Application of fractional order calculus to control theory

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***Abstract*—**The principal goal of this paper is to introduce the fundamentals of the Fractional Order Calculus (FOC), outline its possible application to the field of analysis and synthesis of control systems, and present several existing Matlab toolboxes related to FOC. The basic theoretical concepts of FOC are followed by methodologies for potential fractional order systems description and their stability investigation. Furthermore, the paper offers brief overview of the fractional order controllers which can be found in the scientific literature and highlights the benefits of the fractional approach in comparison with the classical integer one. On top of that, the Matlab toolboxes, useful for the practical design and analyses connected with fractional order control, are also discussed in the paper.

***Keywords*—**Fractional order calculus, differintegration, fractional order controllers, control theory, control systems, Matlab toolboxes.

I. Introduction

THE Fractional Order Calculus (FOC) constitutes the branch of mathematics dealing with differentiation and integration under an arbitrary order of the operation, i.e. the order can be any real or even complex number, not only the integer one [1], [2], [3]. Although the FOC represents more than 300-year-old issue [4], [5], [6], its great consequences in contemporary theoretical research and real world applications have been widely discussed relatively recently. The idea of non-integer derivative was mentioned for the first time probably in a letter from Leibniz to L'Hospital in 1695. Later on, the pioneering works related to FOC have elaborated by personalities such as Euler, Fourier, Abel, Liouville or Riemann. The interested reader can find the more detailed historical background of the FOC e.g. in [1].

According to [4], [7], the reason why FOC remained practically unexplored for engineering applications and why only pure mathematics was "privileged" to deal with it for so long time can be seen in multiple definitions of FOC, missing simple geometrical interpretation, absence of solution methods for fractional order differential equations and seeming adequateness of the Integer Order Calculus (IOC) for majority of problems. However, nowadays the situation is going better and the FOC provides efficient tool for many issues related to fractal dimension, "infinite memory", chaotic behaviour, etc. Thus, the FOC has already came in useful in engineering areas such as bioengineering, viscoelasticity, electronics, robotics, control theory and signal processing [7]. Several control applications are available for example in [8]-[10]. The information on fractional order Proportional-Integral (-Derivative) (PI(D)) controllers can be found e.g. in [11], [12] while the works [13]-[17] deal with their classical fixed order versions.

This paper is the extended version of the contribution [18]. It is not intended to bring any novel theoretical knowledge nor application results. Its main purpose is to aggregate the FOC theory and introduce its utilization in control theory on the basis of literature from "References" Section. Moreover, the paper outlines several toolboxes for fractional order control in Matlab environment.

The work is organized as follows. In Section II, the basic theoretical concepts of FOC and various definitions of differintegral operator are introduced. The Section III then presents the possible ways of description of fractional order systems. The Section IV follows the previous one with opening the problem of stability investigation for this class of systems. Further, the brief survey on existing fractional order controllers is provided in Section V. Subsequently, Section VI overviews four Matlab toolboxes for problems of fractional order control. And finally, Section VII offers some conclusion remarks.

II. Basic Concepts of Fractional Order Calculus

The FOC is based on generalization of differentiation and integration to an arbitrary order, which can be rational, irrational or even complex. This generalization has led to the introduction of basic continuous differintegral operator [1], [2], [4], [7]:

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \operatorname{Re}\alpha > 0 \\ 1 & \operatorname{Re}\alpha = 0 \\ \displaystyle\int_a^t (d\tau)^{-\alpha} & \operatorname{Re}\alpha < 0 \end{cases} \tag{1} $$

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where α is the order of the differintegration (usually $\alpha \in \mathbb{R}$) and a is a constant connected with initial conditions.

There is an array of definitions of differintegral in the literature. The three most frequent definitions bear the names of Riemann-Liouville, Grünwald-Letnikov and Caputo. The most known and used Riemann-Liouville version has the form [4]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{2}$$

under condition $(n-1<\alpha<n)$. The term $\Gamma(\cdot)$ represents so-called Gamma function.

Alternatively, the Grünwald-Letnikov definition is given by [4], [9]:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{3}$$

where

$$ \binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} \tag{4}$$

and where h is the time increment and [.] means the integer part.

