865	-0.2865	4.4944e-8
0.50	0.75	-0.3144	-0.3144	2.7770e-6
0.50	1.00	-0.2707	-0.2707	3.9355e-5
0.75	0.00	-0.0000	0.0000	2.2927e-5
0.75	0.25	-0.1048	-0.1048	5.3034e-7
0.75	0.50	-0.1738	-0.1738	1.9752e-7
0.75	0.75	-0.1907	-0.1907	7.5757e-7
0.75	1.00	-0.1642	-0.1642	4.1819e-7
1.00	0.00	-0.0000	0.0000	2.6724e-6
1.00	0.25	-0.0636	-0.0636	1.6488e-6
1.00	0.50	-0.1054	-0.1054	1.1408e-6
1.00	0.75	-0.1157	-0.1157	3.1583e-6
1.00	1.00	-0.0996	-0.0996	2.3462e-5

TABLE X
COMPUTATIONAL RESULTS WITH $r = 8$ FOR EXAMPLE 4.4. (CONT.)

s_1	s_2	$\partial^2 x(s_1, s_2)/\partial s_1^2$		
		Exact	Approx.	Abs. Error
0.00	0.00	4.0000	4.0001	1.3174e-4
0.00	0.25	3.7577	3.7577	1.3041e-5
0.00	0.50	3.1152	3.1152	6.1249e-6
0.00	0.75	2.2791	2.2791	1.2551e-6
0.00	1.00	1.4715	1.4715	4.8815e-5
0.25	0.00	2.4261	2.4262	6.6066e-5
0.25	0.25	2.2791	2.2791	5.1271e-7
0.25	0.50	1.8895	1.8895	1.8856e-7
0.25	0.75	1.3824	1.3824	1.0243e-6
0.25	1.00	0.8925	0.8926	3.7883e-5
0.50	0.00	1.4715	1.4715	1.6583e-5
0.50	0.25	1.3824	1.3824	5.7666e-7
0.50	0.50	1.1460	1.1460	4.4944e-8
0.50	0.75	0.8384	0.8384	2.7770e-6
0.50	1.00	0.5413	0.5414	3.9355e-5
0.75	0.00	0.8925	0.8925	2.2927e-5
0.75	0.25	0.8384	0.8384	5.3034e-7
0.75	0.50	0.6951	0.6951	1.9752e-7
0.75	0.75	0.5085	0.5085	7.5757e-7
0.75	1.00	0.3283	0.3283	4.1819e-7
1.00	0.00	0.5413	0.5413	2.6724e-6
1.00	0.25	0.5085	0.5085	1.6488e-6
1.00	0.50	0.4216	0.4216	1.1408e-6
1.00	0.75	0.3084	0.3084	3.1583e-6
1.00	1.00	0.1991	0.1991	2.3462e-5

From TABLE XIII and TABLE XIV, it shows the similar results as two dimensional case. That is the numerical errors deceased when order of Taylor expansion increased for each point.

V. CONCLUSION

In this paper, we have developed the new Taylor series method and then applied it to find the numerical solution of multivariate Fredholm integral equations of the second kind. This new method finds an approximation of the solution of the solution pointwise and therefore lends itself to numerical computations which can be done in parallel. The method also computes the derivatives of the solution concurrently. The results of several numerical experiments have shown the efficiency and accuracy of this new method.

TABLE XI
COMPUTATIONAL RESULTS WITH $r = 8$ FOR EXAMPLE 4.4. (CONT.)

s_1	s_2	$\partial^2 x(s_1,s_2)/\partial s_1 \partial s_2$		
		Exact	Approx.	Abs. Error
0.00	0.00	0.0000	0.0001	1.3174e-4
0.00	0.25	0.9394	0.9394	1.3041e-5
0.00	0.50	1.5576	1.5576	6.1249e-6
0.00	0.75	1.7093	1.7093	1.2551e-6
0.00	1.00	1.4715	1.4715	4.8815e-5
0.25	0.00	0.0000	0.0001	6.6066e-5
0.25	0.25	0.5698	0.5698	5.1271e-7
0.25	0.50	0.9447	0.9447	1.8856e-7
0.25	0.75	1.0368	1.0368	1.0243e-6
0.25	1.00	0.8925	0.8926	3.7883e-5
0.50	0.00	0.0000	-0.0000	1.6583e-5
0.50	0.25	0.3456	0.3456	5.7666e-7
0.50	0.50	0.5730	0.5730	4.4944e-8
0.50	0.75	0.6288	0.6288	2.7770e-6
0.50	1.00	0.5413	0.5414	3.9355e-5
0.75	0.00	0.0000	-0.0000	2.2927e-5
0.75	0.25	0.2096	0.2096	5.3034e-7
0.75	0.50	0.3475	0.3475	1.9752e-7
0.75	0.75	0.3814	0.3814	7.5757e-7
0.75	1.00	0.3283	0.3283	4.1819e-7
1.00	0.00	0.0000	-0.0000	2.6724e-6
1.00	0.25	0.1271	0.1271	1.6488e-6
1.00	0.50	0.2108	0.2108	1.1408e-6
1.00	0.75	0.2313	0.2313	3.1583e-6
1.00	1.00	0.1991	0.1991	2.3462e-5

