A modified common set of weights method to complete ranking DMUs

A. Payan, F. Hosseinzadeh Lotfi, A. A. Noora, M. Khodabakhshi

***Abstract*—** Liu and Peng (Computers & Operations Research, 35[18], 1624-1637, 2008) presented a method for obtaining a common set of weights in data envelopment analysis (DEA) and they also provided a system for ranking decision making units (DMUs) using common set of weights. Their method has two main problems. At first, their presented model may have alternative optimal solutions (alternative common set of weights). Alternative optimal solutions may lead to different ranks for each DMU. Second, all criteria for ranking, by their suggested system, may be identical for at least two DMUs and so these DMUs will have the same rank. Therefore, there is no full ranking for DMUs using the suggested method. The aim of this paper is surveying these shortcomings and presenting methods to overcome them. This paper suggests a method to obtain unique common set of weights which can be applied for all methods used linear programs for acquiring common set of weights. Moreover, by definition bad benchmark against benchmark defined by Liu and Peng, a system for full ranking DMUs is proposed. Numerical examples are used to illustrate the proposed method.

***Keywords*—**Alternative optimal solutions, Common set of weights, Data envelopment analysis (DEA), Linear programming, Ranking.

I. Introduction

DATA envelopment analysis (DEA) measures the efficiency of homogeneous decision making units (DMUs) by using mathematical optimization techniques. In this approach, by using the definition of production possibility set (PPS), DMUs are divided into two groups, DMUs lying in the interior of the PPS are called inefficient units and DMUs lying on the frontier of the PPS are called efficient units. A thorough review on DEA up 2009 can be found in Cook and Seiford [1].

Nowadays, DEA is widely used for analyzing units in various systems. Asmild *et al.* [2] utilized DEA for reallocations of police personnel. Assessment of universities through DEA was done by Kuah and Wong [3]. Sancho *et al.* [4] proposed a DEA model for determining the efficiency of wastewater treatment plants in Spanish. A DEA approach was used to evaluate economical and social roles of NOCs [5]. Assessing relative performance of water fabrication operations using DEA was suggested by Chen and Chien [6].

In the main methods of DEA, such as CCR method that is presented by Charnes *et al.* [7] and BCC method that is suggested by Banker *et al.* [8], efficient units have efficiency score one, whereas inefficient units have efficiency scores between zero and one. Using efficiency scores, we have the ability to rank DMUs. Although efficient units will have same ranking, they do not have equal performance in actual practice. To overcome this problem, researchers have presented several approaches for ranking the efficient units.

The methods for ranking are divided into two groups. In a group, there are methods that only can rank vertex efficient DMUs are called super-efficiency methods. The first work of this kind is presented by Andersen and Petersen [9] and after is introduced many methods such as Tone [10] and so on. In another group, all efficient DMUs can be ranked. They are divided into three basic groups: cross-efficiency methods such as work Doyle and Green [11], MCDM methods such as work Jahanshahloo *et al.* [12] and interval DEA methods such as work Entani *et al.* [13]. For a review of ranking methods, see Adler *et al.* [14].

One important stream of ranking methods is utilizing common set of weights that is extended by MCDM methods. Interesting research works in this area can be found, for example, in Cook *et al.* [15] and Roll *et al.* [16]. The common set of weights, which is the normal vector of supporting hyperplane of PPS, is obtained by solving a multiple objective program. To rank DMUs, a norm is defined to measure distance DMUs from the hyperplane. These distances are used to rank DMUs. However, there are different methods for finding the normal vector of supporting hyperplane. Also there are different methods for determining distance DMUs from this hyperplane. One of these methods is Cook and Zhu's method [17].

In this article, a ranking system for efficient DMUs is introduced based on the work Liu and Peng [18]. In the proposed method by Liu and Peng [18], there is no guarantee that the common set of weights is unique. Therefore, the ranking results corresponding to alternative common set of

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weights may not identical. In other words, a specific DMU may obtain a different rank for each case. Also, all criteria for ranking some of the DMUs may be equal. The latter case implies that we cannot obtain a complete ranking for all DMUs. This paper will propose a method to overcome these difficulties.

The paper is organized as follows. Section 2 briefly discusses Liu and Peng's method [18]. Section 3 introduces our proposed approach and states some of its properties. Numerical examples are given in section 4, and section 5 concludes the paper.

II. Liu and Peng's Method

Assume that there are n DMUs to be evaluated, each DMU with m inputs and s outputs. We denote by $x_{ij}\,(i = 1,2,\dots, m)$ and $y_{rj}\,(r = 1,2,\dots, s)$ the values of the inputs and outputs of $\text{DMU}_j\,(j = 1,2,\dots, n)$, which are all known and positive. According to the implication of efficiency, the absolute efficiency of DMU_j is defined as:

$$\theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}},$$

where u_r, v_i are the weights assigned to the rth output and the i th input, respectively. In order to determine the efficiency of DMU_j in relation to the other DMUs, Charnes *et al.* [7] developed the following well-known CCR model as:

$$Max\ \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}}$$

$$s.t\quad \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,2,\dots n,$$

$$u_r \ge \varepsilon, \quad r = 1,2,\dots,s,$$

$$v_i \ge \varepsilon, \quad i = 1,2,\dots,m, \tag{1}$$

where the subscript zero represents the DMU under evaluation and ε is the non-Archimedean number. The optimal value of this problem is considered as the relative efficiency of DMU_0.

In the proposed method by Liu and Peng [18], they considered virtual DMUs $(\sum_{i=1}^{m} v_i x_{ij}, \sum_{r=1}^{s} u_r y_{rj})(j \in E)$ (E is the set of indices of efficient DMUs) and introduced the score unity as a benchmark for these virtual DMUs such that $(\sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij}) \le 1 (j \in E)$. As a geometrical interpretation, $(\sum_{i=1}^{m} v_i x_{ij}, \sum_{r=1}^{s} u_r y_{rj})(j \in E)$ are points in R^2 and the benchmark is a straight line that passes through the origin with gradient one. Then $(v_1, v_2,\dots,v_m, u_1, u_2,\dots,u_s)$ must be determined such that the distance between these points and the benchmark line be as small as possible. This distance is measured by vertical and horizontal gaps Δ_j^O and Δ_j^I as shown in Fig. 1, in which

$$M = (\sum_{i=1}^{m} v_i x_{ij}, \sum_{r=1}^{s} u_r y_{rj}),$$

$$M' = (\sum_{i=1}^{m} v_i x_{ij} - \Delta_j^I, \sum_{r=1}^{s} u_r y_{rj} + \Delta_j^O).$$

