Characteristic Impedances Calculations in Arteries with Atherosclerosis Using MAPLE

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***Abstract*—**Cardiovascular diseases cause deaths every year. For that reason it is important to model these diseases and the troubles that they can cause into the human body, particularly the arteries, in the cardiovascular field. As an effort for achieving the understanding of the phenomena, an electric analogue representation of the arteries and blood flow has been made, where the key part is the characteristic impedance. We present the calculations made for obtaining the characteristic impedances in different cases. The Navier-Stokes in cylindrical coordinates is used with the boundary condition, representing Newtonian and non-Newtonian fluids, with a special interest in Atherosclerosis disease. Laplace transform is used as a classical method for solving the differential equation. The power of the computer algebra system is shown through this work.

***Keywords*—**Arteries, Atherosclerosis, Blood flow, Characteristic impedance, Computer algebra, Non-newtonian fluids.

I. Introduction

THE cardiovascular system has as primary function nutrients and waste transport through all body, which is constituted by the veins, the arteries, the blood vessels and the heart that pump the blood for a network of branching pipes, doing the distribution of oxygen and the collection of carbon dioxide, between others functions.

Blood flow is an unsteady phenomenon (use of differential equation are required), where a normal arterial flow could be considered as laminar with secondary flows in the curves and branches. But in some cases, this flow could turn turbulent, because in the cardiovascular system, the Reynolds number varies from 1 in the small arterioles to 4000 in the largest artery [1]. In the same way, the arteries varies depending of the flow and pressure, these variations create abnormal conditions. This abnormal condition could be produced by problems due to cholesterol, generating the Atherosclerosis disease [2]. Both conditions (normal and abnormal) are preciously studied for helping to reduce the deaths mainly in developed countries where the majority of deaths are the result of cardiovascular diseases [3].

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Similarly, the pressure and flow are pulsated, because of the heart cyclic function, systolic and diastolic periods (in some special occasions the flow goes in the contrary direction of the pressure) [4]-[5]. As the viscosity is not constant throughout the artery, it's necessary to consider the blood (a fluid composed by cells, proteins, lipoproteins) as a non-Newtonian fluid, and non-Newtonian viscosity which is widely studied in the bio-rheology field. This kind of studies will be useful for the prediction of particulars flow in each patient, and the designs of electronic devices that imitate of alter the blood flow [6].

For carrying out the studies in hemodynamic phenomena [7]-[8]-[9], it's easier to bring an electrical model for the fluid system, in which the pressure is analogue of voltage and the flow is analogue of current, consequently, it is required an equivalent impedance for this model [10]. In this work, we present the calculations based in the Navier-Stokes equation for finding the characteristic impedances from different possible cases in arterial phenomena as normal artery comportment or altered artery comportment which is linked with the atherosclerosis disease.

II. Problem

Hemodynamic fluid systems could be described by the Navier-Stokes equation, which is presented in cylindrical coordinates as a facility for the treatment:

$$\rho \frac{\partial}{\partial t} v(r,t) = -\frac{\partial}{\partial z} P(z,t) + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} v(r,t) - \frac{\partial^2}{\partial r^2} v(r,t) \right) \quad (1)$$

where ρ is the density of the blood, μ is the viscosity, both constants. And $v(r,t)$ and $\partial P(z,t)/\partial z$ are the functions of speed and pressure gradient respectively.

The domain of application is an arterial cylindrical section with radius R and length l. It's necessary to have the initial condition and boundary conditions for obtaining the complete solution of a differential equation. As initial condition we will have and null condition for all cases treated in this study,

$$v(r,0) = 0 \quad (2)$$

We have two boundary conditions, which represent a Newtonian or non-Newtonian Fluid, these conditions are presented in the below sections. Moreover, in some cases we should use mathematical tools like the singularity analyze for getting the complete solution.