Finally, Caputo has defined the differintegral as [5], [6]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{5}$$

Each of the definitions of an interpolation of the integer order operations sequence has its advantages and drawbacks and the user choice depends mainly on the purpose and the area of application [3], [11].

The control theory widely exploits the Laplace transform for the sake of analysis and synthesis simplicity. The Laplace transform (denoted as L) of the differintegral can be written as [4], [9]:

$$ \begin{aligned} L\left\{ {}_aD_t^\alpha f(t) \right\} &= \int_0^\infty e^{-st} \, {}_0D_t^\alpha f(t) dt = \\ &= s^\alpha F(s) - \sum_{m=0}^{n-1} s^m (-1)^j \, {}_0D_t^{\alpha-m-1} f(t)\Big|_{t=0} \end{aligned} \tag{6}$$

where integer n lies within $(n-1<\alpha\le n)$.

III. Description of Fractional Order Systems

A fractional order continuous-time linear time-invariant dynamical system can be described by a fractional order differential equation [3]-[6]:

$$ \begin{aligned} & a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) = \\ & = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \cdots + b_0 D^{\beta_0} u(t) \end{aligned} \tag{7}$$

where $u(t)$ is the input signal, $y(t)$ is the output signal, $D^\gamma \equiv {}_0D_t^\gamma$ represents fractional derivative, a_k with $(k=0,\dots,n)$ and b_k with $(k=0,\dots,m)$ denote constants, and α_k with $(k=0,\dots,n)$ and β_k with $(k=0,\dots,m)$ are arbitrary real numbers. According to [4]-[6], one can assume inequalities $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0$ and $\beta_m > \beta_{m-1} > \cdots > \beta_0$ without loss of generality.

Another option for fractional order system description is in the form of incommensurate real orders transfer function [3], [5], [6]:

$$ G(s) = \frac{B(s^{\beta_k})}{A(s^{\alpha_k})} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} \tag{8}$$

The symbols in (8) have the same meaning as in (7).

It has been shown (e.g. in [5], [6], [19]) that every incommensurate order system (8) can be expressed as a commensurate one by means of a multivalued transfer function. The domain of such multivalued functions can be seen as a Riemann surface with Riemann sheets. Graphical interpretations of several functions via Riemann surface are visualized in [5], [6].

If the transfer function (8) is supposed for $\alpha_k = \alpha k$, $\beta_k = \beta k$, $0<\alpha<1$, $k \in \mathbb{Z}$, it represents the specialized case of commensurate order system. Then the corresponding transfer function is:

$$ G(s) = K \frac{B(s^\alpha)}{A(s^\alpha)} = K \frac{\sum_{k=0}^{m} b_k (s^\alpha)^k}{\sum_{k=0}^{n} a_k (s^\alpha)^k} \tag{9}$$

The analytical solution of fractional order differential equations in time domain (e.g. for the purpose of step or impulse responses computation) can be expressed for example with the assistance of functions of Mittag-Leffler type [3], [5], [6].

IV. Stability of Fractional Order Systems

Obviously, the closed-loop stability represents the very fundamental and critical requirement during control system design. It is widely known that an integer order continuous-time linear time-invariant system is stable if and only if all roots of its characteristic polynomial have negative real parts. In other words, the roots must lie in the left half of the complex plane. Investigation of stability of the fractional order systems represents the more complicated issue [5], [6], [20].

For example, the stability of commensurate fractional order systems can by analyzed via the theorem of Matington [20] or the definition from [5], [6], which describes the way of mapping the poles from s^{α}-plane into the *w*-plane. An interesting result is that the poles of the stable fractional order system can be located even in the right half of such complex plane. This fact is illustrated e.g. in Fig. 1 where the stability region for a commensurate fractional order linear time-invariant system with order $0<\alpha<1$ is depicted [4]-[6].

