# Analytical solution for steady state and transient heat processes in a double-fin assembly

Marija Lencmane, Andris Buikis

***Abstract*—**this paper deals with the three dimensional formulation of steady state and transient problems for the heat exchanger consisting of rectangular fins attached on either sides of a plane wall (double-fin assembly). With the help of the conservative averaging method problem was reduced to the two dimensional problem. Analytical solution based on Green function is proposed. This solution is obtained in the form of the 2nd kind Fredholm integral equations. Some solutions for the system of 2nd kind Fredholm integral equations are given.

***Keywords*—** analytical solution, Conservative averaging method, extended surfaces, Green function, heat transfer.

## I. Introduction

Extended surface is used specially to enhance the heat transfer between a solid and surrounding medium. Such an extended surface is termed a fin. Extended surfaces are widely examined in [16]-[18]. The rate of heat transfer is directly proportional to the extent of the wall surface, the heat transfer coefficient and to the temperature difference between solid and the surrounding medium. Finned surfaces are widely used in many applications such as air conditioners, aircrafts, chemical processing plants, etc. In [3] is considered performance of a heat-exchanger consisting of rectangular fins attached to both sides of plane wall. In [2],[3] works one dimensional steady-state double-fin assembly problem is compared with the single-fin assembly. Papers [2]-[7] deals with a numerical solution for the one dimesional problems. We consider analytical solution for the three dimensional problem as in [10]-[15]. In paper [10] mathematical three dimensional formulation of transient problem for one element with one rectangular fin is examined, reduce it by conservative averaging method [9] to the system of three heat equations with linear sink terms. Reference [11] shows exact analytical solution for two-dimensional steady-state process for system with one rectangular fin by the method of Green function [1]. In [12] three dimensional exact analytical solution for the distribution of the temperature field in the wall with one rectangular fin in the form of the 2nd kind Fredholm integral equation is constructed.

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## II. Mathematical formulation of 3D problem

In this section we present mathematical three dimensional formulation of a transient problem one element with two rectangular fins attached to both sides.

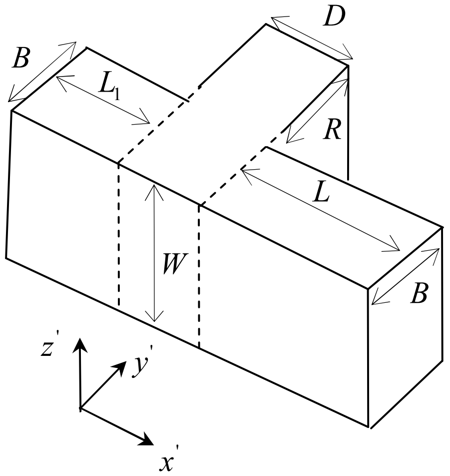


Fig. 1 heat exchanger consisting of rectangular fins attached on either sides of a plane wall

We will use following dimensionless arguments, parameters:

$$x = x'/(B+R),\ y = y'/(B+R),\ z = z'/(B+R),$$
$$l = L/(B+R),\ l_1 = L_1/(B+R),\ w = W/(B+R),$$
$$b = B/(B+R),\ \delta = D/B+R,$$
$$\beta = hk^{-1}(B+R),\ \beta_0 = h_0 k^{-1}(B+R)$$

and dimensionless temperatures:

$$\bar{V}(x,y,z,t) = \frac{\tilde{V}(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{V}_0(x,y,z,t) = \frac{\tilde{V}_0(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{V}_1(x,y,z,t) = \frac{\tilde{V}_1(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{\Theta}(x,y,z,t) = \frac{\tilde{\Theta}(x,y,z,t) - T_a(t)}{T_b(t) - T_a(t)},$$

$$\bar{\Theta}_0(x,y,z,t) = \frac{\tilde{\Theta}_0(x.y,z,t) - T_a(t)}{T_b(t) - T_a(t)}.$$

We have introduced following dimensional thermal and geometrical parameters: $k$ - heat conductivity coefficients for the wall, right fin and left fin, $h(h_0)$ - heat exchange coefficient for the right (left) side, $2B$ – fins width (thickness), $L$ – right fin length, $L_1$ – left fin length, $D$ - thickness of the wall, $W$ - walls' width (length), $2R$ – distance between two fins (fin spacing). Further, $\tilde{\Theta}_0(x.y,z,t)$ is the surrounding (environment) temperature on the left (hot) side (the heat source side) of the wall, $\tilde{\Theta}(x,y,z,t)$ - the surrounding temperature on the right (cold - the heat sink side) of the wall and the fin. Finally $\tilde{V}_0(x,y,z,t)$, $\tilde{V}(x,y,z,t)$, $\tilde{V}_1(x,y,z,t)$ are the dimensional temperatures in the wall, right fin and left fin where $T_a(T_b)$ are integral averaged environment temperatures over appropriate edges:

$$T_a(t) = w(1+l)^{-1}\int_0^w dz\int_b^1 \Theta(D,y,z,t)dy + w(1+l)^{-1}\int_\delta^{\delta+l} dx\int_o^w \Theta(x,B,z,t)dz$$