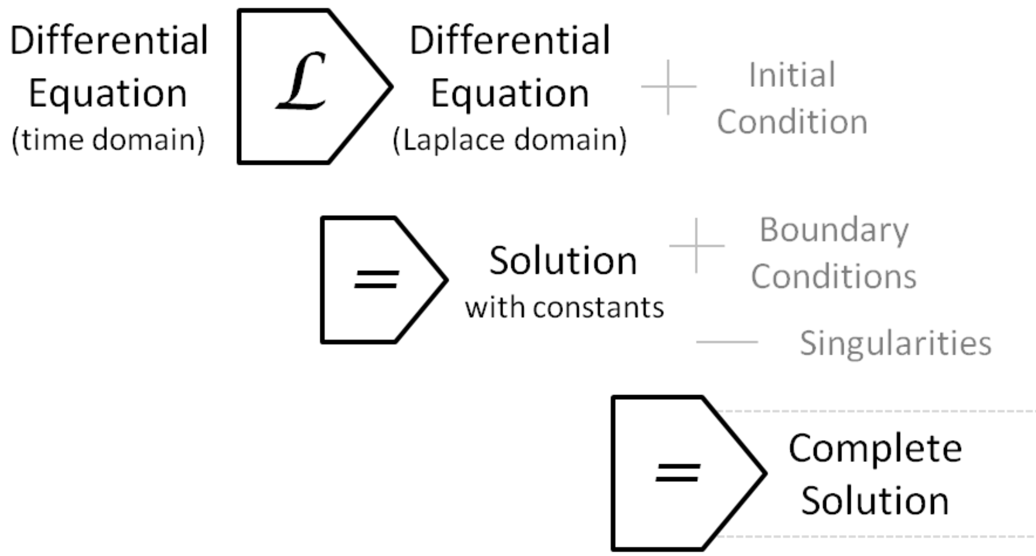

Fig. 1 Method applied for solving the differential Navier-Stokes equations, using the Laplace transform.

A. Normal Newtonian Artery

For an normal artery, where the inertial forces are more important that the viscous forces, we can treat the fluid dynamics as a Newtonian fluid, in the model this is represented by the boundary condition of displacement in the artery border is zero

$$v(R,t)=0 \tag{3}$$

B. Normal Non-Newtonian Artery

In the opposite direction from the previous section, in the case of the viscous forces cannot be neglected, we have to consider the fluid as non-Newtonian. For this case the boundary condition depends on the speed and acceleration of the fluid.

$$\left[\beta v(r,t)+\mu\frac{\partial}{\partial r}v(r,t)\right]_{r=R}=0 \tag{4}$$

Where β is the friction factor between the blood and the artery wall.

C. Altered Artery with Linear Viscosity Shape

For an artery with some kind of disease that is related with the viscosity of the blood, as example the atherosclerosis in which the cholesterol in the blood changes his viscosity in a not constant way, but we consider that cholesterol distribution does not changes the fluid density. Because of the non-constant viscosity, the Navier-Stokes is modified, and we have

$$\begin{aligned}\rho\frac{\partial}{\partial t}v(r,t)= & -\frac{\partial}{\partial z}P(z,t)+\frac{\mu(r)}{r}\frac{\partial}{\partial r}v(r,t)\\ & +\frac{\partial}{\partial r}\mu(r)\frac{\partial}{\partial r}v(r,t)\\ & +\mu(r)\frac{\partial^2}{\partial r^2}v(r,t)\end{aligned} \tag{5}$$

as a first case we are going to consider the viscosity variation as linear, with a viscosity in the centre of the artery ν that increases outwards.

$$\mu(r)=\mu r+\nu \tag{6}$$

We solve this new equation using boundary considerations, Newtonian (3) and non-Newtonian (4) condition. Other kinds of expressions for the atherosclerosis disease are presented by Wang [11].

D. Altered Artery with Quadratic Viscosity Shape

As an extension of the previous situation we extend the viscosity function into a quadratic expression and keep the Navier-Stokes (5). Equally we find the impedance for the Newtonian condition (3) as much as the non-Newtonian condition (4). The viscosity function turns to:

$$\mu(r)=\mu r^2+\nu \tag{7}$$

This situation could represent a worse state of the disease than previous one because the viscosity increases too much at the artery wall and decrease the blood flow.

E. Altered Artery with Two Fluids

Other way to analyze the cardiovascular disease as the atherosclerosis is to suppose that there are two fluids inside the artery, one in the inner part and the other one in the outer, which have different characteristics as density and viscosity, but constant for each one. In this case, we have to solve the Navier-Stokes equation (1) for each fluid (this fluids could be blood and fat or lipid) and use a condition in the interface R_i between them:

$$v_1(R_i,t)=v_2(R_i,t) \tag{8}$$

As previous problems, we make the calculations for the Newtonian (3) and non-Newtonian (4) boundary conditions.

III. Method

The process for solving the differential equation involves the Laplace transform, which take the equation from the time domain to the frequency domain. If we think in the input of our system, the heart, it has a periodic signal, so, the Laplace domain makes the analysis much easier. This method is shown in Fig.1. Solving the differential equation with the help of the initial and boundaries conditions, we obtain the speed function of the blood into the artery, for getting the characteristic impedance, first we proceed to integrated the speed into the artery cross section, obtaining the flow Q and after we make the division between a pressure difference and the flow, which is analogue to the electric scheme.

$$Z=\frac{\Delta P}{Q} \quad \text{,that is analogue to} \quad Z=\frac{\Delta V}{I} \tag{9}$$

IV. Results

We present the characteristic impedances from the previously cases, respecting the same order and having the corresponding numeration.