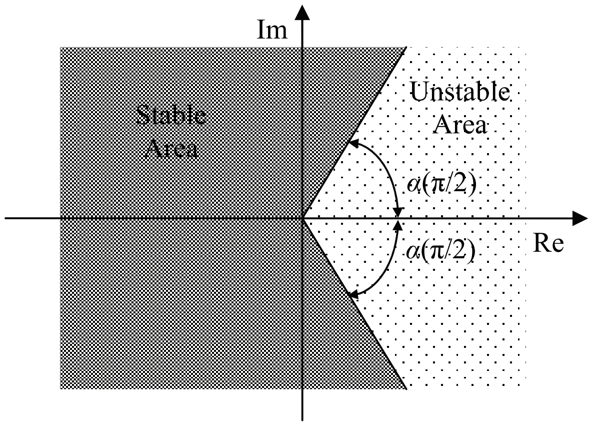

Fig. 1 region of stability for the commensurate fractional order system with $0<\alpha<1$ [4]-[6]

The situation for $\alpha=1$ is then visualized in Fig. 2. In this case, the region of stability corresponds to the classical *s*-plane [5]-[6].

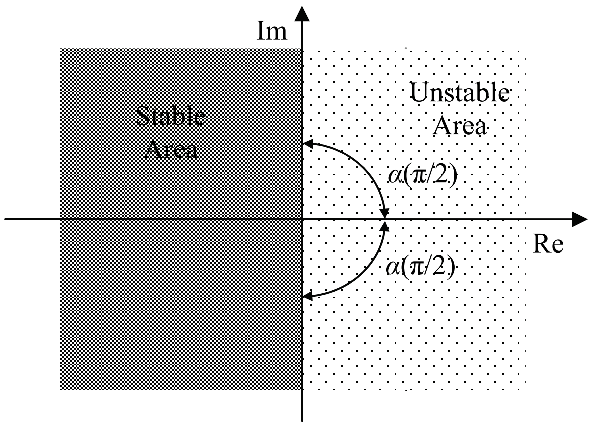

Fig. 2 region of stability for the commensurate fractional order system with $\alpha=1$ [5]-[6]

Finally, Fig. 3 shows the region of stability under assumption of $1<\alpha<2$ [5]-[6].

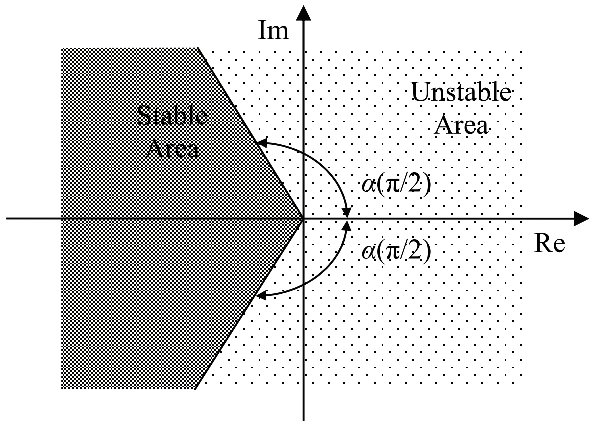

Fig. 3 region of stability for the commensurate fractional order system with $1<\alpha<2$ [5]-[6]

V. Fractional Order Controllers

The nice survey on fractional order control is given e.g. in [21]. This paper has distinguished among four typical combinations of integer/fractional order controlled system vs. integer/fractional order of controller and shown that the fractional algorithms have better results from many points of view.

Usually, the four basic approaches to fractional order control, i.e. four different fractional order controllers are reviewed in the literature [4], [7], [22]. Their overview can be found in the following subsections.

A. Tilted Proportional and Integral (TID) Controller

First, the TID controller has the same structure as classical PID controller, but the proportional gain is replaced with a function $s^{-\alpha}$ with $\alpha \in \mathbb{R}$, which allows wider tuning options and better control behaviour in comparison with the integer order PID controller [23].

B. CRONE

Next popular controller is CRONE [24]-[30]. The abbreviation CRONE stands for French "Commande Robuste d'Ordre Non Entier" (non-integer order robust control) and represents approach inspired by the fractal robustness. Presently, it has been developed three generations of the CRONE methodology [27]:

- The first generation CRONE control – "control of plant with an uncertain magnitude and constant phase with respect to frequency around the desired open loop gain crossover frequency" [28] (real fractional order for controller definition) [27]
- The second generation CRONE control – "control of plant with an uncertain magnitude around the desired open loop gain crossover frequency" [28] (real fractional order for open-loop definition) [27]

- The third generation CRONE control – the most general CRONE methodology (complex fractional order(s) for open-loop definition) [27]

The CRONE controllers have been already applied to many real plants. Besides, the approach has its own Matlab toolbox [28]-[30] which will be briefly introduced in the Section VI-A.