$$+ w(1+l)^{-1}\int_0^w dz\int_0^b \Theta(D+L,y,z,t)dy,$$

$$T_b(t) = w(1+l_1)^{-1}\int_0^w dz\int_b^1 \Theta_0(0,y,z,t)dy + w(1+l_1)^{-1}\int_{-l_1}^0 dx\int_o^w \Theta_0(x,B,z,t)dz$$

$$+ w(1+l)^{-1}\int_0^w dz\int_0^b \Theta_0(-L_1,y,z,t)dy.$$

The one element of the wall (base) is placed in the domain $\{x\in[0,\delta],\ y\in[0,1],\ z\in[0,w]\}$. The rectangular right fin in dimensionless arguments occupies the domain $\{x\in[\delta,\delta+l],\ y\in[0,b],\ z\in[0,w]\}$. The rectangular left fin in dimensionless arguments occupies the domain $\{x\in[-l_1,0],\ y\in[0,b],\ z\in[0,w]\}$. We describe the temperature field by functions $\bar{V}(x,y,z,t)$, $\bar{V}_0(x,y,z,t)$, $\bar{V}_1(x,y,z,t)$ in the wall and fins:

$$\frac{\partial^2 \bar{V}_0}{\partial x^2} + \frac{\partial^2 \bar{V}_0}{\partial y^2} + \frac{\partial^2 \bar{V}_0}{\partial z^2} = \frac{1}{a^2}\frac{\partial \bar{V}_0}{\partial t}, \tag{1}$$

$$\frac{\partial^2 \bar{V}}{\partial x^2} + \frac{\partial^2 \bar{V}}{\partial y^2} + \frac{\partial^2 \bar{V}}{\partial z^2} = \frac{1}{a^2}\frac{\partial \bar{V}}{\partial t}, \tag{2}$$

$$\frac{\partial^2 \bar{V}_1}{\partial x^2} + \frac{\partial^2 \bar{V}_1}{\partial y^2} + \frac{\partial^2 \bar{V}_1}{\partial z^2} = \frac{1}{a^2}\frac{\partial \bar{V}_1}{\partial t}. \tag{3}$$

We must add initial conditions for the heat equations (1)–(3):

$$\bar{V}_0\Big|_{t=0} = V_0^0(x,y,z), \tag{4}$$

$$\bar{V}\Big|_{t=0} = V^0(x,y,z), \tag{5}$$

$$\bar{V}_1\Big|_{t=0} = V_1^0(x,y,z). \tag{6}$$

We assume heat fluxes from the flank surfaces (edges) and from the top and the bottom edges:

$$\frac{\partial \bar{V}_0}{\partial z}\Big|_{z=0} = Q_{0,2}(x,y,t),\ \frac{\partial \bar{V}_0}{\partial z}\Big|_{z=w} = Q_{0,3}(x,y,t), \tag{7}$$

$$\frac{\partial \bar{V}}{\partial z}\Big|_{z=0} = Q_2(x,y,t),\ \frac{\partial \bar{V}}{\partial z}\Big|_{z=w} = Q_3(x,y,t), \tag{8}$$

$$\frac{\partial \bar{V}_1}{\partial z}\Big|_{z=0} = Q_{1,2}(x,y,t),\ \frac{\partial \bar{V}_1}{\partial z}\Big|_{z=w} = Q_{1,3}(x,y,t). \tag{9}$$

In the case of steady state problem all above mentioned functions are time-independent. Three dimensional formulation of a steady state problem can be obtained in the similar way. Instead of (1) –(3) we have:

$$\frac{\partial^2 \bar{V}_0}{\partial x^2} + \frac{\partial^2 \bar{V}_0}{\partial y^2} + \frac{\partial^2 \bar{V}_0}{\partial z^2} = 0,$$

$$\frac{\partial^2 \bar{V}}{\partial x^2} + \frac{\partial^2 \bar{V}}{\partial y^2} + \frac{\partial^2 \bar{V}}{\partial z^2} = 0,$$

$$\frac{\partial^2 \bar{V}_1}{\partial x^2} + \frac{\partial^2 \bar{V}_1}{\partial y^2} + \frac{\partial^2 \bar{V}_1}{\partial z^2} = 0.$$

Initial conditions (4)-(6) are not needed. Conditions (7)-(9) are in the form:

$$\frac{\partial \bar{V}_0}{\partial z}\Big|_{z=0} = Q_{0,2}(x,y),\ \frac{\partial \bar{V}_0}{\partial z}\Big|_{z=w} = Q_{0,3}(x,y),$$

$$\frac{\partial \bar{V}}{\partial z}\Big|_{z=0} = Q_2(x,y),\ \frac{\partial \bar{V}}{\partial z}\Big|_{z=w} = Q_3(x,y),$$

$$\frac{\partial \bar{V}_1}{\partial z}\Big|_{z=0} = Q_{1,2}(x,y),\ \frac{\partial \bar{V}_1}{\partial z}\Big|_{z=w} = Q_{1,3}(x,y).$$

