The Combined Solutions and Exact Solutions of a Fifth Order Ordinary Differential Equation

Chung-Hsien Tsai and Shy-Jen Guo

***Abstract*—**The objective of this paper is to construct combined solutions and to show exact solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3.

***Keywords*—**Bump, Lyapunov's Center Theorem, Schauder fixed point theorem, Steady capillary-gravity wave.

I. Introduction

PROGRESSIVE capillary-gravity waves on an irrotional incompressible inviscid fluid of constant density with surface tension in a two-dimensional channel of finite depth have been studied since nineteen century. Assume that a coordinate system moving with the wave at a speed is chosen so that in reference to it the wave motion is steady. Let H be the depth of water, g the acceleration of gravity, T the coefficient of surface tension, and ρ the constant density of the fluid. Then there are two nondimensional numbers which are important and defined as $F = c^2/(gH)$, the Froude number, and $\tau = T/(\rho g H^2)$, the Bond number.

When F is not close to 1, the linear theory of water waves is applicable. But when F approaches to 1, the solutions of linearized equations of water waves will grow to infinity (Peters and Stoker [12]). Therefore for F close to 1 nonlinear effect must be taken into account and thus $F = 1$ is a critical value. The first study of a solitary wave on water with surface tension is due to Korteweg and DeVries [10] after whom the K-dV equation with surface tension effect is named. A stationary K-dV equation with Bond number not near $1/3$ can also be formally derived by different approaches. However, if τ is close to 1, the formal derivation of the stationary K-dV equation fails. Thus $\tau = 1/3$ is also a critical value.

It becomes apparent that the problems for F near 1 and for τ near $1/3$ depend on each other and are difficult because they are not only strongly nonlinear, but also very delicate. Since the full nonlinear equations for the water waves are too complicated to study, it is of interest to study model equations. In Hunter and Vanden-Broeck's work [8], a fifth order ordinary differential equation considered as a perturbed stationary K-dV equation was obtained with the assumption that $F = 1 + F_2\epsilon^2$, $\tau = 1/3 + \tau_1\epsilon$ and ϵ is a small positive parameter. By integrating the fifth order ordinary differential equation once and set the con-stant of integration to be zero, then the model equation becomes

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = 0 \tag{1}$$

Equation (1) has been studied extensively by many authors [1-8] and several types of solutions have been found, such as periodic solutions [1, 5, 6, 7], solitary wave solutions [2-8], generalized solitary wave solutions (solitary waves with osciallatory tails at infinity) in the parameter region $\tau_1 < 0$ and $F_2 > 0$ [1,8], etc.

II. Derivation of the Model Equation

We consider the two-dimensional flow of an irrotional incompressible inviscid fluid of constant density ρ^* with surface tension T^* in a two-dimensional channel of finite depth. A rectangular coordinate system (x^*, y^*) is chosen such that the flow is bounded above by the free surface $y^* = \eta^*(x^*, t^*)$ and below by the rigid horizontal bottom with a bump $y^* = -H + \mathbf{b}^*(x^*)$.

The governing equations are:

In $-\infty < x^* < \infty, -H + \mathbf{b}^*(x^*) < y^* < \eta^*$

$$\phi^*_{x^*x^*} + \phi^*_{y^*y^*} = 0, \tag{2}$$

at the free surface, $y^* = \eta^*$

$$\eta^*_{t^*} + \phi^*_{x^*}\eta^*_{x^*} - \phi^*_{y^*} = 0, \tag{3}$$

$$\phi^*_t + \frac{1}{2}(\phi^{*2}_{x^*} + \phi^{*2}_{y^*}) + g\eta^* - \frac{T^*\eta^*_{x^*x^*}}{\rho^*(1+\eta^{*2}_{x^*})^{\frac{3}{2}}} = \frac{B^{*2}}{2}. \tag{4}$$

at the bottom, $y^* = -H + b^*(x^*)$

$$\phi^*_{y^*} - \phi^*_{x^*}\mathbf{b}^*_{x^*} = 0 \tag{5}$$

Where $\phi^*(x^*, y^*, t^*)$ is the potential function, B^* is an arbitrary constant, and H is the depth when the bump $\mathbf{b}^*$ is zero. In order to investigate long waves and derive asymptoyic solutions, it is conventient to introduce the following dimensionless variables:

CHUNG-HSIEN TSAI is with the Department of Information Technology, Overseas Chinese University, 407 Taichung, TAIWAN (R.O.C.) (e-mail: ctsai@ocu.edu.tw).

SHY-JEN GUO is with the Department of International Trade, National Taichung University of Science and Technology, 404 Taichung, TAIWAN (R.O.C.) (corresponding author. e-mail: ch861997@yahoo.com).

$$\left.\begin{aligned} &x=\tfrac{x^*}{L}, y=\tfrac{y^*}{H}, t=(\tfrac{H}{L})^4\tfrac{(gH)^{\frac{1}{2}}}{L}t^*,\\ &\eta(x,t)=\tfrac{\eta^*(x^*,t^*)}{A}, B=\tfrac{B^*}{(gH)^{\frac{1}{2}}}\\ &\phi(x,y,t)=\tfrac{H}{LA(gH)^{\frac{1}{2}}}\phi^*(x^*,y^*,t^*),\\ &\tau=\tfrac{T^*}{\rho^* gH^2}, \mathbf{b}(x)=\tfrac{(H/L)^{-2M}}{H}\mathbf{b}^*(x^*), \end{aligned}\right\} \tag{6}$$

where M is a positive integer to be chosen later.
In terms of the nondimensional variables (6), (2)-(5) become:

In $-\infty<x<\infty, -1+\beta^M\mathbf{b}(x)<y<\alpha\eta$

$$\beta\phi_{xx}+\phi_{yy}=0, \tag{7}$$

At the free surface, $y=\alpha\eta$

$$\beta^2\eta_t+\alpha\phi_x\eta_x-\beta^{-1}\phi_y=0, \tag{8}$$

$$\beta^2\phi_t+\frac{\alpha}{2}(\phi_x^2+\beta^{-1}\phi_y^2)+\eta-\frac{\beta\tau\eta_{xx}}{(1+\alpha^2\beta\eta_x^2)^{\frac{3}{2}}}=\frac{B^2}{2\alpha}, \tag{9}$$

at the bottom, $y=-1+\beta^M\mathbf{b}(x)$

$$\phi_y-\beta^{M+1}\phi_x\mathbf{b}_x=0. \tag{10}$$

In (7)-(10), α,β, and τ are nondimensional parameters

$$\alpha=\frac{A}{H},\quad \beta=(\frac{H}{L})^2,\quad \tau=\frac{T^*}{\rho^* gH^2}. \tag{11}$$