C. *Fractional Order PID Controllers*

The elegant and efficient fractional order modification of conventional PID controllers has been introduced in [11]. They are known as $PI^{\lambda}D^{\mu}$ controllers and can be described by transfer function:

$$C(s) = K_P + K_I s^{-\lambda} + K_D s^{\mu} \tag{10}$$

where λ and μ are positive real numbers, and K_P, K_I and K_D denote the proportional, integral and derivative constant, respectively. This embellishment of PID algorithm offers much wider selection of tuning parameters as can be seen in Fig. 4 [31].

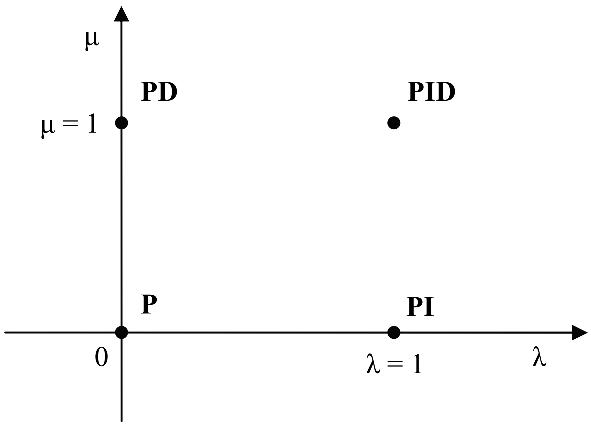

Fig. 4 plane of $PI^{\lambda}D^{\mu}$ controllers [31]

Obviously, this variety of controller parameters can consequently improve the control performance. However, there is a relative lack of rigorous tuning techniques for this type of controllers so far.

D. *Fractional Lead-Lag Controller*

Finally, the paper [32] has introduced the extension of classical lead-lag controllers to its fractional version. Furthermore, self-tuning approach for fractional lead-lag compensators can be found in [33].

VI. Matlab Toolboxes for Fractional Order Control

The calculations and simulations of problems related to fractional order control can be advantageously performed by means of some toolbox in Matlab and Simulink environment. Several of them are going to be introduced under the scope of this Section.

A. *CRONE Toolbox*

The popular CRONE Matlab Toolbox is dedicated to fractional order calculus and applies the original theoretical and mathematical concepts developed by the CRONE research group [24]-[30]. The CRONE toolbox has been developed progressively since the nineties of the last century [29].

Presently, there are two versions of the toolbox available – a classical and an object oriented one. Both of them can be downloaded (after registration) from the Internet [28]. The classical version is embellished with a Graphical User Interface (GUI) and contains three main modules [28]:

- Mathematical module – implementation of fractional calculus algorithm
- System identification module – identification of fractional order models in frequency and time domain
- CRONE control module – implementation of fractional order robust control design

The object oriented version contains various scripts and allow overloading some basic mathematical operators and standard Matlab routines for the fractional order cases. It assumes user who is familiar with the basics of work with Matlab [28].

Just for illustration, several windows briefly outlining the way of work with the classical CRONE toolbox are going to be shown. The main window "CRONE Toolbox – Crone Control-System Design Guided Start" which can be launched by "crone_control" command is depicted in Fig. 5.

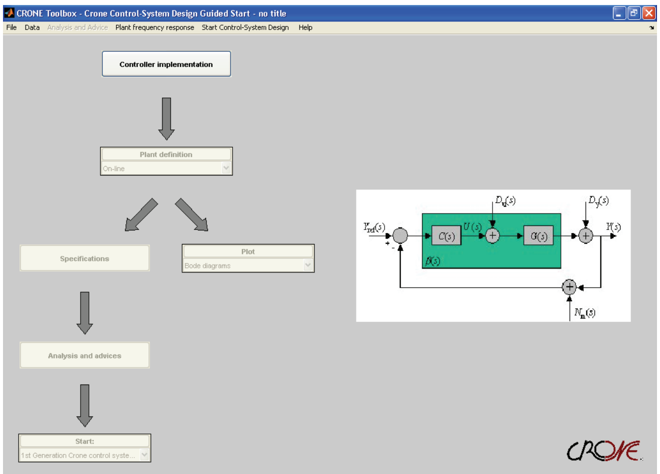

Fig. 5 main window of "CRONE Toolbox – Crone Control-System Design Guided Start" [28]

The bottom pop-up menu allows selection of the generation of the CRONE control design methodology. The Fig. 6 presents the window related to the first generation.