## III. REDUCING TO THE 2D MODEL

Such type of boundary conditions (BC)(7) – (9) allows us to make the exact reducing of this three dimensional problem to the two dimensional problem by conservative averaging method [5]. Let us introduce following integral averaged values:

$$V_0(x,y,t) = w^{-1}\cdot\int_0^w \bar{V}_0(x,y,z,t)dz, \tag{10}$$

$$\vartheta_0(x,y,t) = w^{-1}\cdot\int_0^w \Theta_0(x,y,z,t)dz, \tag{11}$$

$$V(x,y,t) = w^{-1}\cdot\int_0^w \bar{V}(x,y,z,t)dz, \tag{12}$$

$$\vartheta(x,y,t) = w^{-1} \cdot \int_0^w \Theta(x,y,z,t)dz, \quad (13)$$

$$V_1(x,y,t) = w^{-1} \cdot \int_0^w \bar{V}_1(x,y,z,t)dz. \quad (14)$$

Realizing the integration of main equations (1) –(3) by usage of the BC (7) – (12) we obtain:

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + Q_0(x,y,t) = \frac{1}{a^2}\frac{\partial V_0}{\partial t}, \quad (15)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + Q(x,y,t) = \frac{1}{a^2}\frac{\partial V}{\partial t}, \quad (16)$$

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + Q_1(x,y,t) = \frac{1}{a^2}\frac{\partial V_1}{\partial t}. \quad (17)$$

Here

$$Q_0(x,y,t) = w^{-1}(Q_{0,3}(x,y,t) - Q_{0,2}(x,y,t)),$$

$$Q(x,y,t) = w^{-1}(Q_3(x,y,t) - Q_2(x,y,t)),$$

$$Q_1(x,y,t) = w^{-1}(Q_{1,3}(x,y,t) - Q_{1,2}(x,y,t)).$$

We add to the main partial differential equations (15) – (17) needed BC as follows:

$$\left(\frac{\partial V_0}{\partial x} + \beta_0[\vartheta_0(x,y,t) - V_0]\right)\Big|_{x=0} = 0, y \in (b,1), \quad (18)$$

$$\left(\frac{\partial V_0}{\partial x} + \beta[V_0 - \vartheta(x,y,t)]\right)\Big|_{x=\delta} = 0, y \in (b,1), \quad (19)$$

$$\frac{\partial V_0}{\partial y}\Big|_{y=0} = Q_{0,0}(x,t), x \in (0,\delta), \quad (20)$$

$$\frac{\partial V_0}{\partial y}\Big|_{y=1} = Q_{0,1}(x,t), x \in (0,\delta). \quad (21)$$

We assume them as ideal thermal contact between wall and fins - there is no contact resistance:

$$V_0\big|_{x=\delta-0} = V\big|_{x=\delta+0}, \quad (22)$$

$$\frac{\partial V_0}{\partial x}\Big|_{x=\delta-0} = \frac{\partial V}{\partial x}\Big|_{x=\delta+0}, \quad (23)$$

$$V_1\big|_{x=0-0} = V_0\big|_{x=0+0}, \quad (24)$$

$$\frac{\partial V_1}{\partial x}\Big|_{x=0-0} = \frac{\partial V_0}{\partial x}\Big|_{x=0+0}. \quad (25)$$

We have following BC for the right fin:

$$\left(\frac{\partial V}{\partial x} + \beta[V - \vartheta(x,y,t)]\right)\Big|_{x=\delta+l} = 0, y \in (0,b), \quad (26)$$

$$\left(\frac{\partial V}{\partial y} + \beta[V - \vartheta(x,y,t)]\right)\Big|_{y=b} = 0, x \in (\delta,\delta+l), \quad (27)$$

$$\frac{\partial V}{\partial y}\Big|_{y=0} = Q_0(x,t), x \in (\delta,\delta+l). \quad (28)$$

We have following BC for the left fin:

$$\left(\frac{\partial V_1}{\partial x} + \beta_0[\vartheta_0(x,y,t) - V_1]\right)\Big|_{x=-l_1} = 0, y \in (0,b), \quad (29)$$

$$\left(\frac{\partial V_1}{\partial x} + \beta_0[V_1 - \vartheta_0(x,y,t)]\right)\Big|_{y=b} = 0, x \in (-l_1,0), \quad (30)$$

$$\frac{\partial V_1}{\partial y}\Big|_{y=0} = Q_1(x,t), x \in (-l_1,0). \quad (31)$$

Finally, we introduce integral averaged values as (10) – (14) and add initial conditions for the heat equations (15) – (17):

$$V_0\big|_{t=0} = V_0^0(x,y), \quad (32)$$

$$V\big|_{t=0} = V^0(x,y), \quad (33)$$

$$V_1\big|_{t=0} = V_1^0(x,y). \quad (34)$$

In the similar way three dimensional steady state problem can be reduced to the two dimensional problem. Instead of (15)–(17) we have:

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + Q_0(x,y) = 0,$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + Q(x,y) = 0,$$

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + Q_1(x,y) = 0.$$

BC are still in the form (18)-(21), (26)-(31), conjugation conditions are in the form (22)-(25) for time-independent functions $V_0(x,y), V(x,y), V_1(x,y), \vartheta_0(x,y), \vartheta(x,y)$. Initial conditions (32)-(34) are not needed for steady state problem.