TABLE XII
COMPUTATIONAL RESULTS WITH $r = 8$ FOR EXAMPLE 4.4. (CONT.)

s_1	s_2	$\partial^2 x(s_1,s_2)/\partial s_2^2$		
		Exact	Approx.	Abs. Error
0.00	0.00	-2.0000	-1.9999	1.3174e-4
0.00	0.25	-1.6440	-1.6440	1.3041e-5
0.00	0.50	-0.7788	-0.7788	6.1249e-6
0.00	0.75	0.1424	0.1424	1.2551e-6
0.00	1.00	0.7358	0.7357	4.8815e-5
0.25	0.00	-1.2131	-1.2130	6.6066e-5
0.25	0.25	-0.9971	-0.9971	5.1271e-7
0.25	0.50	-0.4724	-0.4724	1.8856e-7
0.25	0.75	0.0864	0.0864	1.0243e-6
0.25	1.00	0.4463	0.4463	3.7883e-5
0.50	0.00	-0.7358	-0.7358	1.6583e-5
0.50	0.25	-0.6048	-0.6048	5.7666e-7
0.50	0.50	-0.2865	-0.2865	4.4944e-8
0.50	0.75	0.0524	0.0524	2.7770e-6
0.50	1.00	0.2707	0.2707	3.9355e-5
0.75	0.00	-0.4463	-0.4463	2.2927e-5
0.75	0.25	-0.3668	-0.3668	5.3034e-7
0.75	0.50	-0.1738	-0.1738	1.9752e-7
0.75	0.75	0.0318	0.0318	7.5757e-7
0.75	1.00	0.1642	0.1642	4.1829e-7
1.00	0.00	-0.2707	-0.2707	2.6724e-6
1.00	0.25	-0.2225	-0.2225	1.6488e-6
1.00	0.50	-0.1054	-0.1054	1.1408e-6
1.00	0.75	0.0193	0.0193	3.1583e-6
1.00	1.00	0.0996	0.0996	2.3462e-5

TABLE XIII
COMPUTATIONAL RESULTS WITH $r = 3$ FOR EXAMPLE 4.5.

s_1	s_2	s_3	Exact	$r = 3$	
				Approx.	Abs. Error
0.00	0.00	0.00	1.0000	0.9603	3.9700e-2
0.00	0.00	0.50	0.6065	0.6022	4.3443e-3
0.00	0.00	1.00	0.3679	0.3675	3.4475e-4
0.00	0.50	0.00	0.6065	0.6022	4.3443e-3
0.00	0.50	0.50	0.3679	0.3675	3.4475e-4
0.00	0.50	1.00	0.2231	0.2230	1.4727e-4
0.00	1.00	0.00	0.3679	0.3675	3.4475e-4
0.00	1.00	0.50	0.2231	0.2230	1.4727e-4
0.00	1.00	1.00	0.1353	0.1350	3.3520e-4
0.50	0.00	0.00	0.6065	0.6022	4.3443e-3
0.50	0.00	0.50	0.3679	0.3675	3.4475e-4
0.50	0.00	1.00	0.2231	0.2230	1.4727e-4
0.50	0.50	0.00	0.3679	0.3675	3.4475e-4
0.50	0.50	0.50	0.2231	0.2230	1.4727e-4
0.50	0.50	1.00	0.1353	0.1350	3.3520e-4
0.50	1.00	0.00	0.2231	0.2230	1.4727e-4
0.50	1.00	0.50	0.1353	0.1350	3.3520e-4
0.50	1.00	1.00	0.0821	0.0816	5.3288e-4
1.00	0.00	0.00	0.3679	0.3675	3.4475e-4
1.00	0.00	0.50	0.2231	0.2230	1.4727e-4
1.00	0.00	1.00	0.1353	0.1350	3.3520e-4
1.00	0.50	0.00	0.2231	0.2230	1.4727e-4
1.00	0.50	0.50	0.1353	0.1350	3.3520e-4
1.00	0.50	1.00	0.0821	0.0816	5.3288e-4
1.00	1.00	0.00	0.1353	0.1350	3.3520e-4
1.00	1.00	0.50	0.0821	0.0816	5.3288e-4
1.00	1.00	1.00	0.0498	0.0492	6.3610e-4