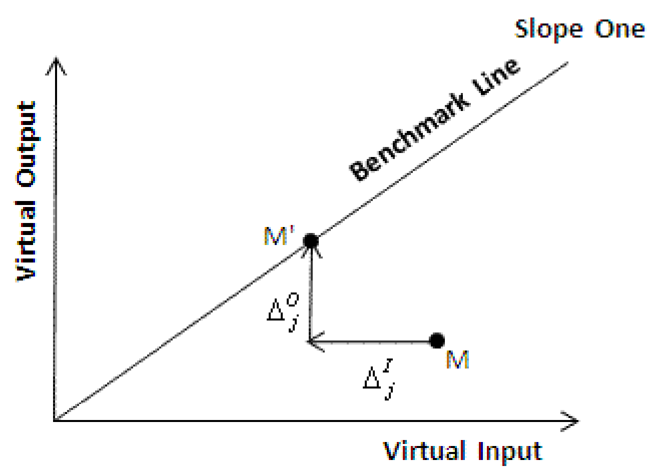

Fig. 1 Analysis of gaps between virtual DMUs and the benchmark

Liu and Peng [18] presented a model as:

$$Min \sum_{j \in E} (\Delta_j^O + \Delta_j^I)$$

$$s.t\quad \frac{\sum_{r=1}^{s} u_r\ y_{rj} + \Delta_j^O}{\sum_{i=1}^{m} v_i\ x_{ij} - \Delta_j^I} = 1, \quad j \in E,$$

$$u_r \geq \varepsilon, \quad r = 1,2,...,s,$$

$$v_i \geq \varepsilon, \quad i = 1,2,...,m,$$

$$(2)\ \Delta_j^O \geq 0\ ,\ \Delta_j^I \geq 0,\ \ j \in E,$$

and with appropriate changes, an equivalent model can be made as:

$$Max \sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_i$$

$$s.t \ \ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \ \ j \in E,$$

$$u_r \geq \varepsilon, \quad r = 1,2,...,s,$$

$$(3)\ v_i \geq \varepsilon, \quad i = 1,2,...,m,$$

The performance of the evaluating DMU with inputs $x_i = \sum_{j \in E} x_{ij}\ (i = 1,2,...,m)$ and outputs $y_r = \sum_{j \in E} y_{rj}\ (r = 1,2,...,s)$ is measured with the above model. The optimal solution of the problem (3), that is $(v_1^*, v_2^*,..., v_m^*, u_1^*, u_2^*,..., u_s^*)$, is considered as the common weights. For ranking the DMUs, Liu and Peng [18] presented a system as follows:

First define $g_j^* = \dfrac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}}, \ \ j \in E.$

1) If $g_j^* > g_i^*$, then the performance of DMU $_j$ is better than that DMU$_i$.

2) If $g_j^* = g_i^* < 1$ and $\Delta_j^* < \Delta_i^*$ then the performance of DMU $_j$ is better than that DMU$_i$, where $\Delta_j^* = \sum_{i=1}^{m} v_i^* x_{ij} - \sum_{r=1}^{s} u_r^* y_{rj}$ is horizontal gap between virtual DMU $_j$ and benchmark line.

3) If $g_j^* = g_i^* = 1$ and $\pi_j^* > \pi_i^*$, then the performance of DMU $_j$ is better than that DMU$_i$, where $\pi_j^*\ (j \in E)$ are the shadow prices of the problem (3).

As mentioned before, this method has two difficulties. First, we do not know whether the problem (3) has a unique optimal solution, and if it has alternative optimal solutions, which common set of weights must be considered? Of course, this imperfection exists in all common set of weights methods. Second, for ranking DMUs that have the same absolute efficiency equal to one, Liu and Peng [18] used shadow prices that may be equal for at least two DMUs. So, these DMUs cannot be differentiated. For overcoming the first difficulty, they considered the optimal solution of the following problem as the common weights.

$$Min \sum_{r=1}^{s} u_r - \sum_{i=1}^{m} v_i$$

$$s.t \ \ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \Delta_j = 0, \quad j \in E,$$

$$\sum_{j \in E} \Delta_j = \Delta^*,$$

$$u_r \geq \varepsilon, \quad r = 1,2,...,s,$$

$$v_i \geq \varepsilon, \quad i = 1,2,...,m,$$

$$(4)\ \Delta_j \geq 0, \quad j \in E,$$

where Δ^* is the optimal value of problem (2). But there is not any guarantee that the optimal solution of the problem (4) is unique. If we submit that the problem (4) has unique optimal solution, the difficulty using of shadow prices for ranking still exists. In the next section, these problems are considered and methods to overcome them are proposed.

III. Methodology

In this section, we consider a new insight of the Liu and Peng's method [18]. To reduce distance between virtual DMUs and the benchmark line, we have to increase $\sum_{r=1}^{s} u_r y_{rj}\ (j \in E)$ and decrease $\sum_{i=1}^{m} v_i x_{ij}\ (j \in E)$. So, for having the total gap for all units as small as possible, we can maximize $\sum_{j \in E} \sum_{r=1}^{s} u_r y_{rj}$ and minimize $\sum_{j \in E} \sum_{i=1}^{m} v_i x_{ij}$. However, the obtained problem is a multiple objective linear program that is not easy to solve. Equivalently, we use fractional programming problem to maximize $\sum_{j \in E} \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{j \in E} \sum_{i=1}^{m} v_i x_{ij}$.

Therefore, we have the following problem for finding common set of weights as:

$$Max \quad \left(\frac{\sum_{j\in E}\sum_{r=1}^{s} u_r y_{rj}}{\sum_{j\in E}\sum_{i=1}^{m} v_i x_{ij}} = \frac{\sum_{r=1}^{s} u_r \; y_r}{\sum_{i=1}^{m} v_i \; x_i}\right)$$

$$s.t \quad \frac{\sum_{r=1}^{s} u_r \; y_{rj}}{\sum_{i=1}^{m} v_i \; x_{ij}} \le 1, \quad j \in E,$$

$$u_r \ge \varepsilon, \quad r = 1,2,...,s,$$

$$v_i \ge \varepsilon, \quad i = 1,2,...,m, \tag{5}$$

The value of the objective function of the problem (5) can be considered as the weighted sum of the absolute efficiencies of efficient DMUs. The problem (5) is equivalent to a linear programming problem as below:

$$Max \sum_{r=1}^{s} u_r y_r$$

$$s.t \quad \sum_{i=1}^{m} v_i x_i = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j \in E,$$

$$u_r \ge \varepsilon, \quad r = 1,2,...,s,$$

$$v_i \ge \varepsilon, \quad i = 1,2,...,m, \tag{6}$$

The optimal value of the objective function of this problem is the relative efficiency of a DMU with inputs $x_i = \sum_{j\in E} x_{ij}$ $(i = 1,2,...,m)$ and outputs $y_r = \sum_{j\in E} y_{rj} \, (r = 1,2,...,s)$. The optimal solution of the problem (6) is considered as the common weights.