## IV. Exact Solution of 2D Steady State Simplified Problem

This section represents solution for the 2D case of periodical system with constant dimensionless environmental temperatures $\vartheta_0 = 1 (\Theta_0 = T_b)$ and $\vartheta = 0 (\Theta = T_a)$. We consider $U(x,y)$ is a temperature of the right fin, $U_0(x,y)$ is a temperature of the wall and $U_1(x,y)$ is a temperature of the left fin. Thus, the main equations are:

$$\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} = 0, \quad (35)$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad (36)$$

$$\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2} = 0. \quad (37)$$

The BC (20), (21), (28), (31) are assumed to be homogeneous:

$$\frac{\partial U_0}{\partial y}\Big|_{y=0} = \frac{\partial U_0}{\partial y}\Big|_{y=1} = \frac{\partial U}{\partial y}\Big|_{y=0} = \frac{\partial U_1}{\partial y}\Big|_{y=0} = 0.$$

Instead of BC (18), (19), (26), (27), (29) and (30) we have:

$$\frac{\partial U_0}{\partial x} + \beta_0\left[1 - U_0\right]\Big|_{x=0} = 0,\ y \in (b, 1), \tag{38}$$

$$\left(\frac{\partial U_0}{\partial x} + \beta U_0\right)\Bigg|_{x=\delta} = 0,\ y \in (b, 1), \tag{39}$$

$$\left(\frac{\partial U}{\partial x} + \beta U\right)\Bigg|_{x=\delta+l} = 0,\ y \in [0, b], \tag{40}$$

$$\left\{\frac{\partial U}{\partial y} + \beta U\right\}\Bigg|_{y=b} = 0,\ x \in [\delta, \delta + l], \tag{41}$$

$$\left(\frac{\partial U_1}{\partial x} + \beta_0\left[1 - U_1\right]\right)\Bigg|_{x=-l_1} = 0,\ y \in [0, b], \tag{42}$$

$$\left\{\frac{\partial U_1}{\partial y} + \beta_0\left[U_1 - 1\right]\right\}\Bigg|_{y=b} = 0,\ x \in [-l_1, 0]. \tag{43}$$

The conjugations conditions on the line between the wall and the left fin are still standing in the form (24), (25) for the functions $U_0(x, y)$ and $U_1(x, y)$. The linear combination of the equations (24), (25) together with BC (38) allow us rewrite them as following BC on the left hand side of the wall:

$$\left(\frac{\partial U_0}{\partial x} - \beta_0 U_0\right)\Bigg|_{x=0+0} = \beta_0 F_1(0, y), \tag{44}$$

where

$$F_1(x, y) = \begin{cases} \dfrac{1}{\beta_0}\dfrac{\partial U_1}{\partial x} - U_1, & 0 \le y \le b,\ 0 \le x \le \delta, \\ -1, & b < y \le 1. \end{cases} \tag{45}$$

In the similar way using the linear combination of the equations (22), (23) together with BC (39) we rewrite following BC on the right hand side of the wall:

$$\left(\frac{\partial U_0}{\partial x} + \beta U_0\right)\Bigg|_{x=\delta-0} = \beta F_0(\delta, y), \tag{46}$$

where

$$F_0(x, y) = \begin{cases} \left(\dfrac{1}{\beta}\dfrac{\partial U}{\partial x} + U\right), & 0 \le y \le b, 0 \le x \le \delta, \\ 0, & , b < y \le 1. \end{cases} \tag{47}$$

On the assumption that the functions $F_1(0, y)$, $F_0(\delta, y)$ are given we can represent solution for the wall in very well known form by the Green function [1]:

$$\begin{aligned} U_0(x, y) = &-\int_0^1 F_1(0, \nu) G_0(x, y, 0, \nu) d\nu \\ &+ \int_0^b F_0(\delta, \nu) G_0(x, y, \delta, \nu) d\nu, \end{aligned} \tag{48}$$

where Green function is:

$$G_0(x, y, \varsigma, \nu) = \sum_{m,n=1}^{\infty} \frac{G_{0,m}^x(x, \varsigma) \cdot G_{0,n}^y(y, \nu)}{\left[(\pi \cdot n)^2 + \mu_m^2\right]},$$

$$G_{0,m}^x(x, \varsigma) = \frac{\varphi_{0,m}(x)\varphi_{0,m}(\varsigma)}{\left\|\varphi_{0,m}\right\|^2},$$

$$G_{0,n}^y(y, \nu) = \cos\left[n\pi(y + \nu)\right] + \cos\left[n\pi(y - \nu)\right],$$

$$\varphi_{0,,m}(x) = \cos(\mu_m x) + \frac{\beta_0}{\mu_m}\sin(\mu_m x),$$

$$\left\|\varphi_{0,m}\right\|^2 = \frac{\beta_0}{2\mu_m^2} + \frac{\beta}{2\mu_m^2}\frac{\mu_m^2 + (\beta_0)^2}{\mu_m^2 + (\beta)^2} + \frac{\delta}{2}\left(1 + \frac{\beta_0^2}{\mu_m^2}\right).$$