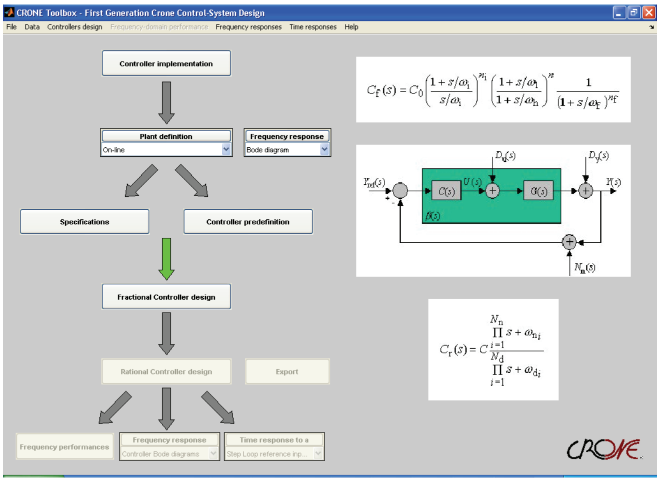

Fig. 6 window for "CRONE Toolbox – First Generation Crone Control-System Design" [28]

The window for second generation would look very similarly and thus the Fig. 7 shows the main setup window for the third generation case.

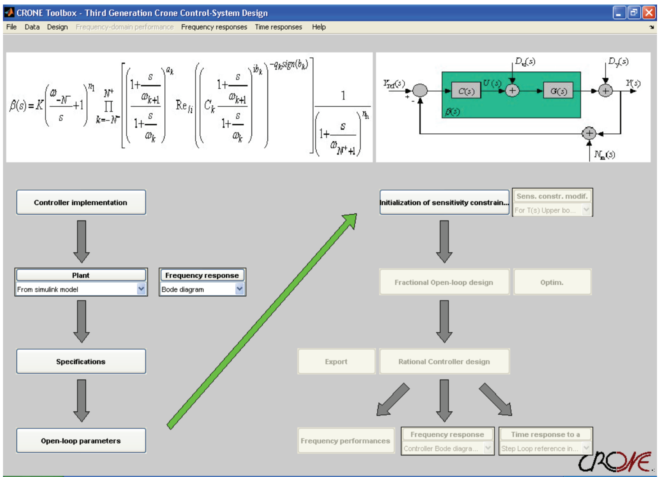

Fig. 7 window for "CRONE Toolbox – Third Generation Crone Control-System Design" [28]

Certainly, this paper does not intend to cover all functionalities of the toolbox. It would not be possible, because the CRONE toolbox is deeply elaborated product which includes many functions and tools related to fractional order control problems. The interested readers and potential users can find the further information, both versions of the CRONE toolbox and rich documentation for example in the web [28].

B. *Toolbox "ninteger" for Matlab*

The main purpose of the "ninteger" toolbox for Matlab [34]-[35] is to assist with the design of fractional order controllers and with their performance assessment. The toolbox was developed by Duarte Pedro Mata de Oliveira Valério and can be downloaded from [34]. It employs over thirty formulas for approximating the non-integer order derivatives. Moreover, the toolbox implements non-integer PID controllers and second and third-generation CRONE controllers. On the top of that, it contains functions for computing norms, model identification and frequency diagrams. Finally, the toolbox is enriched with a GUI and has a Simulink library.