We seek solutions for periodic water waves of wavelength λ^*, and introduce the dimensionless wavelength

$$\lambda=\frac{\lambda^*}{L}, \tag{12}$$

The Froude number F is defined as

$$F=\frac{c}{(gH)^{\frac{1}{2}}}=\frac{\alpha}{\lambda}\int_0^\lambda\phi_x dx. \tag{13}$$

Since we are interested in small amplitude and shallow-water waves with τ near $\frac{1}{3}$, in (7)-(10), we take

$$\alpha=\epsilon^2,\quad \beta=\epsilon. \tag{14}$$

and expand η,ϕ,τ, and B as

$$\left.\begin{aligned} &\eta=\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+\cdots\\ &\phi=\tfrac{Bx}{\epsilon^2}+\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\cdots\\ &\tau=\tfrac{1}{3}+\epsilon\tau_1+\epsilon^2\tau_2+\cdots\\ &B=B_0+\epsilon B_1+\epsilon^2 B_2+\cdots\\ &F=F_0+\epsilon F_1+\epsilon^2 F_2+\cdots \end{aligned}\right\} \tag{15}$$

Substituting (14) and (15) into (7)-(10), taking $M=4$ in (10), and expanding at the boundary condition $y=0$ and $y=-1$, we obtain in $-\infty<x<\infty,\ -1<y<0$

$$\epsilon(\phi_{0_{xx}}+\epsilon\phi_{1_{xx}}+\epsilon^2\phi_{2_{xx}}+O(\epsilon^3))+(\phi_{0_{yy}}+\epsilon\phi_{1_{yy}}+\epsilon^2\phi_{2_{yy}}+O(\epsilon^3))=0, \tag{16}$$

at $y=0$,

$$\begin{aligned} &\epsilon^2(\eta_{0t}+\epsilon\eta_{1t}+O(\epsilon^2))\\ &+\epsilon^2\{(\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2}+\phi_{0_x}(x,0,t)+O(\epsilon))\}\\ &(\eta_{0x}+\epsilon\eta_{1x}+\epsilon^2\eta_{2x}+O(\epsilon^3))\\ &-\epsilon^{-1}\{(\phi_{0y}(x,0,t)+\epsilon^2(\eta_0+\epsilon\eta_1+O(\epsilon^2))\phi_{0yy}(x,0,t)\\ &+\epsilon(\phi_{1y}(x,0,t)+\epsilon^2(\eta_0+\epsilon\eta_1+O(\epsilon^2))\phi_{1yy}(x,0,t)\\ &+O(\epsilon^4))+\epsilon^2(\phi_{2y}(x,0,t)+O(\epsilon^2))+\epsilon^3(\phi_{3y}(x,0,t)\\ &+O(\epsilon^2))+O(\epsilon^4))\}=0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\epsilon^2(\phi_{0t}(x,0,t)+\epsilon\phi_{1t}(x,0,t)+O(\epsilon^2))\\ &+\frac{\epsilon^2}{2}\{\{\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2}\\ &+(\phi_{0x}(x,0,t)+\epsilon^2\eta_0\phi_{0xy}(x,0,t))+\epsilon\phi_{1x}(x,0,t)\\ &+\epsilon^2\phi_{2x}(x,0,t)+O(\epsilon^3)\}^2+\epsilon^{-1}\{(\phi_{0y}+\epsilon^2\phi_{0yy})+\epsilon\phi_{1y}\\ &+\epsilon^2\phi_{2y}+O(\epsilon^3)\}^2\}+(\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+O(\epsilon^3))\\ &-\epsilon(\frac{1}{3}+\epsilon\tau_1+\epsilon^2\tau_2+O(\epsilon^3))(\eta_{0xx}+\epsilon\eta_{1xx}+\epsilon^2\eta_{2xx}\\ &+O(\epsilon^3))(1+O(\epsilon^{10}))\\ &=\frac{(B_0+\epsilon B_1+\epsilon^2B_2+\epsilon^3B_3+\epsilon^4B_4+O(\epsilon^5))^2}{2\epsilon^2}, \end{aligned} \tag{18}$$

at $y=-1$

$$\begin{aligned} &(\phi_{0y}(x,-1,t)+\epsilon\phi_{1y}(x,-1,t)+\epsilon^2\phi_{2y}(x,-1,t)\\ &+\epsilon^3\phi_{3y}(x,-1,t)+O(\epsilon^4))-\epsilon^5(\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2})\\ &+\phi_{0x}(x,-1,t)+O(\epsilon))\mathbf{b}_x=0. \end{aligned} \tag{19}$$

From (16) to (19), we have

$O(\epsilon^{-1})$:

$$\phi_{0y}(x,0,t)=0. \tag{20}$$

$O(1)$:

$$\phi_{0yy}(x,y,t)=0, \tag{21}$$

$$B_0\eta_{0x}-\phi_{1y}(x,0,t)=0, \tag{22}$$

$$B_0\phi_{0x}(x,0,t)+\eta_0=0, \tag{23}$$

$$\phi_{0y}(x,-1,t)=0. \tag{24}$$

From (21) and by (22) or (24), it follows that

$$\phi_{0y}=0,\quad \phi_0(x,y,t)=\phi_0(x,t). \tag{25}$$

$O(\epsilon)$:

$$\phi_{0xx}(x,t)+\phi_{1yy}(x,y,t)=0, \tag{26}$$

$$B_0\eta_{1x}+B_1\eta_{0x}-\phi_{2y}(x,0,t)=0, \tag{27}$$

$$B_0\phi_{1x}(x,0,t)+B_1\phi_{0x}(x,0,t)+\tfrac{\phi_{0y}^2(x,0,t)}{2}+\eta_1-\tfrac{1}{3}\eta_{0xx}=0, \tag{28}$$

$$\phi_{1y}(x,-1,t)=0. \tag{29}$$

From (26) and by (29), we found that

$$\phi_{1y}(x,y,t)=-\phi_{0xx}(x,t)(y+1), \tag{30}$$

and

$$\phi_{1x}(x,y,t)=-\phi_{0xxx}(x,t)(\frac{y^2}{2}+y)+R_{1x}(x,t), \tag{31}$$

From (22), (23), and by (30), we obtain

$$B_0=1, \tag{32}$$

$$\phi_{0x}=-\eta_0. \tag{33}$$

From (28) and by (25), (31), and (32), it follows that

$$\phi_{1xx}(x,0,t)=\frac{1}{3}\eta_{0xxx}-\eta_{1x}+B_1\eta_{0x} \tag{34}$$

$$=R_{1xx}(x,t). \tag{35}$$

$O(\epsilon^2)$:

$$\phi_{1xx}(x,y,t)+\phi_{2yy}(x,y,t)=0, \tag{36}$$

$$\begin{aligned}&\eta_{0t}+B_0\eta_{2x}+B_1\eta_{1x}+(B_2+\phi_{0x})\eta_{0x}\\&-\eta_0\phi_{1yy}-\phi_{3y}=0 \text{ at } y=0,\end{aligned} \tag{37}$$

$$\begin{aligned}&\phi_{0t}+\phi_{2x}+B_1\phi_{1x}-B_2\eta_0+\tfrac{\eta_0^2}{2}+\eta_2-\tfrac{1}{3}\eta_{1xx}\\&-\tau_1\eta_{0xx}=0 \text{ at } y=0,\end{aligned} \tag{38}$$

$$\phi_{2y}(x,-1,t)=0. \tag{39}$$

From (36), (39) and by (31), we found that

$$R_2(x,t)=-\frac{1}{3}\phi_{0xxxx}(x,t)-R_{1xx}(x,t), \tag{40}$$

$$\phi_{2y}(x,y,t)=\phi_{0xxxx}(x,t)(\frac{y^3}{6}+\frac{y^2}{2}+\frac{y}{3})+R_2(x,t)(y+1), \tag{41}$$

and

$$\begin{aligned}\phi_2(x,y,t)&=\phi_{0xxxx}(x,t)(\frac{y^4}{24}+\frac{y^3}{6}+\frac{y^2}{6})\\&+R_2(x,t)(\frac{y^2}{2}+y)+R_3(x,t)\end{aligned} \tag{42}$$

From (27) and by (32),(41)

$$R_2(x,t)=\eta_{1x}+B_1\eta_{0x}. \tag{43}$$

From (37) and by (30),(32) ,(33),

$$\eta_{2x}=-\eta_{0t}-B_1\eta_{1x}-(B_2-2\eta_0)\eta_{0x}+\phi_{3y}(x,0,t) \tag{44}$$

Differentiating (38) about x and by (33) , (35) , (42)

$$\eta_{2x}=\eta_{0t}-R_{3xx}-B_1R_{1xx}+(B_2-\eta_0)\eta_{0x}+\frac{1}{3}\eta_{1xxx}+\tau_1\eta_{0xxx}. \tag{45}$$

By (34), (35), (40), and (43)

$$B_1=0. \tag{46}$$

By (44), (45), and (46)

$$\begin{aligned}\frac{1}{3}\eta_{1xxx}&=-2\eta_{0t}-2B_2\eta_{0x}+3\eta_0\eta_{0x}\\&-\tau_1\eta_{0xxx}+R_{3xx}+\phi_{3y}(x,0,t)\end{aligned} \tag{47}$$

$O(\epsilon^3)$:

$$\phi_{2xx}(x,y,t)+\phi_{3yy}(x,y,t)=0, \tag{48}$$

$$\phi_{3y}(x,-1,t)=B_0b_x. \tag{49}$$

From(48), (49) and by (42), we obtain

$$\begin{aligned}\phi_{3y}(x,-1,t)&=\frac{1}{45}\phi_{0xxxxxx}(x,t)-\frac{1}{3}R_{2xx}(x,t)\\&+R_{3xx}(x,t)+\phi_{3y}(x,0,t)\end{aligned} \tag{50}$$

By (32), (33), (43), (46), and (50), we have

$$2\eta_{0t}+2B_2\eta_{0x}-3\eta_0\eta_{0x}+\tau_1\eta_{0xxx}-\frac{1}{45}\eta_{0xxxxx}=\mathbf{b}_x. \tag{51}$$

The Froude number F is defined and expanded as

$$\begin{aligned}F&=F_0+\epsilon F_1+\epsilon^2F_2+O(\epsilon^3)\\&=\frac{\epsilon^2}{\lambda}\int_0^\lambda(\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2}+\phi_{0x}+O(\epsilon))dx\\&=B_0+\epsilon B_1+\epsilon^2B_2+\frac{\epsilon^2}{\lambda}\int_0^\lambda\phi_{0x}dx+O(\epsilon^3).\end{aligned} \tag{52}$$

By (33) and the mean value of periodic solution over a period is zero, we found that

$$\int_0^\lambda\phi_{0x}dx=-\int_0^\lambda\eta_0dx=0\cdot$$

If η_0 is a solitary wave solution with the properity that

$$\int_0^\infty\phi_{0x}dx=-\int_0^\infty\eta_0dx<\infty, \tag{53}$$

then, with $\lambda=\infty$, the term

$$\frac{1}{\lambda}\int_0^\lambda\phi_{0x}dx$$

in (52) will be zero. We shall see that all the solitary wave solutions discovered in the following chapters will satisfy (53).

Therefore, we have

$$B_0=F_0,\ B_1=F_1,\ B_2=F_2\cdot$$

and then (51) becomes

$$2\eta_{0t}+2F_2\eta_{0x}-3\eta_0\eta_{0x}+\tau_1\eta_{0xxx}-\frac{1}{45}\eta_{0xxxxx}=\mathbf{b}_x\cdot \tag{54}$$

Next, we assume $\eta_{0t}=0$ in equation (54), integrate (54) once and set the constant of integration to be zero, then we have the following model equation

$$2F_2\eta_0-\frac{3}{2}\eta_{0x}^2+\tau_1\eta_{0xx}-\frac{1}{45}\eta_{0xxxx}=\mathbf{b}\cdot \tag{55}$$

In the following sections, we shall use η for η_0 in equation (55), that is,

$$2F_2\eta-\frac{3}{2}\eta^2+\tau_1\eta_{xx}-\frac{1}{45}\eta_{xxxx}=\mathbf{b}\cdot \tag{56}$$

and disscuss the solutions of the model equation (56).