A. Normal Newtonian Artery

In this first case we present all step as example of the method, searching the characteristic impedance. We start taking Laplace transform to the Navier-Stokes equation (1):

$$\begin{aligned}\rho sV(r)-\rho v(r,0)= & -\frac{\partial}{\partial z}P(z)\\ & +\mu\left(\frac{1}{r}\frac{d}{dr}V(r)+\frac{d^2}{dr^2}V(r)\right)\end{aligned} \tag{10}$$

and we apply the initial condition (2)

$$\begin{aligned}\rho sV(r)= & -\frac{\partial}{\partial z}P(z)\\ & +\mu\left(\frac{1}{r}\frac{d}{dr}V(r)+\frac{d^2}{dr^2}V(r)\right)\end{aligned} \tag{11}$$

now, we solve the system:

$$V(r)=\mathcal{J}_0\left(\sqrt{-\frac{\rho s}{\mu}}r\right)C_1+\mathcal{Y}_0\left(\sqrt{-\frac{\rho s}{\mu}}r\right)C_2-\frac{\frac{d}{dz}P(z)}{\rho s} \tag{12}$$

where $\mathcal{J}_\alpha(x)$ is the Bessel function of the first kind and $\mathcal{Y}_\alpha(x)$ is the Bessel function of the second kind. After that, we make a series expansion of (12) around $r=0$, looking for singularities.

$$C_1+\frac{2}{\pi}\left(\ln\left(\frac{r}{2}\sqrt{-\frac{\rho s}{\mu}}\right)+\gamma\right)C_2-\frac{\frac{d}{dz}P(z)}{\rho s}+\mathcal{O}(r^2) \tag{13}$$

here, we see that there is a logarithmic singularity at $r=0$, if the constant C_2 is different of zero, so, $C_2=0$ and

$$V(r)=\mathcal{J}_0\left(\sqrt{-\frac{\rho s}{\mu}}r\right)C_1-\frac{\frac{d}{dz}P(z)}{\rho s} \tag{14}$$

we find the last constant using the boundary condition (3) in (14)

$$C_1=\frac{\frac{d}{dz}P(z)}{\rho s\mathcal{J}_0\left(\sqrt{-\frac{\rho s}{\mu}}\right)R} \tag{15}$$

we rewrite the solution including the found constant, obtaining:

$$V(r)=\frac{\frac{d}{dz}P(z)}{\rho s}\left(\frac{\mathcal{J}_0\left(\sqrt{-\frac{\rho s}{\mu}}r\right)}{\mathcal{J}_0\left(\sqrt{-\frac{\rho s}{\mu}}R\right)}-1\right) \tag{16}$$

the term inside of the Bessel function is related to the Womersley number α which is

$$\alpha=R\sqrt{\frac{\rho\omega}{\mu}} \tag{17}$$

with ω the frequency of pumping, we rewrite

$$V(r)=\frac{\frac{d}{dz}P(z)}{\rho i\omega}\left(\frac{\mathcal{J}_0\left(\alpha y i^{3/2}\right)}{\mathcal{J}_0\left(\alpha i^{3/2}\right)}-1\right) \tag{18}$$

with $y=r/R$ this result is similar to that presented by Womersley [12].

After we have the speed we proceed to calculate the flow, integrating the speed into the artery cross section

$$Q(s)=\int_0^{2\pi}\int_0^R V(r)r\,dr\,d\theta \tag{19}$$

including (18) in (19), we solve:

$$Q(s)=\pi R^2\frac{\frac{d}{dz}P(z)}{\rho s}\left(1-\frac{2}{\sqrt{\frac{\rho s}{\mu}}R}\frac{\mathcal{I}_1\left(\sqrt{\frac{\rho s}{\mu}}R\right)}{\mathcal{I}_0\left(\sqrt{\frac{\rho s}{\mu}}R\right)}\right) \tag{20}$$

where $\mathcal{I}_{\alpha}(x)$ is the modified Bessel function of the first kind and it could be simplified using $X=\left(\sqrt{\rho s/\mu}\right)R$ (the same $X=\alpha\sqrt{i}$)

$$Q(s)=\pi R^{2}\frac{\frac{d}{dz}P(z)}{\rho s}\left(1-\frac{2}{X}\frac{\mathcal{I}_{1}(X)}{\mathcal{I}_{0}(X)}\right) \tag{21}$$

the gradient of pressure is supposed constant, along the artery distance L

$$\frac{d}{dz}P(z)=\frac{\Delta P}{L} \tag{22}$$

we obtain so:

$$Q(s)=\pi R^{2}\frac{\Delta P}{\rho sL}\left(1-\frac{2}{X}\frac{\mathcal{I}_{1}(X)}{\mathcal{I}_{0}(X)}\right) \tag{23}$$

using the relation (9) we obtain the characteristic impedance

$$Z(s)=\frac{L}{R^{2}}\frac{\rho s}{\pi}\left(\frac{X\mathcal{I}_{0}(X)}{X\mathcal{I}_{0}(X)-2\mathcal{I}_{1}(X)}\right) \tag{24}$$

using a recurrence relation between Bessel function

$$X\mathcal{I}_{0}(X)-2\mathcal{I}_{1}(X)=X\mathcal{I}_{2}(X) \tag{25}$$

it is simplified to

$$Z(s)=\frac{L}{R^{2}}\frac{\rho s}{\pi}\frac{\mathcal{I}_{0}(X)}{\mathcal{I}_{2}(X)} \tag{26}$$