More specifically, the "ninteger" toolbox for Matlab (in version 2.3) consists of the files with following purposes [34]:

Approximations of fractional order controllers:
- nid.m – approximation of a fractional derivative
- nipid.m – approximation of a fractional PID
- crone1.m – first-generation CRONE approximation
- newton.m – generalization of Carlson method
- matsudaCFE.m – Matsuda method

Functions for controllers assuring a constant phase open loop:
- crone2.m – second-generation CRONE approximation (in the frequency domain)
- crone2z.m – second-generation CRONE approximation (in the discrete-time domain)

Functions for controllers assuring a constant slope phase open loop:
- crone3.m – computation of parameters for a third-generation CRONE

Identification of fractional models from frequency response data:
- hartley.m – identification of a model with unit numerator or unit denominator using the method of Hartley-Lorenzo
- levy.m – identification of a model using a variation of Levy's method
- vinagre.m – improvement of the Levy's method with better low-frequency fit
- sanko.m – iterative improvement of the Levy's method
- lawro.m – incorporation of additional data into a pre-existent model

Functions for analysis of fractional systems:
- freqrespFr.m – frequency response of a fractional system
- bodeFr.m – Bode plots of a fractional system
- nyquistFr.m – Nyquist plot of a fractional system
- nicholsFr.m – Nichols plot of a fractional system
- sigmaFr.m – singular values plot of a fractional system
- normh2Fr.m – H_2 norm of a fractional system
- normhinfFr.m – H_∞ norm of a fractional system

Functions for continued fractions:
- contfrac.m – computation of a continued fraction expansion of a real number

- contfracf.m – computation of a continued fraction expansion of a rational function
- contfraceval.m – evaluation of a continued fraction expansion of a real number
- contfracfeval.m – evaluation of a continued fraction expansion of a rational function

Furthermore, the "ninteger" toolbox contains also the Simulink library (namely nintblocks.mdl) and several files related to GUI (the main GUI files are ninteger.m and ninteger.fig).

The main window of the toolbox GUI (only with pre-defined values and parameters), which appears after start of the program, is shown in Fig. 8.

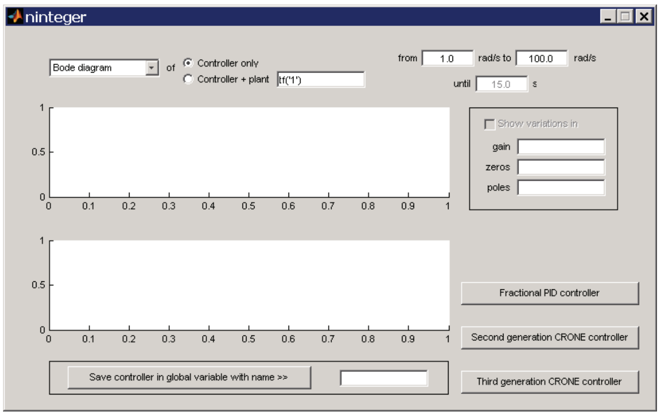

Fig. 8 empty main GUI of "ninteger" toolbox [34]

The three buttons in the right down corner allow choosing a type of controller to devise. The first possibility, hidden behind "Fractional PID controller" button, opens the window depicted in Fig. 9, where an array of additional options can be selected.

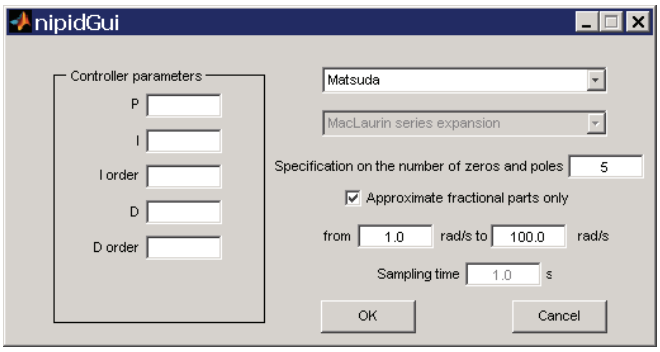

Fig. 9 window "Fractional PID controller" [34]

Next, the dialogue window which appears after pressing the "Second generation CRONE controller" is provided in Fig. 10.