Here $\mu_m$ are the positive roots of the transcendental equation:

$$\tan(\mu_m \delta) = \frac{\mu_m(\beta_0 + \beta)}{\mu_m^2 - \beta_0\beta}.$$

Unfortunately the representation (48) is unusable as solution for the wall because of unknown functions $F_1(0, y), F_0(\delta, y)$ i.e. temperature in the fins. That is why we will pay attention to the solution for the fins now. In the same way we can rewrite the conjugations conditions (22), (23) in the form of BC on the left side of the right rectangular fin:

$$\left(\frac{\partial U}{\partial x} - \beta U\right)\Bigg|_{x=\delta+0} = \beta F(\delta, y), \tag{49}$$

where

$$F(x, y) = \left(\frac{1}{\beta}\frac{\partial U_0}{\partial x} - U_0\right), 0 \le y \le b, \delta \le x \le \delta + l. \tag{50}$$

Then, similar as for the wall we can represent solution for the right fin in following form:

$$U(x, y) = -\int_0^b F(\delta, \eta) G(x, y, \delta, \eta) d\eta, \tag{51}$$

where Green function is:

$$G(x, y, \xi, \eta) = \sum_{i,j=1}^{G_m^{(x)}} \frac{G_i^{(x)}(x, \xi) \cdot G_j^{(y)}(y, \eta)}{\lambda_i^2 + k_j^2},$$

$$G_i^x(x, \xi) = \frac{\phi_i(x)\phi_i(\xi)}{\left\|\phi_i\right\|^2},$$

$$G_j^{(y)}(y, \eta) = \frac{\psi_j(y, \eta)}{2\left\|\psi_j\right\|^2},$$

$$\phi_i(x) = \cos\left[\lambda_i(x - \delta)\right] + \frac{\beta}{\lambda_i}\sin\left[\lambda_i(x - \delta)\right],$$

$$\left\|\phi_i\right\|^2 = \frac{\beta}{\lambda_i^2} + \frac{l}{2}\left(1 + \frac{\beta^2}{\lambda_i^2}\right),$$

$$\psi_j(y, \eta) = \cos\left[\kappa_j(y + \eta)\right] + \cos\left[\kappa_j(y - \eta)\right],$$

$$\left\|\psi_j\right\|^2 = \frac{1}{2}\left(b + \frac{\beta}{k_j^2 + \beta^2}\right).$$

Here $\lambda_i, k_j$ are the positive roots of the transcendental equations:

$$\tan(\lambda_i l) = \frac{2\lambda_i \beta}{\lambda_i^2 - \beta^2},$$

$$\tan(\kappa_j b) = \frac{\beta}{\kappa_j}.$$

Finally, we rewrite the conjugations conditions in the form of BC on the right side of the left rectangular fin:

$$\left(\frac{\partial U_1}{\partial x} + \beta_0 U_1\right)\Big|_{x=0+0} = \beta_0 F_2(0, y), \tag{52}$$

where

$$F_2(x, y) = \frac{1}{\beta_0}\frac{\partial U_0}{\partial x} + U_0,\ 0 \le y \le b. \tag{53}$$

Thus, solution for the left fin we can represent in following form:

$$U_1(x, y) = \beta_0 \int_0^b G_1(x, y, -l_1, \nu) d\nu$$
$$+ \int_0^b F_2(0, \nu) G_1(x, y, 0, \nu) d\nu \tag{54}$$
$$+ \beta_0 \int_{-l_1}^0 G_1(x, y, \xi, b) d\xi,$$

where

$$G_1(x, y, \xi, \eta) = \sum_{i,j=1}^{G_m^{(x)}} \frac{G_{1,i}^{(x)}(x, \xi) \cdot G_{1,j}^{(y)}(y, \eta)}{\mu_i^2 + \lambda_j^2},$$

$$G_{1,i}^{x}(x, \xi) = \frac{\varphi_{1,i}(x)\varphi_{1,i}(\xi)}{\|\varphi_{1,i}\|^2},$$

$$G_{1,j}^{(y)}(y, \eta) = \frac{\psi_{1,j}(y, \eta)}{2\|\psi_{1,j}\|^2},$$

$$\varphi_{1,i}(x) = \cos[\mu_i(x + l_1)] + \frac{\beta_0}{\mu_i}\sin[\mu_i(x + l_1)],$$

$$\|\varphi_{1,i}\|^2 = \frac{\beta_0}{\mu_i^2} + \frac{l_1}{2}\left(1 + \frac{\beta_0^{\,2}}{\mu_i^2}\right),$$

$$\psi_{1,j}(y) = \cos[\lambda_j(y - \eta)] + \cos[\lambda_j(y + \eta)],$$

$$\|\psi_{1,j}\|^2 = \frac{1}{2}\left(b + \frac{\beta_0}{\lambda_j^2 + \beta_0^{\,2}}\right).$$