III. PROBLEM FORMULATION

We follow Zufiria [19] to construct a Hamiltonian associated to (56). When $\mathbf{b}=0$, we rewrite (56) as

$$\eta_{xxxx}-45\tau_1\eta_{xx}-90F_2\eta+\frac{135}{2}\eta^2=0. \tag{57}$$

We multiply $-\eta_x$ to (57) and integrate the resulting equation, then equation (57) has first integral as

$$H=45F_2\eta^2+\frac{1}{2}\eta_{xx}^2-\eta_{xxx}\eta_x+\frac{45}{2}\tau_1\eta_x^2-\frac{45}{2}\eta^3, \tag{58}$$

where H is a constant. Introducing the change of variables

$$\left.\begin{aligned} q_1=\eta,\quad & p_1=\eta_{xxx}-45\tau_1\eta_x,\\ q_2=\eta_{xx},\quad & p_2=\eta_x,\end{aligned}\right\}$$

then (58) becomes

$$H(q_1,q_2,p_1,p_2)=45F_2q_1^2+\frac{1}{2}q_2^2-p_1p_2-\frac{45}{2}\tau_1p_2^2-\frac{45}{2}q_1^3, \tag{59}$$

and we have

$$\frac{dz}{dx}=J\nabla_z H(z)=Az+g(z)\equiv f(z,\mu), \tag{60}$$

where $\mu=(\tau_1,F_2)\in\mathbf{R}^2$,

$$z=\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\in\mathbf{R}^4,\quad J=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}, \tag{61}$$

and

$$A=\begin{pmatrix}0&0&0&-1\\0&0&-1&-45\tau_1\\-90F_2&0&0&0\\0&-1&0&0\end{pmatrix},\ g(z)=\begin{pmatrix}0\\0\\\frac{135}{2}q_1^2\\0\end{pmatrix}. \tag{62}$$

Therefore (59) is a two degree of freedom Hamiltonian with two parameters τ_1 and F_2. Because different parameters (τ_1,F_2) in (59) give rise to different eigenvalues λ for the linearized system of (60) at the origin, we divide the parameter plane (τ_1,F_2) into following nine cases

Case 0 $(\tau_1=0,F_2=0)$: $\lambda=0,0,0,0$.

Case 1 $(\tau_1\in\mathbf{R},F_2>0)$: $\lambda=\pm r,\pm wi;\ r,w>0$.

Case 2 $(\tau_1<0,F_2=0)$: $\lambda=0,0,\pm wi;\ w>0$.

Case 3 $(\tau_1<0,F_2<0,\ (45\tau_1)^2+360F_2>0)$: $\lambda=\pm w_1 i,\pm w_2 i;\ w_1>w_2>0$.

Case 4 $(\tau_1<0,F_2<0,\ (45\tau_1)^2+360F_2=0)$: $\lambda=\pm wi,\pm wi;\ w>0$

Case 5 $(\tau_1\in\mathbf{R},F_2<0,\ (45\tau_1)^2+360F_2<0)$: $\lambda=\pm a\pm bi;\ a,b>0$

Case 6 $(\tau_1>0,F_2<0,\ (45\tau_1)^2+360F_2=0)$: $\lambda=\pm r,\pm r;\ r>0$

Case 7 $(\tau_1>0,F_2<0,\ (45\tau_1)^2+360F_2>0)$: $\lambda=\pm r_1,\pm r_2;\ r_1>r_2>0$

Case 8 $(\tau_1>0,F_2=0)$: $\lambda=0,0,\pm r;\ r>0$.

We rewrite (56) as follows,

$$\eta_{xxxx}-45\tau_1\eta_{xx}-90F_2\eta=-45(\mathbf{b}(x))+\frac{3}{2}\eta^2)\equiv f, \tag{63}$$

IV. PROBLEM SOLUTION

A. Combined solutions for Case 1

In this section, we shall construct a combined solution for equation (63) in **Case 1**. We construct this half-periodic and half-solitary-wave solution as follows : On interval $(-\infty,x_1)$, we let $\mathbf{b}(x)=0$ and use Lyapunov's Center Theorem to show that a periodic solution $\eta_P(x)$ exists initiating at $x=x_1$ to the left. On $[x_1,x_2]$, we shall use Schauder fixed point theorem to prove there exist a bounded solution $\eta_C(x)$ for equation (9) subject to initial values $(\eta_P(x_1),\ \eta_P'(x_1),\ \eta_P''(x_1),\ \eta_P'''(x_1))$ at $x=x_1$. On (x_2,∞), we also let $\mathbf{b}(x)=0$ and show that equation (9) with initial values at $x=x_2$ has a solution $\eta_R(x)$, which decay to zero exponentially at positive infinity by using a theorem from [6]. Then we combine $\eta_P(x),\eta_C(x)$ and $\eta_R(x)$ to have a solution of equation (63). Since the proof of existence of bounded solutions $\eta_C(x)$ and $\eta_R(x)$ on $[x_1,x_2]$ and (x_2,∞) are the same as in [13], in the following, we shall focus on the existence of $\eta_P(x)$ on interval $(-\infty,x_1)$. First, we state Lyapunov's Center Theorem :

Theorem 1 *Assume that a system with a non-degenerate integral has an equilibrium point with exponents $\pm wi,\ \lambda_3,\cdots,\lambda_m$ where $iw\neq 0$ is pure imaginary. If λ_j/iw is never an integer for $j=3,\cdots,m$, then there exists a one-parameter family of periodic orbits emanating from the equilibrum point. Moreover, when approaching the equilibrum point along the family, the periods tend to $2\pi/w$.*

When $\mathbf{b}(x)=0$, equation (56) possesses a Hamiltonian (59) H and an equilibrium at the origin. In **Case 1**, the engenvalues of the linearized systems of (63) are $\pm wi$ and $\pm r$ where

$$w=(-(45\tau_1-((45\tau_1)^2+360F_2)^{\frac{1}{2}})/2)^{\frac{1}{2}}>0$$

and

$$r=((45\tau_1+((45\tau_1)^2+360F_2)^{\frac{1}{2}}/2)^{\frac{1}{2}}>0.$$

Thus, by Theorem, there exists a periodic motion of period close to $2\pi/w$ in the nonlinear system of differential equations with the Hamiltonian H. Since the amplitude of the periodic motions are small and depends on initial conditions, we can write the periodic solutions in the form [11]

$$z_P(x;\epsilon)=\epsilon e^{Ax}a+O(\epsilon^2) \tag{64}$$

where ϵ is a small parameter, A is the same as in (8), and a is a fixed nonzero vector such that $z_P(0;\epsilon)/\epsilon\ \to a$ when $\epsilon\to 0$. We rewrite (64) in eigenvector coordinates as

$$\hat{z}_P(x;\epsilon) = \epsilon e^{\Lambda x}\hat{a} + (\epsilon^2) \quad (65)$$

where $\hat{z}_P(x;\epsilon) = P^{-1}z_P(x;\epsilon), \hat{a} = P^{-1}a, \Lambda = diag(-wi, wi, -r, r)$, and P is a 4×4 matrix with the column vectors ξ_1, ξ_2, ξ_3, and ξ_4 corresponding to the unit eigenvectors of eigenvalues $-wi, wi, -r$ and r. We see that $\hat{a}$ must be in the form $(\hat{a}_1, \hat{a}_2, 0, 0)$, otherwise (65) will not be periodic. Therefore, vector $a \in R^4$ lies in the two dimensional eigenspace $S_a = \{\xi_1, \xi_2\}$ where ξ_2 is the conjugate of ξ_1.