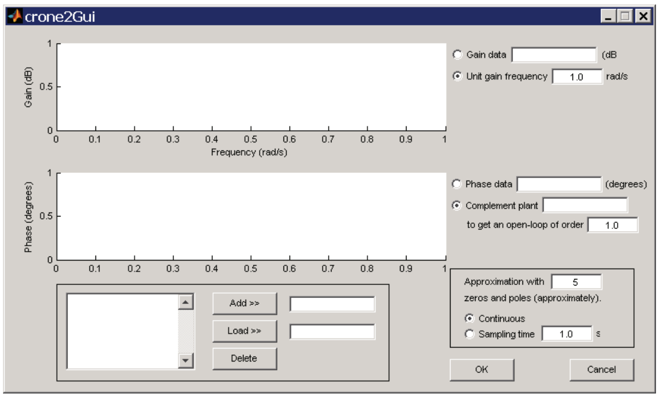

Fig. 10 window "Second generation CRONE controller" [34]

And finally, the Fig. 11 shows the window connected with "Third generation CRONE controller".

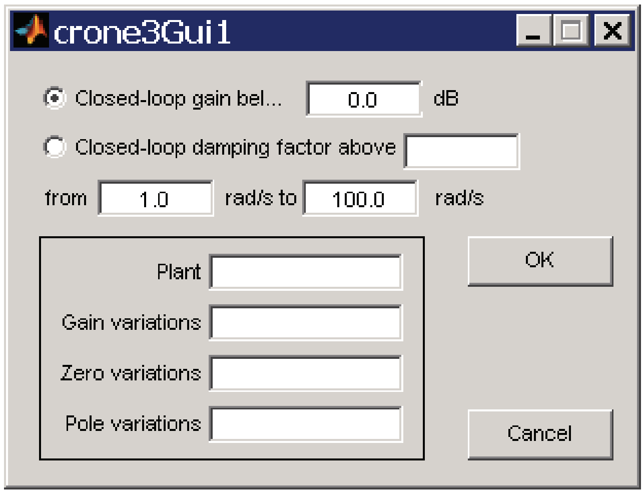

Fig. 11 window "Third generation CRONE controller" [34]

As in the previous case, the main objective of the part has not been to provide the comprehensive description of the toolbox facilities, but just the basic overview. The toolbox requires Matlab in version 7.0 or above, Control Toolbox, Optimisation Toolbox and at least functions rad2deg.m and deg2rad.m from the Map Toolbox. Almost 100-page user manual is available at [34].

C. *FOTF Matlab Toolbox*

The Fractional Order Transfer Function (FOTF) Matlab Toolbox was presented in the tutorial paper [4]. It utilizes the numerical computation of fractional order operators (3). The FOTFs are supposed in the form (8). The toolbox consists of the functions located in the @fotf folder (in order to define the FOTF class). Some files overload the original ones. The names of the files and their purposes are as follows:

- foft.m – definition of the FOTF-class
- display.m – display of the class

- mtimes.m – multiplication of two FOTF objects (in serial connection)
- plus.m – addition of two FOTF objects (in parallel connection)
- unique.m – simplification of FOTF model (using collection of polynomial terms)
- feedback.m – calculation of feedback connection transfer function
- minus.m – subtraction of two FOTF objects
- uminus.m – negation of FOTF object
- inv.m – inversion of FOTF object
- isstable.m – stability test
- lsim.m – simulation of time response of FOTF to arbitrary input signal
- step.m – step response of FOTF
- bode.m – Bode plot of FOTF
- nyquist.m – Nyquist plot of FOTF
- nichols.m – Nichols plot of FOTF

The interested reader can find the full codes of the mentioned functions as well as an array of illustrative examples of their application in [4].

D. Matlab Scripts by Ivo Petráš

Several useful scripts related to fractional order control can be found also at the webpage [36]. They cover for example frequency characteristics, step and impulse responses, or digital fractional order differentiators/integrators.

VII. Conclusion

The paper has been focused on introduction to FOC with emphasis to potential application to engineering, especially analysis and synthesis of control systems. It has offered the basic theoretical aspects of FOC, dealt with description and stability of fractional order systems, overviewed the possible fractional order control approaches, and briefly presented the basic facilities and GUIs of several fractional order control toolboxes under Matlab + Simulink environment.

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