TABLE XIV
COMPUTATIONAL RESULTS WITH $r = 3$ FOR EXAMPLE 4.5.

s_1	s_2	s_3	Exact	$r = 4$	
				Approx.	Abs. Error
0.00	0.00	0.00	1.0000	1.0160	1.5989e-2
0.00	0.00	0.50	0.6065	0.6079	1.3360e-3
0.00	0.00	1.00	0.3679	0.3680	7.3607e-5
0.00	0.50	0.00	0.6065	0.6079	1.3360e-3
0.00	0.50	0.50	0.3679	0.3680	7.3607e-5
0.00	0.50	1.00	0.2231	0.2231	2.3300e-5
0.00	1.00	0.00	0.3679	0.3680	7.3607e-5
0.00	1.00	0.50	0.2231	0.2231	2.3300e-5
0.00	1.00	1.00	0.1353	0.1352	8.7379e-5
0.50	0.00	0.00	0.6065	0.6079	1.3360e-3
0.50	0.00	0.50	0.3679	0.3680	7.3607e-5
0.50	0.00	1.00	0.2231	0.2231	2.3300e-5
0.50	0.50	0.00	0.3679	0.3680	7.3607e-5
0.50	0.50	0.50	0.2231	0.2231	2.3300e-5
0.50	0.50	1.00	0.1353	0.1352	8.7379e-5
0.50	1.00	0.00	0.2231	0.2231	2.3300e-5
0.50	1.00	0.50	0.1353	0.1352	8.7379e-5
0.50	1.00	1.00	0.0821	0.0819	1.7637e-4
1.00	0.00	0.00	0.3679	0.3680	7.3607e-5
1.00	0.00	0.50	0.2231	0.2231	2.3300e-5
1.00	0.00	1.00	0.1353	0.1352	8.7379e-5
1.00	0.50	0.00	0.2231	0.2231	2.3300e-5
1.00	0.50	0.50	0.1353	0.1352	8.7379e-5
1.00	0.50	1.00	0.0821	0.0819	1.7637e-4
1.00	1.00	0.00	0.1353	0.1352	8.7379e-5
1.00	1.00	0.50	0.0821	0.0819	1.7637e-4
1.00	1.00	1.00	0.0498	0.0495	2.5609e-4

ACKNOWLEDGMENT

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

REFERENCES

[1] K. E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, *Cambridge University Press, Cambridge, UK*. (1997).

[2] Y. Ren, B. Zhang, and H. Qiao, A Simple Taylor-Series Expansion Method for a Class of Second Kind Integral Equations, *J. Comp. and Appl. Math.*, Vol. 110, pp. 15–24, (1999).

[3] H. Lechoslaw, D. Konrad, Integral Modeling and Simulation in Some Thermal Problems, *Proceedings of the 5th IASME/WSEAS Int. Conference on Heat Transfer, Thermal Engineering and Enviroment*, Athens, Greece, Augst 25-27, pp. 42–47, (2007).

[4] P. Huabsomboon, B. Novaprateep, and H. Kaneko, On Taylor-Series Expansion Method for the Second Kind Integral Equations, *J. Comp. and Appl. Math.*, Vol. 234, pp. 1466–1472, (2010).

[5] P. Huabsomboon, B. Novaprateep, and H. Kaneko, On Taylor-Series Expansion Method for Volterra Integral Equations of the Second Kind, *Scientiae Mathematicae Japonicae*, Vol. 73, pp. 541–551, (2011).

[6] L.Phusanisa, P. Huabsomboon, and H. Kaneko, A Modifiled Taylor Series Method to the Classical Love's Equation, *Recent Researches in Mathematical in Electrical Engineering and Computer Science*,Angers,France, November 17-19, pp. 68–72, (2011).

[7] M.Tavassoli Kajani and N.Akbari Shehni, Solutions of two-dimensional integral equations systems by using differential transform method, *Applications of Mathematics and computer Engineeriong*,Puerto Morelos, Mexico, January 29-31, pp. 74–77, (2011).

[8] P. Huabsomboon, B. Novaprateep, and H. Kaneko, Taylor-Series Expansion Methods for Nonlinear Hammerstein Equations, *Scientiae Mathematicae Japonicae*, to appear in (2012).

[9] B. Novaprateep, K. Neamprem and H. Kaneko, An Expansion Methods for Multivariate Fredholm Integral Equations, *Latest Advances in Information Science and Application*, Singapore City, Singapore, May 11–13, pp 91–94, (2012).