A. Unique Common Weights

The problem (6) may have alternative optimal solutions. To obtain the unique optimal solution of the problem (6), we suggest the following method. This approach can be applied for all the methods that use linear programming models for obtaining common set of weights. Consider a linear programming problem and its dual as:

$$\begin{aligned} & Min \;\; cx \\ & s.t \;\; Ax = b, \\ & \qquad x \ge 0, \end{aligned} \tag{7}$$

$$\begin{aligned} & Max \;\; wb \\ & s.t \;\; wA \le c, \\ & \qquad w \;\; free, \end{aligned} \tag{8}$$

where A is an $m \times n$ matrix and rank (A) = m. All definitions and theorems are taken from Murty [19].

Definition 1 A basic feasible solution of the problem (7) is degenerate if at least one of its basic variables be zero. A basic feasible solution is nondegenerate if it is not degenerate.

Definition 2 The linear programming problem (7) is totally nondegenerate if each basic feasible solution is nondegenerate.

Theorem 1 The linear programming problem (7) is totally nondegenerate iff each feasible solution has at least m non zero components.

Theorem 2 If a linear programming problem has alternative optimal solutions, its dual has a degenerate optimal solution.

So, from the mathematical standpoint, alternative optimal solutions of the primal cause the degeneracy in the optimal solution of the dual problem. Therefore, if we avoid the degeneracy in optimal solutions of the dual, we can remove the alternative optimal solutions of the primal model. To do so, we proceed as follows.

Theorem 3 For each $b \in R^m$ there is a positive real number ε_1, such that for all $\bar{\varepsilon}$ and $0 < \bar{\varepsilon} < \varepsilon_1$ the problem

$$\begin{aligned} & Min \;\; cx \\ & s.t \;\; Ax = b(\varepsilon) = (b_1 + \bar{\varepsilon}^1, ..., b_m + \bar{\varepsilon}^m), x \ge 0, \end{aligned} \tag{9}$$

is totally nondegenerate.

By contra positive of theorem (2), if a linear programming problem does not have a degenerate optimal solution, then its dual does not have alternative optimal solutions (has unique optimal solution).

Result 1 The problem (9) is a nondegenerate linear programming problem, so it does not have a degenerate optimal solution and therefore the dual of problem (9) does not have alternative optimal solutions.

Now, consider the dual of problem (6) as:

$$Min \quad \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$

$$s.t \quad \sum_{j\in E} \lambda_j x_{ij} + s_i^- = \theta x_i, \quad i = 1,2,...,m,$$

$$\sum_{j\in E}\lambda_j y_{rj} - s_r^+ = y_r, \quad r=1,2,...,s,$$

$$\lambda_j \geq 0, \quad j\in E,$$

$$s_i^- \geq 0, \quad i=1,2,...,m,$$

$$s_r^+ \geq 0, \quad r=1,2,...,s,$$

$$\theta \quad free, \tag{10}$$

Using theorem (3), there is an ε_1 , such that for all $0<\bar{\varepsilon}<\varepsilon_1$, the following problem does not have a degenerate optimal solution.

$$Min \quad \theta - \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+)$$

$$s.t \quad \sum_{j\varepsilon E}\lambda_j x_{ij} + s_i^- - \theta x_i = \bar{\varepsilon}^i, \quad i=1,2,...,m,$$

$$\sum_{j\in E}\lambda_j y_{rj} - s_r^+ = y_r + \bar{\varepsilon}^{m+r}, \quad r=1,2,...,s,$$

$$\lambda_j \geq 0, \quad j\in E,$$

$$s_i^- \geq 0, \quad i=1,2,...,m,$$

$$s_r^+ \geq 0, \quad r=1,2,...,s,$$

$$\theta \quad free, \tag{11}$$

So, the dual of the problem (11) does not have alternative optimal solutions. The dual is as:

$$Max \sum_{r=1}^{s} u_r y_r + \sum_{i=1}^{m} v_i \bar{\varepsilon}^i + \sum_{r=1}^{s} u_r \bar{\varepsilon}^{m+r}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_i = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j\in E,$$

$$u_r \geq \varepsilon, \quad r=1,2,...,s,$$

$$v_i \geq \varepsilon, \quad i=1,2,...,m, \tag{12}$$

Now, the unique optimal solution of this problem is considered as the unique common weights. But this is the theoretical part of the issue. To solve the problem (12), $m+s+1$ phases must be considered. In the first phase, the problem (6) is solved. If θ^* be the optimal value of the problem (6), then the following m problems must be solved.

$$v_i^* = Max \ v_i$$

$$s.t \sum_{i=1}^{m} v_i x_i = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \ j\in E,$$

$$\sum_{r=1}^{s} u_r y_r = \theta^*,$$

$$v_k = v_k^*, \ k=1,2,..., i-1,$$

$$u_r \geq \varepsilon, \quad r=1,2,...,s,$$

$$v_i \geq \varepsilon, \quad i=1,2,...,m, \tag{13}$$

$$(i=1,2,...,m)$$

and then, the following s problems are solved.

$$u_r^* = Max \ u_r$$

$$s.t \sum_{i=1}^{m} v_i x_i = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j\in E,$$

$$\sum_{r=1}^{s} u_r y_r = \theta^*,$$

$$v_k = v_k^*, \quad i=1,2,..., m,$$

$$u_l = u_l^*, \quad l=1,2,..., r-1,$$

$$u_r \geq \varepsilon, \quad r=1,2,...,s, \tag{14}$$

$$(r=1,2,...,s)$$

According to the models (13) and (14), the smaller index for weight of input, the preferable weight of input for optimization and the smaller index for weight of output, the preferable weight of output for optimization. Also weights for inputs have preference to weights for outputs. These preferences can be changed by the suggestions of decision maker.

B. A System for Ranking

The aim of solving the problem (6) is to find $(v_1, v_2, ..., v_m, u_1, u_2, ..u_s)$ such that the points $(\sum_{i=1}^{m} v_i x_{ij}, \sum_{r=1}^{s} u_r y_{rj})$ $(j \in E)$ can be as close to the benchmark line, as possible. Then DMUs are ranked according to their gaps from the benchmark line. But observe that there are often more than one virtual DMU on the benchmark (such that this case happens in the work of Liu and Peng [18]). In other words, the gap between virtual DMUs and the benchmark line is zero. So, these DMUs have the same rank, which is not desirable from managerial view point. In what follows, we try to remove this difficulty.