B. *Normal Non-Newtonian Artery*

As the previous result, we use the method applying the non-Newtonian boundary condition (4), and we obtain:

$$Z(s)=\frac{L}{R^{2}}\frac{\rho s}{\pi}\left(\frac{\mathcal{I}_{0}(X)\beta R+\mathcal{I}_{1}(X)\mu X}{\mathcal{I}_{2}(X)\beta R+\mathcal{I}_{1}(X)\mu X}\right) \tag{27}$$

C. *Altered Artery with Linear Viscosity Shape*

Now, we solve the Navier-Stokes equation with viscosity variable (5) with the viscosity function (6), and we find the impedance from its solution. This first result is made taking in account the Newtonian fluid consideration (3):

$$Z(s)=\frac{L}{2}\frac{\rho s}{\pi}\frac{\mathrm{HeunC}(0,0,0,\delta,0,zR)}{\Theta} \tag{28}$$

$$\Theta=\int_{0}^{R}\begin{pmatrix}\mathrm{HeunC}(0,0,0,\delta,0,zR)\\-\mathrm{HeunC}(0,0,0,\delta,0,zr)\end{pmatrix}r\,dr$$

with $\delta=\rho s\nu/\mu^{2}$ and $z=-\mu/\nu$. This solution has an integral inside which is not resolved yet, but for the application we can use numerical solutions which will provide results. Equally, the impedance is described by the Heun Confluent function HeunC which could be reviewed in [13]-[14].

For the non-Newtonian case (non-Newtonian condition) (4), the impedance expression we find is presented below:

$$Z(s)=\frac{L}{2}\frac{\rho s}{\pi}\frac{A}{\int_{0}^{R}\left(A-\mathrm{HeunC}(0,0,0,\delta,0,zr)\beta\nu\right)r\,dr} \tag{29}$$

with A as follow:

$$\begin{aligned}A= {} & \mathrm{HeunC}(0,0,0,\delta,0,zR)\beta\nu\\ & -\mathrm{HeunCPrime}(0,0,0,\delta,0,zR)\left(\mu^{2}R+\mu\nu\right)\end{aligned} \tag{30}$$

in the last equation appears HeunCPrime which is derivative of the Heun Confluent function.

D. *Altered Artery with Quadratic Viscosity Shape*

As the previous result, this one is obtained from the Navier-Stokes equation with viscosity variable (5), but in this case with the viscosity function (7). As first result we present the characteristic impedance for a fluid considered as Newtonian (3):

$$Z(s)=\frac{L}{2}\frac{\rho s}{\pi}\frac{\mathcal{P}_{n}(W(R))}{\int_{0}^{R}\left(\mathcal{P}_{n}(W(R))-\mathcal{P}_{n}(W(r))\right)r\,dr} \tag{31}$$

here appears the Legendre Polynomials $\mathcal{P}_{n}(x)$ where the degree is defined by

$$n=-\frac{1}{2}\left(\sqrt{1+\frac{\rho s}{\nu}}+1\right) \tag{32}$$

and the argument, which is function of the radius r

$$W(r)=1+\frac{2\nu}{\mu}r^{2} \tag{33}$$

For the non-Newtonian fluid consideration (4), the characteristic impedance we obtain is

$$Z(s)=\frac{L}{2}\frac{\rho s}{\pi}\frac{B}{\int_{0}^{R}\left(B+\mathcal{P}_{n}(W(r))2R\beta\right)r\,dr} \tag{34}$$

with B as follow:

$$\begin{aligned}B= {} & \left(4\nu R^{2}m-2\beta R+2\mu m\right)\mathcal{P}_{n}(W(R))\\ & +2\mu m\mathcal{P}_{m}(W(R))\end{aligned} \tag{35}$$

with a new degree for the Legendre polynomials

$$m=-\frac{1}{2}\left(\sqrt{1+\frac{\rho s}{\nu}}-1\right) \tag{36}$$

E. *Altered Artery with Two Fluids*

In this subsection, we present the characteristic impedance, which expression is big so, we have to decompose into multiple parts, we start redefining the arguments of the Bessel functions that appears:

$$X=\sqrt{\frac{\rho_{2}s}{\mu_{2}}}R\quad Y=\sqrt{\frac{\rho_{2}s}{\mu_{2}}}R_{1}\quad Z=\sqrt{\frac{\rho_{1}s}{\mu_{1}}}R_{1} \tag{37}$$

also we present some expression of Bessel functions compressed to get space:

$$
\begin{aligned}
\mathcal{D}_1 &= 2\left[\mathcal{I}_0(X)i\mathcal{Y}_1(iY)+\mathcal{I}_1(Y)\mathcal{Y}_0(iX)\right]\\
\mathcal{D}_2 &= 2\left[\mathcal{I}_0(X)\mathcal{Y}_0(iY)-\mathcal{I}_0(Y)\mathcal{Y}_0(iX)\right]\\
\mathcal{D}_3 &= 2\left[\mathcal{I}_1(X)i\mathcal{Y}_1(iY)+\mathcal{I}_1(Y)i\mathcal{Y}_1(iX)\right]\\
\mathcal{D}_4 &= 2\left[\mathcal{I}_0(Y)i\mathcal{Y}_1(iX)+\mathcal{I}_1(X)\mathcal{Y}_0(Y)\right]
\end{aligned} \tag{38}
$$

for the Newtonian condition (3), we obtain:

$$
Z(s) = \frac{L}{2}\frac{\rho_1\rho_2 s}{\pi}\,\frac{C}{C\dfrac{\left(R^2\rho_1+R_1^2(\rho_2-\rho_1)\right)}{2}+C'R+C''R_1} \tag{39}
$$

where C is defined as

$$
C=-\left(\sqrt{\mu_2\rho_2}\,\mathcal{D}_1\mathcal{I}_0 Z+\sqrt{\mu_1\rho_1}\,\mathcal{D}_2\mathcal{I}_1 Z\right)\left(\rho_1\rho_2 s\right)^{3/2} \tag{40}
$$

C' is defined as follow

$$
\begin{aligned}
C' = {} & \mathcal{D}_3\mathcal{I}_0(Z)\left(\mu_2\rho_2\right)^{3/2}\rho_1^{5/2}s\\
&+\mathcal{I}_1(Z)\left(\mathcal{I}_1(X)\mathcal{Y}_0(iX)+\mathcal{I}_0(X)i\mathcal{Y}_1(iX)\right)\\
&\cdot\left(2\sqrt{\mu_1\mu_2}\,\rho_2^2\rho_1^2 s\right)\\
&+\mathcal{I}_1(Z)\mathcal{I}_1(X)\left(\mathcal{Y}_0(iY)-\mathcal{Y}_0(iX)\right)\\
&\cdot\left(2\sqrt{\mu_1\mu_2}\,\rho_2\rho_1^3 s\right)\\
&+\mathcal{I}_1(Z)i\mathcal{Y}_1(iX)\left(\mathcal{I}_0(Y)-\mathcal{I}_0(X)\right)\\
&\cdot\left(2\sqrt{\mu_1\mu_2}\,\rho_2\rho_1^3 s\right)
\end{aligned} \tag{41}
$$

C'' is described by

$$
\begin{aligned}
C'' = {} & \mathcal{D}_1\mathcal{I}_1(Z)\left(\sqrt{\mu_1\mu_2}\,\rho_2^3\rho_1 s\right)\\
&+\mathcal{I}_1(Z)i\mathcal{Y}_1(iY)\left(\mathcal{I}_0(X)-\mathcal{I}_0(Y)\right)\\
&\cdot\left(2\sqrt{\mu_1\mu_2}\,\rho_2\rho_1^3 s\right)\\
&+\mathcal{I}_1(Z)\mathcal{I}_1(Y)\left(\mathcal{Y}_0(iX)-\mathcal{Y}_0(iY)\right)\\
&\cdot\left(2\sqrt{\mu_1\mu_2}\,\rho_2\rho_1^3 s\right)\\
&-\mathcal{I}_1(Z)\left(\mathcal{I}_1(Y)\mathcal{Y}_0(iX)+\mathcal{I}_0(X)i\mathcal{Y}_1(iY)\right)\\
&\cdot\left(4\sqrt{\mu_1\mu_2}\,\rho_2^2\rho_1^3 s\right)\\
&+\mathcal{I}_1(Z)\left(\mathcal{I}_0(Y)i\mathcal{Y}_1(iY)+\mathcal{I}_1(Y)\mathcal{Y}_0(iY)\right)\\
&\cdot\left(2\sqrt{\mu_1\mu_2}\,\rho_2^2\rho_1^3 s\right)
\end{aligned} \tag{42}
$$