On interval $(-\infty, x_1)$, by Theorem and the discussion above, there is a one-parameter family of periodic solutions in the form (65) with initial values $z_P(x_1;\epsilon)$ having the properties that $z_P(x_1;\epsilon)/\epsilon \to e^{Ax_1}a$ as $\epsilon \to 0$ and $a \in S_a \cap R^4$. The solution η_R on (x_2, ∞) can be found by Theorem as in [13]. As in [13], the bounded solution $\eta_C(x)$ on (x_1, x_2) is obtained by Schauder fixed point theorem and it is required that the initial values at $x = x_1$ and the bump **b** both must be sufficiently small. Now, we write the first component of $z_P(x;\epsilon^*;x_1)$ as $\eta_P(x;\epsilon^*;x_1)$ to obtain the solution of (9) on $(-\infty, x_1)$ .As in [13], we combine $\eta_P(x;\epsilon^*;x_1)$, $\eta_C(x;x_1)$, and $\eta_R(x;T_2)$ to be a solution of equation (63) in **Case 1**, which is periodic on interval $(-\infty, x_1)$ and decays to zero exponentially at positive infinity on interval (x_2, ∞).

B. Combined solutions for Case 3

In this section, we would like to discuss the solutions of equation (63) for the parameters τ_1, F_2 corresponding to **Case 3**. As in previous sections, the idea is to investgate the solutions of equation (63) on three different intervals $(-\infty, x_1), [x_1, x_2]$, and (x_2, ∞) .On $[x_1, x_2]$, we shall prove there exists bounded solutions of equation (63) with initial values at $x = x_1$ by Schauder fixed point theorem. On intervals $(-\infty, x_1)$ and (x_2, ∞), we let **b**(x) = 0 and show that equation (63) has periodic or bounded solutions. Then these solutions can be combined to become a C^4 solution of equation (63).

From Section III, we know that the eigenvalues of the linearized systems of equation (63) in **Case 3** are two pairs of pure imaginaries, $\pm w_1 i$ and $\pm w_2 i$, with $w_1 > w_2 > 0$. When w_1/w_2 is irrational, **Theorem 1** (Lyapunov's Center Theorem) can be used to construct periodic solutions on intervals $(-\infty, x_1)$ and (x_2, ∞). There exist two one-parameter families of periodic orbits emanating from the fixed point $z = 0$. If we let $w = w_1$ in **Theorem 1**, then the periods of this one-parameter periodic family tend to $2\pi/w_1$ when the fixed point is approached along the family. We call this family as short-period family since $w_1 > w_2$. If we let $w = w_2$ in **Theorem 1**, then the periods of this one-parameter periodic family tend to $2\pi/w_2$ when the fixed point is approached along the family. We call this family as long-period family.

We write the short-period family of periodic solutions in the form

$$z_{w_1}(x;\epsilon) = \epsilon e^{Ax}a_{w_1} + O(\epsilon^2) \quad (66)$$

where ϵ is a small parameter, A is the same as in (86), and a_{w_1} is a fixed nonzero vector such that $z_{w_1}(0;\epsilon)/\epsilon \to a_{w_1}$ when $\epsilon \to 0$. We rewrite (66) in eigenvector coordinates as

$$\hat{z}_{w_1}(x;\epsilon) = \epsilon e^{\Lambda x}\hat{a}_{w_1} + O(\epsilon^2) \quad (67)$$

where $\hat{z}_{w_1}(x;\epsilon) = P^{-1}z_{w_1}(x;\epsilon), \hat{a}_{w_1} = P^{-1}a, \Lambda = diag(-w_1 i, w_1 i, -w_2 i, w_2 i)$, and P is a 4×4 matrix with the column vectors ξ_1, ξ_2, ξ_3, and ξ_4 corresponding to the unit eigenvectors of eigenvalues $-w_1 i, w_1 i, -w_2 i$, and $w_2 i$ respectively. We see that $\hat{a}_{w_1}$ must be in the form $(\hat{a}_{w_{11}}, \hat{a}_{w_{12}}, 0, 0)$ since the periods tend to $2\pi/w_1$ when we approach the fixed point zero along the family. Therefore, vector $a_{w_1} \in R^4$ lies in the two dimensional eigenspace $S_{w_1} = \{\xi_1, \xi_2\}$ where ξ_2 is the conjugate of ξ_1 .Thus, on intevral $(-\infty, x_1)$, we have periodic solutions (67) in short-period family with initial values $z_{w_1}(x_1;\epsilon)$ having the properties that $z_{w_1}(x_1;\epsilon)/\epsilon \to e^{Ax_1}a_{w_2}$ as $\epsilon \to 0$ and $a_{w_2} \in S_{w_2} \cap R^4$ On intevral (x_2, ∞), we also obtain periodic solutions (67) in short-period family with initial values $z_{w_2}(x_2;\epsilon)$ having the properties that $z_{w_1}(x_2;\epsilon)/\epsilon \to e^{Ax_2}a_{w_1}$ as $\epsilon \to 0$ and $a_{w_1} \in S_{w_1} \cap R^4$.

By the same arguments as above on periodic solutions of short-period family, we have periodic solutions $z_{w_2}(x;\epsilon) = \epsilon e^{Ax}a_{w_2} + O(\epsilon^2)$ in long-period family on intevral $(-\infty, x_1)$ with initial values $z_{w_2}(x_1;\epsilon)$ having the properties that $z_{w_2}(x_1;\epsilon)/\epsilon \to e^{Ax_1}a_{w_2}$ as $\epsilon \to 0$ and $a_{w_2} \in S_{w_2} \cap R^4$ where $S_{w_2} = \{\xi_3, \xi_4\}$ and ξ_4 is the conjugate of ξ_3. On intevral (x_2, ∞), we also have periodic solutions $z_{w_2}(x;\epsilon)$ in long-period family with initial values $z_{w_2}(x_2;\epsilon)$ having the properties that $z_{w_2}(x_2;\epsilon)/\epsilon \to e^{Ax_2}a_{w_2}$ as $\epsilon \to 0$ and $a_{w_2} \in S_{w_2} \cap R^4$.