The benchmark line is a good benchmark, because the virtual DMUs try to reach it. On the contrary, a bad benchmark can be defined, which the DMUs try to keep away from it. Geometrically, a good benchmark is a line with gradient one that passes through the origin and all virtual DMUs are below it. We define bad benchmark as a line that passes through the origin with a gradient such that all virtual DMUs are lie above it. In Fig. 2, four DMUs A, B, C and D exist which all of them are below the good benchmark and over the bad benchmark.

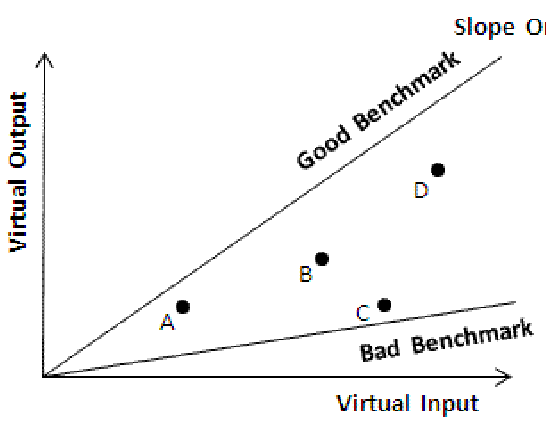

Fig. 2 Analysis of bad benchmark against good benchmark

As we know, each movement toward the good benchmark is equivalent to reducing the distance from it, and is equivalent to increasing the distance from the bad benchmark. We use the gaps between the virtual DMUs (vertical or horizontal gaps) and the bad benchmark for ranking DMUs that have the same distance from the good benchmark. In Fig. 3, we consider the two virtual DMUs M and N that are on the good benchmark, where $M = (x_M, y_M)$ and $N = (x_N, y_N)$. γ_M and γ_N are the horizontal gaps from the bad benchmark and g is the gradient of the bad benchmark line. We have $\gamma_M = \frac{1}{g} y_M - x_M$ and $\gamma_N = \frac{1}{g} y_N - x_N$ and $\gamma_N > \gamma_M$.

The gap between virtual DMU N and the bad benchmark is greater than that of virtual DMU M, so DMU N rank above DMU M. As shown in Fig. 3, DMU N has higher position than DMU M on good benchmark and so output DMU N is more than that of DMU M. This means that the value of virtual output is a criterion for ranking DMUs that their corresponding virtual DMUs are on good benchmark. From the economical point of view, this criterion is justifiable.

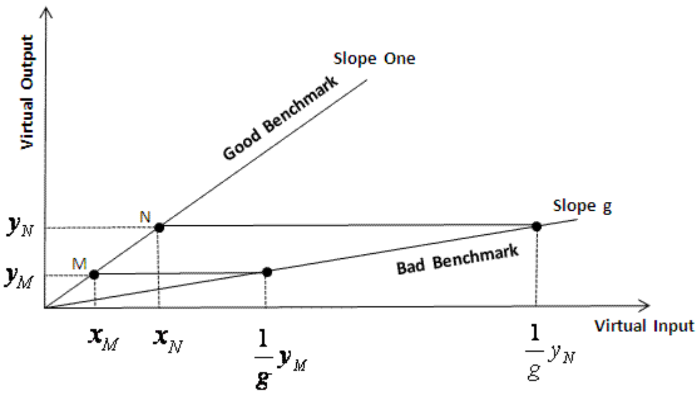

Fig. 3 Comparing two DMUs that are on the good benchmark

Definition 3 Suppose virtual DMU $_j$ and DMU $_i$ are on the good benchmark. If the gap between virtual DMU $_j$ and the bad benchmark is more than that of virtual DMU $_i$, then the performance of DMU $_j$ is better than that of DMU $_i$.

Note that between virtual DMU *j* and *i,* the greater the DMU to the benchmark line, the better the input-output combination. In fact, input for the DMU that is closer to the benchmark line is increased, whereas its output is decreased

According to this definition, the proposed ranking system by Liu and Peng [18] is modified as follows:

First define

$$g_j^* = \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}}, \quad j \in E,$$

$$0 < g < \underset{j \in E}{Min}\{g_j^*\},$$

where $(v_1^*, v_2^*, ..., v_m^*, u_1^*, u_2^*, ..., u_s^*)$ is the optimal solution of the problem (12) and $g \in (0, \underset{j \in E}{Min}\{g_j^*\})$ can be considered as the bad benchmark. Also,

$\Delta_j^* = \sum_{i=1}^{m} v_i^* x_{ij} - \sum_{r=1}^{s} u_r^* y_{rj}$, $\gamma_j^* = \frac{1}{g}\sum_{r=1}^{s} u_r^* y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij}$,

where γ_j^*, Δ_j^* are the horizontal gaps between virtual DMU_j and the bad and good benchmarks, respectively. Now, the ranks of DMUs are determined as follows:

1) If $g_j^* > g_i^*$, then the performance of DMU_j is better than that of DMU_i.

2) If $g_j^* = g_i^*$ and $\Delta_j^* < \Delta_i^*$, then the performance of DMU_j is better than that of DMU_i.

3) If $g_j^* = g_i^*$ and $\Delta_j^* = \Delta_i^*$ and $\gamma_j^* > \gamma_i^*$, then the performance of DMU_j is better than that of DMU_i.

Note that, when $g_j^* > g_i^*$, or $\Delta_j^* < \Delta_i^*$ in the case $g_j^* = g_i^*$, our proposed method will produce the same ranks as those of the Liu and Peng's method [18]. But, only when $g_j^* = g_i^* = 1$ and therefore $\Delta_j^* = \Delta_i^*$=0, we consider a bad benchmark g that $0 < g < \underset{j\in E}{Min}\{g_j^*\}$, and a DMU has a better rank than another when it has a larger distance than another to the bad benchmark.

Theorem 4 Two DMUs have equal rank iff their corresponding virtual DMUs have the same position.