Here $\mu_i$, $\lambda_j$ are the positive roots of the transcendental equations:

$$\tan(\mu_i l_1) = \frac{2\mu_i \beta_0}{\mu_i^2 - \beta_0^{\,2}},$$

$$\tan(\lambda_j b) = \frac{\beta_0}{\lambda_j}.$$

Using notation (51) and representation (47) we can easy obtain the following equation:

$$F_0(x, y) = -\int_0^b F(\delta, \eta) \cdot \Gamma(x, y, \delta, \eta) d\eta, \tag{55}$$

where

$$\Gamma(x, y, \xi, \eta) = \left(\frac{\partial}{\partial x} + \beta\right) G(x, y, \xi, \eta).$$

In the similar way we find equation for $F_1(0, y)$ by using (45) and (54):

$$F_1(0, y) = \beta_0 \int_0^b \Gamma_1(0, y, -l_1, \nu) d\nu$$
$$- \beta_0 \int_{-l_1}^0 \Gamma_1(0, y, \xi, b) d\xi \tag{56}$$
$$+ \int_0^b F_2(0, \nu) \Gamma_1(0, y, 0, \nu) d\nu,$$

where

$$\Gamma_1(x, y, \varsigma, \nu) = \left(\frac{\partial}{\partial x} - \beta_0\right) G_1(x, y, \varsigma, \nu).$$

Next,we find equation for $F(\delta, y)$ by using (50) and (48):

$$F(\delta, \eta) = \int_0^b F_0(\delta, \nu) \Gamma_0(\delta, \eta, \delta, \nu) d\nu$$
$$- \int_0^1 F_1(0, \nu) \Gamma_0(\delta, \eta, 0, \nu) d\nu, \tag{57}$$

where

$$\Gamma_0(x, y, \varsigma, \nu) = \left(\frac{\partial}{\partial x} - \beta\right) G_0(x, y, \varsigma, \nu).$$

Finally, using (53) and (48) we get equation for $F_2(0, y)$:

$$F_2(0, y) = -\int_0^1 F_1(0, \nu) \Gamma_2(0, y, 0, \nu) d\nu$$
$$+ \int_0^b F_0(\delta, \nu) \Gamma_2(\delta, y, \delta, \nu) d\nu, \tag{58}$$

where

$$\Gamma_2(x, y, \varsigma, \nu) = \left(\frac{\partial}{\partial x} + \beta_0\right) G_0(x, y, \varsigma, \nu).$$

When a system of Fredholm integral equations of the second kind (58) – (61) is solved, we obtain the temperatures fields in the wall (48), left fin (54) and right fin (51).

## V. Exact solution of 2D transient simplified problem

In this section we explain the main idea of solution for the 2D case of periodical system with constant dimensionless environmental temperatures $\vartheta_0 = 1\left(\Theta_0 = T_b\right)$ and $\vartheta = 0\,(\Theta = T_a)$. We consider $U(x, y, t)$ is a temperature of the right fin, $U_0(x, y, t)$ is a temperature of the wall and $U_1(x,y,t)$ is a temperature of the left fin. Thus, the main equations are:

$$\frac{\partial^2 U_0}{\partial x^2}+\frac{\partial^2 U_0}{\partial y^2}=\frac{1}{a^2}\frac{\partial U_0}{\partial t}, \tag{59}$$

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\frac{1}{a^2}\frac{\partial U}{\partial t}, \tag{60}$$

$$\frac{\partial^2 U_1}{\partial x^2}+\frac{\partial^2 U_1}{\partial y^2}=\frac{1}{a^2}\frac{\partial U_1}{\partial t}. \tag{61}$$

The BC (20), (21), (28), (31) are assumed to be homogeneous:

$$\left.\frac{\partial U_0}{\partial y}\right|_{y=0}=\left.\frac{\partial U_0}{\partial y}\right|_{y=1}=\left.\frac{\partial U}{\partial y}\right|_{y=0}=\left.\frac{\partial U_1}{\partial y}\right|_{y=0}=0.$$

Instead of BC (18), (19), (26), (27), (29) and (30) we have:

$$\left.\frac{\partial U_0}{\partial x}+\beta_0[1-U_0]\right|_{x=0}=0,\ y\in(b,1), \tag{62}$$

$$\left.\left(\frac{\partial U_0}{\partial x}+\beta U_0\right)\right|_{x=\delta}=0,\ y\in(b,1), \tag{63}$$

$$\left.\left(\frac{\partial U}{\partial x}+\beta U\right)\right|_{x=\delta+l}=0,\ y\in[0,b], \tag{64}$$

$$\left.\left\{\frac{\partial U}{\partial y}+\beta U\right\}\right|_{y=b}=0,\ x\in[\delta,\delta+l], \tag{65}$$

$$\left.\left(\frac{\partial U_1}{\partial x}+\beta_0[1-U_1]\right)\right|_{x=-l_1}=0,\ y\in[0,b], \tag{66}$$

$$\left.\left\{\frac{\partial U_1}{\partial y}+\beta_0[U_1-1]\right\}\right|_{y=b}=0,\ x\in[-l_1,0]. \tag{67}$$