As in [19], the bounded solution $\eta_C(x)$ on $[x_1, x_2]$ is obtained by Schauder fixed point theorem and it is required that the initial values at $x = x_1$ and the bump **b** both must be sufficiently small such that M_Y and $M_{\mathbf{b}}$ satisfy (98) and (100) in [19]. These requirements could be met by choosing a small bump **b** and sufficiently small ϵ, say ϵ^*. Now, we write the first component of $z_{w_1}(x;\epsilon_1^*;x_1)$ or $z_{w_2}(x;\epsilon_2^*;x_1)$ as $\eta_L(x;\epsilon_L;x_1)$ to be the solution of (63) on $(-\infty, x_1)$. In [15], we showed that the zero solution is stable for **Case 3**, thus bounded $\eta_R(x;\epsilon_R;x_2)$ on interval (x_2, ∞) can be obtained if $(\eta_c(x_2), \eta_c'(x_2), \eta_c''(x_2), \eta_c'''(x_2))$ is small and this could be done as disscussed in [13]. As in [13], we combine $\eta_L(x;\epsilon_L;x_1)$, $\eta_C(x;x_1)$, and $\eta_R(x;\epsilon_R;x_2)$ to obtain a solution of equation (63) in **Case 3** with w_1/w_2 irrational, which is periodic on interval $(-\infty, x_1)$ and bounded on $[x_1, \infty]$.

C. Combined solutions in Case 4 for large x

In this section, we would like to discuss the solutions of equation (63) for the parameters τ_1, F_2 corresponding to Case 4. As in previous sections, we shall show the existence of solutions of equation (63) on three different intervals $(-\infty, x_1), [x_1, x_2]$, and (x_2, ∞). Then combine these solutions to become a C^4 solution of equation (63).

First we show there exist peroidic solutions of equation (63) with b(x) = 0 by a theorem from Meyer [11]. In [11], Meyer discussed the bifurcation occurring in restricted 3-body problem. The Hamiltonian he concerned depends on a parameter $\bar{\mu}$ and has the properties that the eigenvalues of the associated linearized operator are (I) $\pm iw_2, \pm iw_2$ if $\bar{\mu} > 0$ where $w_1, w_2 \in \mathbf{R}, w_1 \neq w_2$, and $w_1 w_2 \neq 0$. (e.g., In Case 3) (II) $\pm iw, \pm iw$ if $\bar{\mu} = 0$ where $w \in \mathbf{R}$ and $w \neq 0$, with two two-dimensional Jordan blocks. (e.g., In Case 4)(III) $\pm a \pm ib$ if $\bar{\mu} < 0$ where $a, b \in \mathbf{R}$ and $ab \neq 0$.(e.g., In Case 5). Meyer transforms the perturbed Hamiltonian to Sokol'skii's normal form

$$H(\bar{\mu}) = w\Gamma_1 + \delta\Gamma_2 + \bar{\mu}(a\Gamma_1 + b\Gamma_3) + \frac{1}{2}(c\Gamma_1^2 + 2d\Gamma_1\Gamma_3 + e\Gamma_3^2) + \cdots, \tag{68}$$

where

$$\Gamma_1 = i(z_2 z_4 - z_1 z_3), \quad \Gamma_2 = z_1 z_2, \quad \Gamma_3 = z_3 z_4. \tag{69}$$

With higher-order terms in H are functions of Γ_1, Γ_2 and Γ_3 only. Then he proved the following results.

Theorem 3 Consider a Hamiltonian of the form (68) with $w \neq 0,\ \delta = \pm 1,\ b \neq 0,\ e \neq 0$. Assume $\delta e > 0$. There exist two Lyapunov families of periodic orbits emanating from the origin when $\delta b \bar{\mu}$ is small and positive. These families persist when $\bar{\mu} = 0$ as two distinct families of periodic orbits emanating from the origin. As $\delta b \bar{\mu}$ becomes negative, the two families detach from the origin as a single family and recede from the origin.

Theorem 3 can be used to show a Hamiltonian Holf biurcation near Case 4 since the Hamiltonian system (59) has the same properties as Meyer [11] discussed if we assume

$$F_2 = -\frac{45}{8}\tau_1^2 + \bar{\mu},$$

with $\tau_1 < 0$ and the parameter $\bar{\mu} \approx 0$. After transforming (59) to Sokol'skii's normal form (68), we have

$$w = (-\frac{45}{2}\tau_1)^{\frac{1}{2}}, \quad \delta = -1, \quad b = \frac{90}{w^2\delta}, \quad e = -\frac{38475}{32w^8}.$$

Since $\delta e > 0$, we have the same results as Theorem 1 described. When $\bar{\mu} > 0$ is small and we are in the region Case 3 but near Case 4, there are two Lyapunov periodic families emanating from the origin. These families persist when $\bar{\mu} = 0$ as two distinct families of periodic orbits emanating from the origin. As $\bar{\mu} < 0$ is small and we are in the region Case 5 but near Case 4, the two families detach from the origin as a single family and recede from the origin.

The periodic solutions derived by Theorem 1 with $\bar{\mu} = 0$ which corresponds to Case 4 can be used as $\eta_L(x)$ in interval $(-\infty, x_1]$. The existence of $\eta_C(x)$ on $[x_1, x_2]$ can also be proved by the same arguments in [13]. On interval $(x_2, \infty]$, since the zero solution of equation (63) with $\mathbf{b}(x) = 0$ is almost stable, bounded $\eta_R(x)$ for large x is obtained provided that $(\eta_R(x_2) = \eta_C(x_2), \eta_R'(x_2) = \eta_C'(x_2), \eta_R''(x_2) = \eta_C''(x_2),\ \eta_R'''(x_2) = \eta_C'''(x_2))$ is small and this could be done as we disscussed in [16].

As in [13], we match $\eta_L(x), \eta_C(x)$, and $\eta_R(x)$ at $x = x_1$ and $x = x_2$ to obtain a solution of equation (63) in Case 4 which is periodic on interval $(-\infty, x_1)$ and bounded on $[x_1, \infty)$ for large x.