Proof Let $(v_1^*, v_2^*, \ldots, v_m^*, u_1^*, u_2^*, \ldots, u_s^*)$ be the optimal solution of the problem (12). First, consider DMU_j and DMU_k have equal ranks. According to our proposed ranking system, we have:

$$A) \quad g^* = \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}} = \frac{\sum_{r=1}^{s} u_r^* y_{rk}}{\sum_{i=1}^{m} v_i^* x_{ik}}$$

$$B) \quad \Delta^* = \sum_{i=1}^{m} v_i^* x_{ij} - \sum_{r=1}^{s} u_r^* y_{rj} = \sum_{i=1}^{m} v_i^* x_{ik} - \sum_{r=1}^{s} u_r^* y_{rk}$$

$$C) \quad \gamma^* = \frac{1}{g}\sum_{r=1}^{s} u_r^* y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} = \frac{1}{g}\sum_{r=1}^{s} u_r^* y_{rk} - \sum_{i=1}^{m} v_i^* x_{ik}$$

Regarding equations B, we have

$$\sum_{r=1}^{s} u_r^* y_{rj} = \sum_{i=1}^{m} v_i^* x_{ij} - \Delta^*,$$

$$\sum_{r=1}^{s} u_r^* y_{rk} = \sum_{i=1}^{m} v_i^* x_{ik} - \Delta^*$$

$$\Rightarrow \frac{\sum_{i=1}^{m} v_i^* x_{ij} - \Delta^*}{\sum_{r=1}^{s} u_r^* y_{rj}} = \frac{\sum_{i=1}^{m} v_i^* x_{ik} - \Delta^*}{\sum_{r=1}^{s} u_r^* y_{rk}}$$

$$\Rightarrow \sum_{r=1}^{s} u_r^* y_{rk} \sum_{i=1}^{m} v_i^* x_{ij} - \Delta^* \sum_{r=1}^{s} u_r^* y_{rk} = \sum_{r=1}^{s} u_r^* y_{rj} \sum_{i=1}^{m} v_i^* x_{ik} - \Delta^* \sum_{r=1}^{s} u_r^* y_{rj}$$

$$\overset{from\ A}{\Rightarrow} \Delta^* \left(\sum_{r=1}^{s} u_r^* y_{rj} - \sum_{r=1}^{s} u_r^* y_{rk}\right) = 0$$

If $\sum_{r=1}^{s} u_r^* y_{rj} = \sum_{r=1}^{s} u_r^* y_{rk}$, then, from equations A, we have $\sum_{i=1}^{m} v_i^* x_{ij} = \sum_{i=1}^{m} v_i^* x_{ik}$.

So, $(\sum_{i=1}^{m} v_i^* x_{ij}, \sum_{r=1}^{s} u_r^* y_{rj}) = (\sum_{i=1}^{m} v_i^* x_{ik}, \sum_{r=1}^{s} u_r^* y_{rk})$. In this case, proof is completed. Otherwise, $\Delta^* = 0$. According to equations C, we have

$$\frac{1}{g}\sum_{r=1}^{s} u_r^* y_{rj} = \sum_{i=1}^{m} v_i^* x_{ij} + \gamma^*,$$

$$\frac{1}{g}\sum_{r=1}^{s} u_r^* y_{rk} = \sum_{i=1}^{m} v_i^* x_{ik} + \gamma^*$$

$$\Rightarrow \frac{\sum_{i=1}^{m} v_i^* x_{ij} + \gamma^*}{\sum_{r=1}^{s} u_r^* y_{rj}} = \frac{\sum_{i=1}^{m} v_i^* x_{ik} + \gamma^*}{\sum_{r=1}^{s} u_r^* y_{rk}}$$

$$\Rightarrow \sum_{r=1}^{s} u_r^* y_{rk} \sum_{i=1}^{m} v_i^* x_{ij} + \gamma^* \sum_{r=1}^{s} u_r^* y_{rk} = \sum_{r=1}^{s} u_r^* y_{rj} \sum_{i=1}^{m} v_i^* x_{ik} + \gamma^* \sum_{r=1}^{s} u_r^* y_{rj}$$

$$\overset{from\ A}{\Rightarrow} \gamma^* \left(\sum_{r=1}^{s} u_r^* y_{rj} - \sum_{r=1}^{s} u_r^* y_{rk}\right) = 0$$

If $\gamma^* = 0$, we have $g = \sum_{r=1}^{s} u_r^* y_{rj} \Big/ \sum_{i=1}^{m} v_i^* x_{ij}$. As a result of $\Delta^* = 0$, from equations B, we have $(\sum_{r=1}^{s} u_r^* y_{rj} \Big/ \sum_{i=1}^{m} v_i^* x_{ij}) = 1$. Therefore, $g = 1$ and this contradicts the definition of g that $0 < g < 1$. It means that $\gamma^* \neq 0$, and so $\sum_{r=1}^{s} u_r^* y_{rj} = \sum_{r=1}^{s} u_r^* y_{rk}$. Using this result and equations A, we have $\sum_{i=1}^{m} v_i^* x_{ij} = \sum_{i=1}^{m} v_i^* x_{ik}$. Thus,

$(\sum_{i=1}^{m} v_i^* x_{ij}, \sum_{r=1}^{s} u_r^* y_{rj}) = (\sum_{i=1}^{m} v_i^* x_{ik}, \sum_{r=1}^{s} u_r^* y_{rk})$ and the proof is completed.

Conversely, suppose that virtual DMUs be coincide. So, $(\sum_{i=1}^{m} v_i^* x_{ij}, \sum_{r=1}^{s} u_r^* y_{rj}) = (\sum_{i=1}^{m} v_i^* x_{ik}, \sum_{r=1}^{s} u_r^* y_{rk})$. It results that equations A, B and C are hold. Consequently, using the proposed ranking method, DMU $_j$ and DMU $_k$ have equal rank.

Using this theorem, two DMUs have the same ranks when the position of their corresponding virtual DMUs in R^2 is coincide. From the probabilistic point of view, the probability of this event is very poor. Thus, this method can be considered as a complete ranking method.

IV. Numerical Examples

In example 1, using the data provided in Table I, the difficulties of Liu and Peng's method [18] are shown. However, our method ranks units with data reported in Table I, respectively. In example 2, using the data listed in Table IV, our method is compared to some of earlier methods in the literature. Then, in empirical example 3, using our proposed method, real world banking data is evaluated. The data was previously analyzed by Jahanshahloo *et al.* [20] and is listed in Table VII.

A. Example 1

In order to survey the performance Liu and Peng's method [18], the following example is provided. Ten DMUs with three inputs and two outputs are presented in Table I. The model (3) with data in Table I has alternative optimal solutions. Two optimal solutions of this model are considered as common set of weights that are presented in the first row of Table II. The proposed system by Liu and Peng [18] for ranking is applied with these two common set of weights. The results of applying this method are presented in Table II.