Initial conditions are still standing in the form (32) – (34). The conjugations conditions on the line between the wall and the left fin are still standing in the form (24), (25) for the functions $U_0(x,y,t)$ and $U_1(x,y,t)$. The linear combination of the equations (24), (25) together with BC (62) allow us rewrite them as following BC on the left hand side of the wall:

$$\left.\left(\frac{\partial U_0}{\partial x}-\beta_0 U_0\right)\right|_{x=0+0}=\beta_0 F_1(0,y,t), \tag{68}$$

where

$$F_1(x,y,t)=\begin{cases}\dfrac{1}{\beta_0}\dfrac{\partial U_1}{\partial x}-U_1, & 0\le y\le b, 0\le x\le\delta,\\ -1, & b<y\le 1.\end{cases} \tag{69}$$

In the similar way using the linear combination of the equations (22), (23) together with BC (63) we rewrite following BC on the right hand side of the wall:

$$\left.\left(\frac{\partial U_0}{\partial x}+\beta U_0\right)\right|_{x=\delta-0}=\beta F_0(\delta,y,t), \tag{70}$$

where

$$F_0(x,y,t)=\begin{cases}\left(\dfrac{1}{\beta}\dfrac{\partial U}{\partial x}+U\right), & 0\le y\le b, 0\le x\le\delta,\\ 0, & ,b<y\le 1.\end{cases} \tag{71}$$

On the assumption that the functions $F_1(x,y,t)$, $F_0(x,y,t)$are given we can represent solution for the wall in very well known form by the Green function:

$$\begin{aligned}U_0(x,y,t)&=\int_0^\delta\int_0^1 U_0^0(\xi,\eta)G_0(x,y,\xi,\eta,t)d\eta d\xi\\ &-a^2\beta_0\int_0^t\int_0^1 F_1(0,\eta,\tau)G_0(x,y,0,\eta,t-\tau)d\eta d\tau\\ &+a^2\beta\int_0^t\int_0^1 F_0(\delta,\eta,\tau)G_0(x,y,\delta,\eta,t-\tau)d\eta d\tau,\end{aligned} \tag{72}$$

where Green function is:

$$G_0(x,y,\xi,\eta,t)=\sum_{m,n=1}^{\infty}G_{0,m}^x(x,\xi,t)\cdot\sum_{m=1}^{\infty}G_{0,n}^y(y,\eta,t),$$

$$G_{0,m}^x(x,\xi,t)=\frac{\varphi_{0,m}(x)\varphi_{0,m}(\xi)}{\|\varphi_{0,m}\|^2}e^{-a^2\mu_n^2 t},$$

$$G_{0,n}^y(y,\eta,t)=e^{-a^2n^2\pi^2 t}(\cos[n\pi(y+\eta)]+\cos[n\pi(y-\eta)]),$$

$$\varphi_{0,,m}(x)=\cos(\mu_m x)+\frac{\beta_0}{\mu_m}\sin(\mu_m x),$$

$$\|\varphi_{0,m}\|^2=\frac{\beta_0}{2\mu_m^2}+\frac{\beta}{2\mu_m^2}\frac{\mu_m^2+\beta_0^2}{\mu_m^2+\beta^2}+\frac{\delta}{2}\left(1+\frac{\beta_0^2}{\mu_m^2}\right).$$

Here $\mu_m$ are the positive roots of the transcendental equation:

$$\tan(\mu_m\delta)=\frac{\mu_m(\beta_0+\beta)}{\mu_m^2-\beta_0\beta}.$$

Unfortunately the representation (72) is unusable as solution for the wall because of unknown functions $F_1(x,y,t)$, $F_0(x,y,t)$ i.e. temperature in the fins. That is why we will pay attention to the solution for the fins now. In the same way we can rewrite the conjugations conditions (22), (23) in the form of BC on the left side of the right rectangular fin:

$$\left.\left(\frac{\partial U}{\partial x}-\beta U\right)\right|_{x=\delta+0}=\beta F(\delta,y,t), \tag{73}$$

where

$$F(x,y,t)=\left(\frac{1}{\beta}\frac{\partial U_0}{\partial x}-U_0\right), 0\le y\le b, \delta\le x\le\delta+l. \tag{74}$$