D. Exact solutions

We rewrite (63) as follows,

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta + \frac{135}{2}\eta^2 = -45b \tag{70}$$

In this section, we would like to discuss solutions of the model equation (70) in the form

$$\eta(x) = A\sec h^n(Bx) + C\cos^m(Dx), \tag{71}$$

where $A, B, C, D \in R$ and $m, n \in N$. (71) could be expressed as the solutions that we are interested in, such as periodic solutions, solitary wave solutions, and generalized solitary wave solutions. To construct soultions in the form (71), we substitute (71) in (70) to obtain

$$\begin{aligned} b(x) = -\frac{1}{45}\{&\{m(m-1)(m-2)(m-3)CD^4 \\ &\cos^{m-4}(Dx) - m(m-1)(2D^2 \\ &(m^2-2m+2) + 45\tau_1)CD^2 \\ &\cos^{m-2}(Dx) + (D^4m^4 + 45\tau_1 D^2 m^2 \\ &-90F_2)C\cos^m(Dx) + \frac{135}{2}C^2 \\ &\cos^{2m}(Dx)\} + \{(B^4n^4 - 45\tau_1 B^2 n^2 \\ &-90F_2)A\sec h^n(Bx) + \frac{135}{2}A^2 \\ &\sec h^{2n}(Bx) - n(n+1)(2B^2 \\ &(n^2+2n+2) - 45\tau_1)AB^2\sec h^{n+2} \\ &(Bx) + n(n+1)(n+2)(n+3)AB^4 \\ &\sec h^{n+4}(Bx)\} + \frac{135}{2}AC\cos^m(Dx) \\ &\sec h^n(Bx))\}, \end{aligned} \tag{72}$$

then (70) has solutions in the form (71) if (72) holds.

In Tsai [13-19], we only proved the existence of solutions for each of the nine cases but did not provide explicit form for the soultions. With the special non-compact bump (72) in each of the nine cases, we shall see that equation (70) has periodic solutions if $A = 0$ and $CD \neq 0$, solitary wave solutions if $C = 0$ and $AB \neq 0$, and generalized solitary wave solutions if $ABCD \neq 0$, In the following, we shall discuss the solutions we mentioned above.

(I) $\eta(x) = A\sec h^{n}(Bx)$

It is clear that (70) has solitary wave solutions $\eta(x) = A\sec h^{n}(Bx)$ when $C = 0$ in bump (71), i.e.

$$b(x) = -\frac{1}{45}\{\{(B^4n^4 - 45\tau_1 B^2 n^2 - 90F_2)A\sec h^{n}(Bx) + \frac{135}{2}A^2 \sec h^{2n}(Bx) - n(n+1)(2B^2(n^2+2n+2) - 45\tau_1)AB^2\sec h^{n+2}(Bx) + n(n+1)(n+2)(n+3)AB^4 \sec h^{n+4}(Bx)\}\}, \tag{73}$$

which is in a solitary-wave shape. Is it possible that $b(x) = 0$ in (73)? When $n = 4$ and let $b(x) = 0$, that is, let the cofficients of $\sec h(Bx)^4$, $\sec h(Bx)^6$, and $\sec h(Bx)^8$ vanish, then we have $A = -(1575/169)\tau_1^{\ 2}$, $B = \sqrt{45\tau_1/13}/2$, and $F_2 = -(810/169)\tau_1^{\ 2}$ with $\tau_1 > 0$. This means that (70) with $b(x) = 0$ has an explicit solitary wave solution

$$\eta(x) = -\frac{1575}{169}\tau_1^{\ 2}\sec h^4(\frac{1}{2}\sqrt{\frac{45\tau_1}{13}}x)$$

When $F_2 = -(810/169)\tau_1^{\ 2}$ with $\tau_1 > 0$.

(II) $\eta(x) = C\cos^{m}(Dx)$

It is clear that (70) has periodic solutions $\eta(x) = C\cos^{m}(Dx)$ when $A = 0$ in bump (71), i.e.

$$b(x) = -\frac{1}{45}\{\{m(m-1)(m-2)(m-3)CD^4 \cos^{m-4}(Dx) - m(m-1)(2D^2(m^2-2m+2) + 45\tau_1)CD^2 \cos^{m-2}(Dx) + (D^4m^4 + 45\tau_1 D^2 m^2 - 90F_2)C\cos^{m}(Dx) + \frac{135}{2}C^2 \cos^{2m}(Dx)\}\}, \tag{74}$$

which is also periodic. The period of (74) is $2\pi/D$ when $m > 1$ and the amplitude is $|b(0)|$ which increases as D increases, that is , as the period of $\eta(x)$ decreases.

(III) $\eta(x) = A\sec h^{n}(Bx) + C\cos^{m}(Dx)$

In this situation, we require that the amplitude C of the periodic part is much smaller than the amplitude A of the solitary part in order to have a generalized solitary wave.

V. NUMERICAL EXPERIMENT

In this section, we shall give combined solutions numerically of equation (63) by using classical fourth-order Runge-Kutta method.

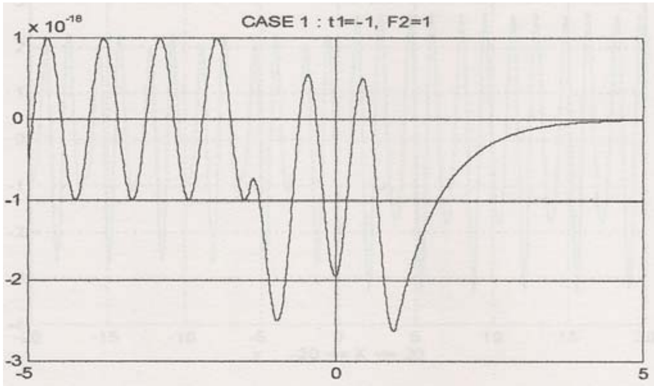

Figure 1: A combined solution of equation (63) obtained by using classical fourth-order Runge-Kutta method for **Case 1** with compact bump $b(x) = 10^{-19}\exp(1/(x^2-1))$ on interval $(-1,1)$.

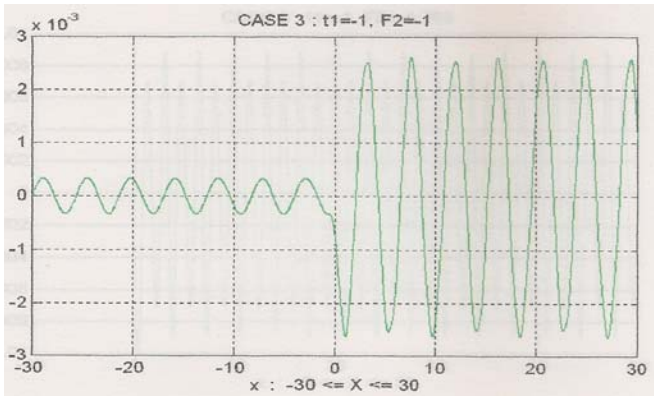

Figure 2: A combined solution of equation (63) obtained by using classical fourth-order Runge-Kutta method for **Case 3** with $\tau_1 = -1$, $F_2 = -1$, and compact bump $b(x) = 10^{-2}\exp(1/(x^2-1))$ on interval $(-1,1)$.