Table I Data for example 1

DMU	Input1	Input2	Input3	Output 1	Output 2
1	2	6	3	1	3
2	1	2	5	1	1.25
3	2	3	4	1	2
4	3	1	4	1	1
5	2	5	4	1	0.75
6	2	4	3	1	1.5
7	2	6	5	1	0.5
8	2.5	4	4	1	1.75
9	2	4	5	1	0.25
10	1.5	6	3	1	1

According to Table II, for weights $(v_1, v_2, v_3, u_1, u_2) = (0.0125, 0.01, 0.015, 0.095, 0.01)$, we have $D_2 \approx D_6 \succ D_3 \succ D_1 \succ D_4 \succ D_8 \succ D_{10} \succ D_5 \succ D_9 \succ D_7$. We cannot rank D_2 and D_6 using Liu and Pong's method [18]. As shown in Table II, D_2 and D_6 have efficiency one and so distance to benchmark is zero for these DMUs. Also, shadow prices for these DMUs are equal to one. We see that all criteria to compare D_2 and D_6 are the same. Thus, Liu and Peng's method [18] fails to rank D_2 and D_6. This is one of the serious weaknesses of this method.

For weights $(v_1, v_2, v_3, u_1, u_2) = (0.01, 0.01, 0.0125, 0.08, 0.01)$, we have

$D_2 \approx D_3 \approx D_4 \succ D_6 \succ D_1 \succ D_8 \succ D_{10} \succ D_5 \succ D_9 \succ D_7$. Comparing these two ranking, we see that DMUs utilizing different common set of weights have different ranks. This is another imperfection of this method.

Table II Results of ranking with Liu and Peng's method

(v^*, u^*)	(0.0125,0.01,0.015,0.095,0.01)				(0.01,0.01,0.0125,0.08,0.01)			
DMU	θ^*	Δ^*	π^*	rank	θ^*	Δ^*	π^*	Rank
D_1	0.96	-	-	4	0.93	-	-	3
D_2	1	0	1	1	1	0	0	1
D_3	1	0	0	2	1	0	0	1
D_4	0.97	-	-	3	1	0	0	1
D_5	0.75	-	-	7	0.72	-	-	6
D_6	1	0	1	1	0.97	-	-	2
D_7	0.62	-	-	9	0.59	-	-	8
D_8	0.85	-	-	5	0.84	-	-	4
D_9	0.69	-	-	8	0.67	-	-	7
D_{10}	0.84	-	-	6	0.80	-	-	5

We can rank DMUs using the position of their virtual DMUs in R^2. The greater the gradient of the line passes through the origin and virtual DMU, the better the DMU. Figs. 4 and 5 show the position of the virtual DMUs in R^2 using different weights. From these Figures, it is obvious that the position of the virtual DMU_j $(j = 1,2,...,10)$ is changed according to weights. This difference in position of the virtual DMU_j $(j = 1,2,...,10)$ is reflected on rank DMU_j $(j = 1,2,...,10)$. For example, D_7 has the last rank, in Fig. 4, using the position of its virtual DMU. On the other hand, using another set of weights, the last rank is devoted to D_9 which is shown in Fig. 5.

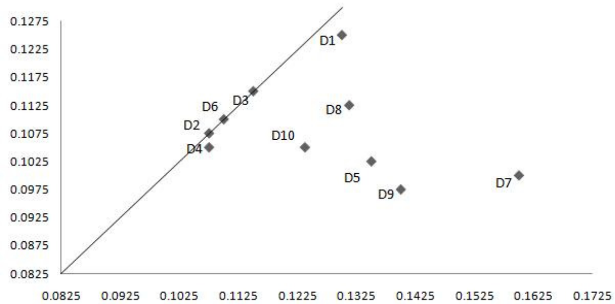

Fig. 4 Positions of DMUs using
$(v_1, v_2, v_3, u_1, u_2) = (0.0125, 0.01, 0.015, 0.095, 0.01)$

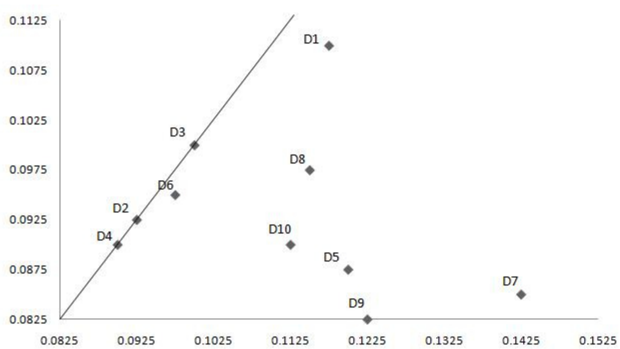

Fig. 5 Positions of DMUs using
$(v_1, v_2, v_3, u_1, u_2) = (0.01, 0.01, 0.0125, 0.08, 0.01)$

As mentioned in Liu and Peng's method [18], because the model (3) has alternative optimal solutions, the optimal solution of the model (4) is considered as common weights. For this example, the common set of weights is $(v_1, v_2, v_3, u_1, u_2) = (0.01, 0.01, 0.0125, 0.08, 0.01)$. As shown in Table II, with this common weights D_2 and D_3 and D_4 have equal ranks. Although, common set of weights is unique, we cannot have full ranking for the DMUs. So, this ranking system using common weights is failed.

In continue, our proposed ranking system with common set of weights is applied for data in Table I. Numerical Results of this method are presented in Table III. The unique optimal solution of the problem (12) for data in Table I is $(v_1, v_2, v_3, u_1, u_2) = (0.017, 0.004, 0.012, 0.085, 0.001)$ that is considered as common weights. Using our proposed ranking system, we have a full ranking for DMUs as $D_6 \succ D_2 \succ D_{10} \succ D_1 \succ D_3 \succ D_5 \succ D_4 \succ D_8 \succ D_9 \succ D_7$.

Table III Results of ranking with our proposed method

DMU	θ^*	Δ^*	γ^*	Rank
D_1	0.93	-	-	4
D_2	1	0	0.03724	2
D_3	0.91	-	-	5
D_4	0.82	-	-	7
D_5	0.83	-	-	6
D_6	1	0	0.03737	1
D_7	0.71	-	-	10
D_8	0.80	-	-	8
D_9	0.76	-	-	9
D_{10}	1	0	0.03715	3

As a result, DMUs can be ranked by the position of their virtual DMUs in R^2. This is an intuitional aspect of the Liu and Peng's method [18], but there is no this aspect for units that their virtual DMUs are on a line. For units that the corresponding virtual DMUs are on a line, the more distance virtual DMU from the origin, the better DMU. As shown in Fig. 6, all DMUs except D_2, D_6 and D_{10} can be ranked according to the gradients of their corresponding lines. D_2, D_6 and D_{10} can ranked using the distances of their corresponding virtual DMUs from the origin and so we have $D_6 \succ D_2 \succ D_{10}$.