Now, we present the solution for the characteristic impedance for a non-Newtonian fluid (condition (4)):

$$
Z(s)=\frac{L}{2}\frac{\rho_1\rho_2 s}{\pi}\frac{D}{\Theta} \tag{43}
$$

$$
\Theta=-D\frac{\left(R^2\rho_1+R_1^2(\rho_2-\rho_1)\right)}{2}+C'\frac{\beta R}{\sqrt{s}}+\left(C''\frac{\beta}{\sqrt{s}}+D'\right)R_1
$$

where D is defined as

$$
\begin{aligned}
D = {} & -\frac{\sqrt{\mu_2\rho_2}}{R}\left(\mathcal{D}_1\beta R+\mathcal{D}_3\mu X\right)\mathcal{I}_0(Z)\left(\rho_1\rho_2^{3/2}s\right)\\
&-\frac{\sqrt{\mu_1\rho_1}}{R}\left(\mathcal{D}_2\beta R+\mathcal{D}_4\mu X\right)\mathcal{I}_1(Z)\left(\rho_1\rho_2^{3/2}s\right)
\end{aligned} \tag{44}
$$

D' is defined as follow

$$
\begin{aligned}
D' = {} & \mathcal{I}_1(Z)\mathcal{D}_3\left(\rho_1^2+\rho_2^2\right)\left(\mu_2\sqrt{\mu_1}\,\rho_2^{3/2}\rho_1 s\right)\\
&+\mathcal{I}_1(Z)\left[\mathcal{I}_1(Y)i\mathcal{Y}_1(iX)-\mathcal{I}_1(X)i\mathcal{Y}_1(iY)\right]\\
&\cdot\left(4\mu_2\sqrt{\mu_1}\,\rho_2^{5/2}\rho_1^2 s\right)
\end{aligned} \tag{45}
$$

V. MAPLE ALGORITHM

We have been presented through all the paper; the differential equations that model the problem, with their initial and boundary conditions, and also, the results obtained via MAPLE, but it is still missing the commands we used in the process for getting the characteristic impedance for the arteries in normal conditions and others conditions.

We are going to present the complete process for getting the first result (26), with the steps as they are presented in MAPLE (we have used the mode: classic worksheet):

We start opening two special packages from MAPLE, because we need them for our calculations. In this chapter we will show the commands in italics with a minor " > " at the beginning.

> *with(inttrans)*
> *with(VectorCalculus)*

With these packages, we can work vectorial equations and we can used integral transformation as the well known Laplace transform. After starting the package we define the main equation, Navier-Stokes equation for a laminar flow of a Newtonian fluid into the arteries.

> *eq1:=rho*diff(v(r,t),t)=-diff(P(z,t),z)+mu*Laplacian(v(r,t), cylindrical[r,theta,z]);*

$$
eq1 := \rho\frac{\partial}{\partial t}v(r,t) = -\frac{\partial}{\partial z}P(z,t)+\frac{\mu\left(\dfrac{\partial}{\partial r}v(r,t)+r\dfrac{\partial^2}{\partial r^2}v(r,t)\right)}{r} \tag{46}
$$

This equation (46) is the same as (1), except for the "order". After that, we have included the initial condition (2):

> *eq2:=v(r,0)=0;*

$$eq2 := v(r,0) = 0 \tag{47}$$

Now, we take Laplace transform from "$eq1$" (46), using the following command:

> *eq3:=laplace(eq1,t,s);*

$$eq3 := \rho s laplace(v(r,t),t,s) - \rho v(r,0) = -\frac{\partial}{\partial z} laplace(P(z,t),t,s) + \frac{\mu \frac{\partial}{\partial r} laplace(v(r,t),t,s)}{r} + \mu \frac{\partial^2}{\partial r^2} laplace(v(r,t),t,s) \tag{48}$$

Here, we can see some changes of the mathematical notation for the Laplace transform, this equation (48) is the same as (10). We continue replacing the initial condition "$eq2$" (47) in the "$eq3$" (48):

> *eq4:=subs(eq2,eq3);*

$$eq4 := \rho s laplace(v(r,t),t,s) = -\frac{\partial}{\partial z} laplace(P(z,t),t,s) + \frac{\mu \frac{\partial}{\partial r} laplace(v(r,t),t,s)}{r} + \mu \frac{\partial^2}{\partial r^2} laplace(v(r,t),t,s) \tag{49}$$

Next, we change the "laplace" expression for a simplified expression, what allow us to work easily.

> *eq5:=subs(laplace(v(r,t),t,s)=V(r),laplace(P(z,t),t,s)=P(z), eq4);*

$$eq5 := \rho s V(r) = -\frac{d}{dz}P(z) + \frac{\mu \frac{d}{dr}V(r)}{r} + \mu \frac{d^2}{dr^2}V(r) \tag{50}$$

Now, as we have the Laplace version for our problem, we are going to include the boundary condition in Laplace domain.