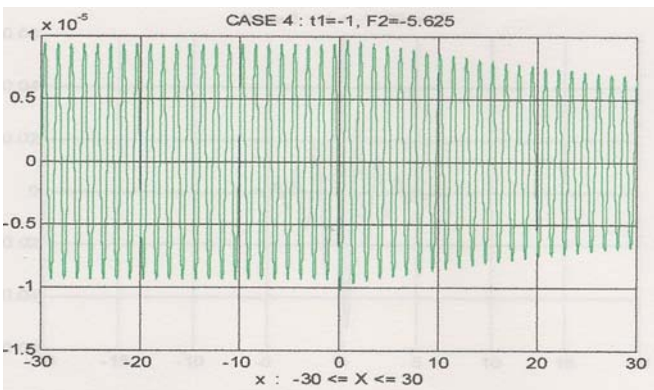

Figure 3: A combined solution of equation (63) obtained by using classical fourth-order Runge-Kutta method for **Case 3** with $\tau_1 = -1$,

$F_2 = -45/8$, and compact bump $b(x) = 10^{-5} \exp(1/(x^2 - 1))$ on interval $(-1,1)$.

VI. Conclusion

We constructed combined solutions of model equation (63) for a sufficiently smooth compact bump $\mathbf{b}(x)$ and has a compact support on the inteval $[x_1, x_2]$ with $\mathbf{b}(x_1) = \mathbf{b}(x_2) = 0$.

We also showed the exact solutions of a fifth order model equation (70) for steady capillary-gravity waves over a non-compact bump with the Bond number near 1/3.

References

[1] Amick, C.J. & Mcleod, J.B., ”A singular per- turbation problem in waterwaves,” *Stability Appl. Anal. Continuous Media* **1**, 127-148 (1991)

[2] Amick, C.J. & Toland, J.F., ”Solitary waves with surface tension I: Trajectories homoclinic to periodic orbits in four dimensions,” *Arch. Rat. Mech. Anal.* **118**, 37-69 (1992)

[3] Amick, C.J. & Toland, J.F., ”Homoclinic orbits in the dynamic phasespace analogy of an elastic strut,” *Euro. Jnl of Applied Mathematics* **3**, 97-114 (1992)

[4] B. Buffoni, A.R. Champneys, A.R., Toland, J.F., ”Bifurcation and colescence of a plethora of homoclinic orbits for a Hamiltonian system,” *J.Dyn. Differential Equations* **8,** 221-281 (1996)

[5] Champneys, A.R. ”Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics,” *Physica D* **112**, 158-186 (1998)

[6] Coppel, W.A. ”Stability and Asymptotic Behavior of Differential equations,” *D.C. HEATH AND COMPANY*, (1965)

[7] Coppola, V.T. & Rand, R.H. ”Computer Algebra Implemention of Lie Transforms for Hamiltonian Systems: Application to the Nonlinear Stability of *L4*,” *ZAMM.Z.angew. Math. Mech.* **69***,* 275-284 (1989)

[8] Hunter J.K., and Vanden-Broeck, J.M ., ”Solitary and periodic gravity-capillary waves of finite amplitude,” *J. Fluid Mech.* **134**, 205-219 (1983)

[9] Iooss, G. & Kirchgässner, K. ”Water waves for small surface tension: an approach via normal form,” *Proceedings of the Royal of Edinburgh* **122A**, 267-299 (1992)

[10] Korteweg, P.J. and de Vries, G. ”On the change of the form of long waves advancing in a rectangular canal, and on a new type of long stationary waves,” *Phil. Mag.* **39**, 422-443 (1895)

[11] Meyer, K.R. & Hall, G.R. ”Intrduction to Hami- ltonian Dynamical Systems and the N-Body Problem,” *Springer-Verlag*, **90**, (1990)

[12] Peters, A.D.; Stoker, J.J. ”Solitary waves in liquids having non-constant density.,” *Comm. Pure Appl. Math.* **13**, 115-164 (1960)

[13] Tsai, Chung-Hsien & Guo, Shy-Jen ” Unsymmetric Solitary Wave Solutions of a Fifth Order Model Equation for Steady Capillary-Gravity Waves over a Bump with the Bond Number Near 1/3.,” *Proceedings of the 6th WSEAS International Conference on FLUID MECHANICS (FLUIDS'09)*, pp.94-99, Ningbo, China January 10-12 (2009).

[14] Tsai, Chung-Hsien & Guo, Shy-Jen (2009) ” The existence of generalized solitary wave solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3.,” *Proceedings of the 6th WSEAS International Conference on FLUID MECHANICS (FLUIDS'09)*, pp.86-89 Ningbo, China January 10-12 (2009).

[15] Tsai, Chung-Hsien & Guo, Shy-Jen ” The stability of zero solution of a model equation for steady capillary-gravity waves with the bond number close to 1/3.,” *Proceedings of the 6th WSEAS International Conference on FLUID MECHANICS (FLUIDS'09),* pp.90-93, Ningbo, China, January 10-12 (2009).

[16] Tsai, Chung-Hsien & Guo, Shy-Jen ” Solitary Wave Solutions of a Fifth Order Model Equation for Steady Capillary - Gravity Waves over a Bump with the Bond Number Near 1/3.,” *Proceedings of the 7th WSEAS International Conference on FLUID MECHANICS (FLUIDS'10),* pp. 236-240, University of Cambridge, UK, February 23-25 (2010).

[17] Tsai, Chung-Hsien & Guo, Shy-Jen ” The existence of generalized solitary wave solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3.,” *International Conference on APPLIED and COMPUTATIONAL MATHEMATICS (ICACM'11),* pp.49-53, Lanzarote, Canary Islands, Spain, May 27-29 (2011).

[18] Tsai, Chung-Hsien & Guo, Shy-Jen (2011) ” Asymptotic Solutions and Unsymmetric Solutions of a Fifth Order Ordinary Differential Equation,” *WSEAS TRANSACTIONS on FLUID MECHANICS* **6**, 1-11 (2011)

[19] Tsai, Chung-Hsien & Guo, Shy-Jen (2012) ”Bounded solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3,” *Proceedings of the AMERICAN CONFERENCE on APPLIED MATHEMATICS (AMERICAN-MATH '12)* ,pp. 145-148, Harvard, Cambridge, January 25-27 (2012)

[20] Zufiria, J.A. ”Weakly nonlinear non-symmetric gravity waves on water of finite depth,” *J. fluid Mech.* **180**, 371-385 (1987)