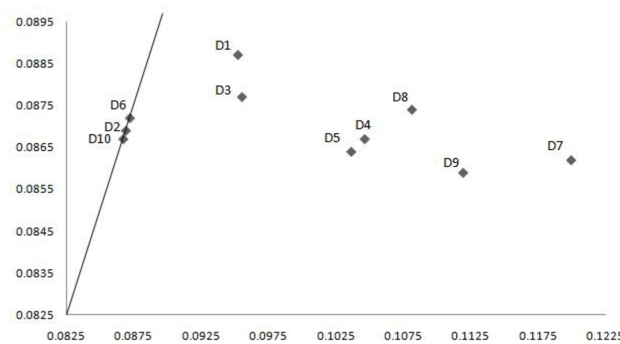

Fig. 6 Positios of DMUs using
$(v_1, v_2, v_3, u_1, u_2) = (0.017, 0.004, 0.012, 0.085, 0.001)$

B. Example 2

In Table IV, there are six DMUs A, B, C, D, E, and F with two inputs and two outputs. Using the CCR method, the efficiencies of DMUs are obtained that are shown in the second column of Table V. Using CCR efficiency, units A, B, C, and D are efficient. Using Maximin efficiency ratio and MOLP Minimax and MOLP Minsum methods, the performance of these DMUs are measured that are presented in the last three columns of Table V, respectively. In MOLP Minimax and MOLP Minsum methods, the efficiency scores of units A and D are equal to one, and so there is not distinction between performances units A and D. Other units can be ranked according to their efficiency scores. Using

MOLP Minmax method, we have $DMU\ A \approx DMU\ D \succ DMU\ E \succ DMU\ B \succ DMU\ C \succ DMU\ F$. Using MOLP Minsum method, we have $DMU\ A \approx DMU\ D \succ DMU\ E \succ DMU\ F \succ DMU\ B \succ DMU\ C$. According to the efficiency scores of DMUs using Minimax efficiency ratio that are presented in the third column of Table V, we have $DMU\ D \succ DMU\ C \succ DMU\ E \succ DMU\ A \succ DMU\ B \succ DMU\ F$, and so unit D has the best performance. Using efficiency scores obtained by our proposed method which are provided in the second column of Table VI, units A, B, and D are efficient. These units can be ranked by their distance to bad benchmark that is presented in the third column of Table VI. The rest of the units can be ranked according to their efficiency scores. So by applying the proposed method in this paper, we have $DMU\ D \succ DMU\ B \succ DMU\ A \succ DMU\ E \succ DMU\ F \succ DMU\ C$. Similar to other methods mentioned above, unit D has the best rank.

Table IV DMUs data

DMU	Input 1	Input 2	output 1	Output 2
A	150	0.0200	14000	3500
B	400	0.7000	14000	21000
C	320	1.2000	42000	10500
D	520	2.0000	28000	42000
E	350	1.2000	19000	25000
F	320	0.7000	14000	15000

Table V Ranking DMUs in Table IV using several methods

DMU	CCR	Maximin efficiency ratio	MOLP- minimax	MOLP-min sum
A	1.0000	0.7110	1.000	1.000
B	1.0000	0.6500	0.953	0.864
C	1.0000	0.9990	0.883	0.830
D	1.0000	1.0000	1.000	1.000
E	0.9780	0.9210	0.974	0.977
F	0.8680	0.648	0.864	0.867

Table VI Our results from Table V

DMU	Efficiency score	Distance from bad benchmark
A	1.000	0.010
B	1.000	0.031
C	0.829	-
D	1.000	0.061
E	0.978	-
F	0.864	-

C. Example 3

In order to have a better understanding of robustness proposed method, an empirical study about twenty branches of banks in Iran is presented in this section. Six factors are considered for evaluation branches, three inputs and three outputs. Staff, computer terminals, and area of branch are considered as inputs. Deposits, loans, and charge are considered as outputs. The data of these twenty branches are presented in Table VII. The last column of Table VII reports the CCR efficiency of branches. We only consider CCR efficient branches for ranking by proposed method. The results of applying our proposed method are presented in Table VIII.

Table VII DMUs' data (Jahanshahloo *et al.* [20])

Branch	Input 1 Staff	Input 2 Computer terminals	Input 3 Area(m2)	Output 1 deposits	Output 2 Loans	Output 3 charge	CCR efficiency
1	0.950	0.700	0.155	0.190	0.521	0.293	1.000
2	0.796	0.600	1.000	0.227	0.627	0.462	0.833
3	0.798	0.750	0.513	0.228	0.970	0.261	0.991
4	0.865	0.550	0.210	0.193	0.632	1.000	1.000
5	0.815	0.850	0.268	0.233	0.722	0.246	0.899
6	0.842	0.650	0.500	0.207	0.603	0.569	0.748
7	0.719	0.600	0.350	0.182	0.900	0.761	1.000
8	0.785	0.750	0.120	0.125	0.234	0.298	0.798
9	0.476	0.600	0.135	0.080	0.364	0.244	0.789
10	0.678	0.550	0.510	0.082	0.184	0.049	0.289
11	0.711	1.000	0.305	0.212	0.318	0.403	0.604
12	0.811	0.650	0.255	0.123	0.923	0.628	1.000
13	0.659	0.850	0.340	0.176	0.645	0.261	0.817
14	0.976	0.800	0.540	0.144	0.514	0.243	0.470
15	0.685	0.950	0.450	1.000	0.262	0.098	1.000
16	0.613	0.900	0.525	0.115	0.402	0.464	0.639
17	1.000	0.600	0.205	0.090	1.000	0.161	1.000
18	0.630	0.650	0.235	0.059	0.349	0.068	0.473
19	0.372	0.700	0.238	0.039	0.190	0.111	0.408
20	0.583	0.550	0.500	0.110	0.615	0.764	1.000

Using this method, the branches four, seven, twelve, fifteen, and seventeen have the same score one and so their corresponding virtual DMUs lies on good benchmark. According to our proposed method the branch fifteen with the highest virtual output has the first rank. Also, branches four, seven, twelve, and seventeen can be ranked using the value of virtual outputs and so we have $Branch\ 7 \succ Branch\ 12 \succ Branch\ 17 \succ Branch\ 4$. The branches one and twenty with scores 0.738 and 0.632 can be differentiated. The branch twenty with score 0.632 has the last rank among CCR efficient branches.