> *eq6:=v(R,t)=0;*

$$eq6 := v(R,t) = 0 \tag{51}$$

but in Laplace domain, it becomes:

> *eq6A:=V(R)=0;*

$$eq6A := V(R) = 0 \tag{52}$$

We solve the differential equation "$eq5$" (50):

> *eq7:=dsolve(eq5,V(r));*

$$eq7 := V(r) = \mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}r\right)\_C2 + \mathrm{BesselY}\left(0, \sqrt{-\frac{\rho s}{\mu}}r\right)\_C1 - \frac{\frac{d}{dz}P(z)}{\rho s} \tag{53}$$

We are going to expand in series this equation, looking for some singularities. We expand around $r = 0$:

> *eq8:=series(rhs(eq7),r=0,2);*

$$eq8 := \_C2 + \left(\frac{2\ln\left(\frac{1}{2}\sqrt{-\frac{\rho s}{\mu}}r\right)}{\pi} + \frac{2\gamma}{\pi}\right)\_C1 - \frac{\frac{d}{dz}P(z)}{\rho s} + O(r^2) \tag{54}$$

We can see a logarithmic singularity at $r = 0$, so we are going to eliminate it.

> *eq9:=_C1=0;*

$$eq9 := \_C1 = 0 \tag{55}$$

> *eq10:=subs(eq9,eq7);*

$$eq10 := V(r) = \mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}r\right)\_C2 - \frac{\frac{d}{dz}P(z)}{\rho s} \tag{56}$$

Now we use the boundary condition "$eq6A$" (52) in (56), for find the constant which is missing:

> *eq11:=subs(r=R,rhs(eq10))=0;*

$$eq11 := \mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}R\right)\_C2 - \frac{\frac{d}{dz}P(z)}{\rho s} = 0 \tag{57}$$

> *eq12:=isolate(eq11,_C2);*

$$eq12 := \_C2 = \frac{\frac{d}{dz}P(z)}{\rho s \left(\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}R\right)\right)} \tag{58}$$

We replace the constant for obtaining the complete expression of velocity:

> *eq13:=subs(eq12,eq10);*

$$eq13 := V(r) = \frac{\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, r\right) \frac{d}{dz} P(z)}{\rho s \left(\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, R\right)\right)} - \frac{\frac{d}{dz} P(z)}{\rho s} \tag{59}$$

> *eq14:=factor(eq13);*

$$eq14 := V(r) = \left(\frac{d}{dz} P(z)\right) * \frac{\left(\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, r\right) - \mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, R\right)\right)}{\rho s \left(\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, R\right)\right)} \tag{60}$$

Equation (60) represents the Laplace transformation of the blood velocity profile in the artery. Now we are going to calculate the flow that it is defined as the integral of the velocity across the circular cross section.

> *eq15:= Q(s)=int(int(V(r)*r,r=0..R),theta=0..2*Pi);*

$$eq15 := Q(s) = 2\pi \int_0^R V(r)\, r\, dr \tag{61}$$

solving the integral

> *eq16:=subs(eq14,eq15);*

$$eq16 := Q(s) = 2\pi \int_0^R \left(\frac{d}{dz} P(z)\right) * \frac{\left(\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, r\right) - \mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, R\right)\right)}{\rho s \left(\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, R\right)\right)} r\, dr \tag{62}$$

We simplify a little bit the expression (62):

> *eq17:=simplify(eq16);*

$$eq17 := Q(s) = \left(\frac{d}{dz} P(z)\right) R \frac{\left(2\mathrm{BesselJ}\left(1, \sqrt{-\frac{\rho s}{\mu}}\, R\right) - R\,\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, R\right) \sqrt{-\frac{\rho s}{\mu}}\right) \pi}{\rho s \left(\mathrm{BesselJ}\left(0, \sqrt{-\frac{\rho s}{\mu}}\, R\right)\right) \sqrt{-\frac{\rho s}{\mu}}} \tag{63}$$

Now, we convert the BesselJ into BesselI, for simplifying the equation more.

> *eq18:=simplify(convert(eq17,BesselI)) assuming rho>0 and s>0 and mu>0 and R>0;*

$$eq18 := Q(s) = \left(\frac{d}{dz} P(z)\right) R \frac{\left(2\mathrm{BesselI}\left(1, \frac{\sqrt{\rho}\sqrt{s} R}{\sqrt{\mu}}\right) \sqrt{\mu} - R\,\mathrm{BesselI}\left(0, \frac{\sqrt{\rho}\sqrt{s} R}{\sqrt{\mu}}\right) \sqrt{\rho}\sqrt{s}\right) \pi}{\rho^{3/2} s^{3/2} \left(\mathrm{BesselI}\left(0, \frac{\sqrt{\rho}\sqrt{s} R}{\sqrt{\mu}}\right)\right)} \tag{64}$$

We introduce a new variable, for representing some characteristics of the artery and blood.