Table VIII System for ranking efficient DMUs

Efficient branch	$\sum_{i=1}^{m} v^*_i x_{ij}$	$\sum_{r=1}^{s} u^*_r y_{rj}$	$\sum_{r=1}^{s} u^*_r y_{rj} / \sum_{i=1}^{m} v^*_i x_{ij}$	Δ^*_j	γ^*_j	Ranking
1	0.126	0.930	0.738	-	-	6
4	0.119	0.119	1.000	0.000	0.069	5
7	0.142	0.142	1.000	0.000	0.082	2
12	0.134	0.134	1.000	0.000	0.077	3
15	0.189	0.189	1.000	0.000	0.109	1
17	0.128	0.128	1.000	0.000	0.077	4
20	0.158	0.100	0.632	-	-	7

V. Conclusion

This research presented a method for ranking all efficient DMUs based on the altered version the work proposed by Liu and Peng [18]. Liu and Peng's method [18] has two basic problems that we discussed in this paper. At first, there is no guarantee that the common set of weights is unique. This is a problem, basically, because there is vagueness in choosing one common set of weights. In this article, we presented a method to overcome this difficulty that can be applied for all methods that use linear programming problem to obtain common set of weights. Whereas being computationally burdensome, this method serves to obtain a unique common set of weights. Second, the proposed ranking system by Liu and Peng [18] cannot fully rank two efficient DMUs when they have equal shadow prices. Using definition a bad benchmark, this research presented a system for full ranking efficient DMUs that have no difficulty using the shadow prices. The proposed method was illustrated by several numerical examples. In future, we will propose utilizing other norms to measure gap between virtual DMUs and benchmarks for further research. Also, applying this method for ranking DMUs with imprecise data is a good direction for research.

REFERENCES

[1] W. D. Cook, L. M. Seiford, "Data envelopment analysis (DEA)-thirty years on", *European Journal of Operational Research*, vol. 192, pp. 1-17, 2009.

[2] M. Asmild, J. C. Paradi, J. T. Pastor, "DEA based models for reallocations of police personnel", *OR Spectrum*, doi: 10.1007/s0029101102436.

[3] C. T. Kuah,K. L. Wong, "Efficiency assessment of universities through data envelopment analysis", *Procedia Computer Science*, vol. 3, pp. 499-506, 2011.

[4] F. H. Sancho, M. M. Senante, R. S. Garrido, "Energy efficiency in Spanish wastewater treatment plants: A non-radial DEA approach", *Science of the total environment*, doi:10.1016/j.scitotenv201104018.

[5] A. Xun, B. Hanrui, Z. Xiaoyang, "A DEA approach to evaluate economical and social roles of NOCs", *Energy Procedia*, vol. 5, pp. 763-767, 2011.

[6] W. C. Chen, C. F. Chien, "Measuring relative performance of water fabrication operations using DEA", *Journal of Intelligent Manufacturing*, vol. 22, pp. 447-457, 2011.

[7] A. Charnes, W. W. Cooper, E. Rhodes, "Measuring the efficiency of decision making units", *European Journal of Operational Research*, vol. 2, pp. 429-444, 1978.

[8] R. D. Banker, A. Charnes, W. W. Cooper, "Some methods for estimating technical and scale inefficiencies in data envelopment analysis", *Management Science*, vol. 30, pp. 1078-1092, 1984.

[9] P. Anderson, N. C. Petersen, "A procedure for ranking efficient units in data envelopment analysis", *Management Science*, vol. 39, pp. 1261-1264, 1993.

[10] K. Tone, "A slack-based measure of super-efficiency in data envelopment analysis", *European Journal of Operational Research*, vol. 143, pp. 32-41, 2002.

[11] J. R. Doyle, R. H. Green, "Efficiency and cross-efficiency in DEA: Derivations, meanings and uses", *Journal of the Operational Research Society*, vol. 45, pp. 567-578, 1994.

[12] G. R. Jahanshahloo, A. Memariani, F. Hosseinzadeh Lotfi, H. Z. Rezai, "A note on some of DEA models and finding efficiency and complete ranking using common set of weights", *Applied Mathematics and Computation*, vol. 166, pp. 265-281, 2005.

[13] T. Entani, Y. Maeda, H. Tanaka, "Dual models of interval DEA and its extension to interval data", *European Journal of Operational Research*, vol. 136, pp. 32-45, 2002.

[14] N. Adler, L. Friedman, Z. Sinuany-Stern, "Review of ranking methods in data envelopment analysis context", *European Journal of Operational Research*, vol. 140, pp. 219-265, 2002.

[15] W. D. Cook, Y. Roll, A. Kazakov, "A DEA model for measuring the relative efficiency of highway maintenance patrols", *Information Systems and Operational Research*, vol. 28, pp. 113-124, 1990.

[16] Y. Roll, W. D. Cook, B. Golany, "Controlling factor weighs in data envelopment analysis", *IIE Transactions*, vol. 24, pp. 1-9, 1991.

[17] W. D. Cook, J. Zhu, "Within-group common weights in DEA: An analysis of power plant efficiency", *European Journal of Operational Research*, vol. 178, pp. 207-216, 2007.

[18] F. H. F. Liu, H. H. Peng, "Ranking of units on the DEA frontier with common weights", *Computers & Operations Research*, vol. 35, pp. 1624-1637, 2008.

[19] K.M. Murty, "Linear Programming", 3 rd edn, Jane Wiley & Sons, New York, 2002.

[20] G. R. Jahanshahloo, H. V. Junior, F. Hosseinzadeh Lotfi, D. Akbarian, "A new DEA ranking system based on changing the reference set", *European Journal of Operational Research*, vol. 181, pp. 331-33, 2007.

[21] N. E. Mastrokis, "Singular value decomposition in multidimensional arrays", *International Journal of System Science*, vol. 27, no. 7, pp. 647-650, 1996.

[22] N. E. Mastrokis,"The Singular Value Decomposition (SVD) in Tensors (Multidimensional Arrays) as an Optimization Problem. Solution via Genetic Algorithms and method of Nelder-Mead", in *Proc. 6th WSEAS Int. Conf. Systems Theory & Scientific Computation*, Elounda, Greece, 2006, pp 7-13.

[23] N. E. Mastrokis, "Genetic algorithms with Nelder-Mead optimization in the variational methods of boundary value problems", *WSEAS Transactions on Mathematics*, vol. 8, no. 3, 2009.
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