> *eq19:=1/mu^(1/2)*rho^(1/2)*s^(1/2)*R=X;*

$$eq19 := \frac{\sqrt{\rho}\sqrt{s} R}{\sqrt{\mu}} = X \tag{65}$$

> *eq20:=isolate(eq19,s);*

$$eq20 := s = \frac{X^2 \mu}{\rho R^2} \tag{66}$$

We substitute the new variable into the flow expression (64):

> *eq21:=lhs(eq18)=subs(eq20,rhs(eq18));*

$$eq21 := Q(s) = \frac{\left(\frac{d}{dz} P(z)\right) R \pi}{\rho^{3/2} \left(\frac{X^2 \mu}{\rho R^2}\right)^{3/2} \left(\mathrm{BesselI}\left(0, \sqrt{\rho}\sqrt{\frac{X^2\mu}{\rho R^2}} R \frac{1}{\sqrt{\mu}}\right)\right)} * \left(2\mathrm{BesselI}\left(1, \sqrt{\rho}\sqrt{\frac{X^2\mu}{\rho R^2}} R \frac{1}{\sqrt{\mu}}\right) \sqrt{\mu} - R\,\mathrm{BesselI}\left(0, \sqrt{\rho}\sqrt{\frac{X^2\mu}{\rho R^2}} R \frac{1}{\sqrt{\mu}}\right) \sqrt{\rho}\sqrt{\frac{X^2\mu}{\rho R^2}}\right) \tag{67}$$

After simplification:

> *eq22:=simplify(eq21,power,symbolic);*

$$eq22 := Q(s) = -\left(\frac{d}{dz}P(z)\right)R^4 * \frac{(-2\text{BesselI}(1,X)+\text{BesselI}(0,X)X)\pi}{\mu X^3\text{BesselI}(0,X)} \quad (68)$$

In the simple case of a linear gradient of pressure we have:

> *eq23:=diff(P(z),z)=-Delta(P)/L;*

$$eq23 := \frac{d}{dz}P(z) = -\frac{\Delta(P)}{L} \quad (69)$$

Then, the expression (68) could be rewrite:

> *eq24:=subs(eq23,eq22);*

$$eq24 := Q(s) = \frac{\Delta(P)R^4\pi}{L\mu X^3\text{BesselI}(0,X)} * (-2\text{BesselI}(1,X)+\text{BesselI}(0,X)X) \quad (70)$$

> *eq25:=isolate(eq20,mu);*

$$eq25 := \mu = \frac{R^2\rho s}{X^2} \quad (71)$$

> *eq26:=subs(eq25,eq24);*

$$eq26 := Q(s) = \frac{\Delta(P)R^2\pi}{LsX\rho\text{BesselI}(0,X)} * (-2\text{BesselI}(1,X)+\text{BesselI}(0,X)X) \quad (72)$$

Finally, the characteristic impedance is defined as the relation between the pressure and the flow as follow:

> *eq27:=Z(s)=Delta(P)/Q(s);*

$$eq27 := Z(s) = \frac{\Delta(P)}{Q(s)} \quad (73)$$

After the substitution:

> *eq28:=subs(eq26,eq27);*

$$eq28 := Z(s) = \frac{LsX\rho\text{BesselI}(0,X)}{R^2\pi} * \frac{1}{(-2\text{BesselI}(1,X)+\text{BesselI}(0,X)X)} \quad (74)$$

With a mathematical *recurrence* relation, as a property of the Bessel functions, we have:

> *eq29:=-2*BesselI(1,X)+BesselI(0,X)*X=X*BesselI(2,X);*

$$eq29 := -2\text{BesselI}(1,X)+\text{BesselI}(0,X)X = X\text{BesselI}(2,X) \quad (75)$$

Finally, we obtain the expression for the characteristic impedance, as we have presented in (26).

> *eq30:=subs(eq29,eq28);*

$$eq30 := Z(s) = \frac{Ls\rho\text{BesselI}(0,X)}{R^2\text{BesselI}(2,X)\pi} \quad (76)$$

VI. Discussion and Conclusions

We have supposed the pressure gradient is constant in the artery length (22), but if we don't do it, the characteristic impedance will rest invariable with the exception of the appearance of L (as example the result (26) is similarly presented by Jager et al. [10]). We include L, which is along z, because we could changed it for a differential of length dL and make the radius R depending of this: $R(z)$, obtaining in this direction an impedance for the arteries with variable section what it is the most frequently case. Using the integration and mixing the impedance results we could obtain the characteristic impedance of the complex arterial system.

MAPLE as other CAS (Computer Algebra System) have the last mathematical tools for helping in the development of our analytical models, but in some cases the results are presented in a simple way, so it is missing a capacity of factorize.

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