Then, similar as for the wall we can represent solution for the right fin in following form:

$$\begin{aligned}U(x,y,t)&=\int_\delta^{\delta+l}\int_0^1 U^0(\xi,\eta)G(x,y,\xi,\eta,t)d\eta d\xi\\ &-a^2\beta\int_0^t\int_0^b F(\delta,\eta,\tau)G(x,y,\delta,\eta,t-\tau)d\eta d\tau\end{aligned} \tag{75}$$

where Green function is:

$$G(x,y,\xi,\eta)=\sum_{n=1}^{\infty}G_m^x(x,\xi,t)\sum_{m=1}^{\infty}G_n^y(y,\eta,t),$$

$$G_i^x(x,\xi,t)=\frac{\phi_i(x)\phi_i(\xi)}{\|\phi_i\|^2}\cdot e^{-a^2\mu_i^2 t},$$

$$G_j^{(y)}(y,\eta,t) = \cdot e^{-a^2\lambda_j^2 t}\frac{\psi_j(y,\eta)}{2\|\psi_j\|^2},$$

$$\phi_i(x) = \cos[\mu_i(x-\delta)] + \frac{\beta}{\mu_i}\sin[\mu_i(x-\delta)],$$

$$\|\phi_i\|^2 = \frac{\beta}{\mu_i^2} + \frac{l}{2}\left(1+\frac{\beta^2}{\mu_i^2}\right),$$

$$\psi_j(y,\eta) = \cos[\lambda_j(y+\eta)] + \cos[\lambda_j(y-\eta)],$$

$$\|\psi_j\|^2 = \frac{1}{2}\left(b+\frac{\beta}{\lambda_j^2+\beta^2}\right).$$

Here $\mu_i, \lambda_j$ are the positive roots of the transcendental equations:

$$\tan(\mu_i l) = \frac{2\mu_i\beta}{\mu_i^2-\beta^2},$$

$$\tan(\lambda_j b) = \frac{\beta}{\lambda_j}.$$

Finally, we rewrite the conjugations conditions in the form of BC on the right side of the left rectangular fin:

$$\left.\left(\frac{\partial U_1}{\partial x} + \beta_0 U_1\right)\right|_{x=0+0} = \beta_0 F_2(0,y,t), \tag{76}$$

where

$$F_2(0,y,t) = \frac{1}{\beta_0}\frac{\partial U_0}{\partial x} + U_0,\ 0 \le y \le b. \tag{77}$$

Thus, solution for the left fin we can represent in following form:

$$\begin{aligned}U_1(x,y,t) &= \int_{-l_1}^{0}\int_0^b U_1^0(\xi,\eta) G_1(x,y,\xi,\eta,t)\,d\eta d\xi \\ &+ a^2\beta_0\int_0^t\int_0^b G_1(x,y,-l_1,\eta,t-\tau)\,d\eta d\tau \\ &+ a^2\beta_0\int_0^t\int_0^b F_2(0,\eta,\tau) G_1(x,y,0,\eta,t-\tau)\,d\eta d\tau \\ &+ a^2\beta_0\int_0^t\int_{-l_1}^0 G_1(x,y,\xi,b,t-\tau)\,d\xi d\tau\end{aligned} \tag{78}$$

where

$$G_1(x,y,\xi,\eta) = \sum_{m,n=1}^{\infty} G_{1,m}^x(x,\xi,t)\sum_{m=1}^{\infty} G_{1,n}^y(y,\eta,t),$$

$$G_{1,i}^x(x,\xi,t) = \frac{\varphi_{1,i}(x)\varphi_{1,i}(\xi)}{\|\varphi_{1,i}\|^2}\cdot e^{-a^2\mu_i^2 t},$$

$$G_{1,j}^{(y)}(y,\eta,t) = \frac{\psi_{1,j}(y,\eta)}{2\|\psi_{1,j}\|^2}\cdot e^{-a^2\lambda_n^2 t},$$

$$\varphi_{1,i}(x) = \cos[\mu_i(x+l_1)] + \frac{\beta_0}{\mu_i}\sin[\mu_i(x+l_1)],$$

$$\|\varphi_{1,i}\|^2 = \frac{\beta_0}{\mu_i^2} + \frac{l_1}{2}\left(1+\frac{\beta_0^2}{\mu_i^2}\right),$$

$$\psi_{1,j}(y,\eta) = \cos[\lambda_j(y-\eta)] + \cos[\lambda_j(y+\eta)],$$

$$\|\psi_{1,j}\|^2 = \frac{1}{2}\left(b+\frac{\beta_0}{\lambda_j^2+\beta_0^2}\right).$$

Here $\mu_i, \lambda_j$ are the positive roots of the transcendental equations:

$$\tan(\mu_i l_1) = \frac{2\mu_i\beta_0}{\mu_i^2-\beta_0^2},$$

$$\tan(\lambda_j b) = \frac{\beta_0}{\lambda_j}.$$

Using notation (75) and representation (71) we can easy obtain the following equation:

$$\begin{aligned}F_0(\delta,y,t) &= -a^2\int_0^t\int_0^b F(\delta,\eta,\tau)\Gamma(\delta,y,\delta,\eta,t-\tau)\,d\eta d\tau \\ &+ C_0(y,t),\end{aligned} \tag{79}$$

where

$$C_0(y,t) = \frac{1}{\beta}\int_\delta^{\delta+l}\int_0^1 U^0(\xi,\eta)\Gamma(x,y,\xi,\eta,t)\,d\eta d\xi,$$

$$\Gamma(x,y,\xi,\eta,t) = \left(\frac{\partial}{\partial x}+\beta\right)G(x,y,\xi,\eta,t).$$

In the similar way we find equation for $F_1(0,y,t)$ by using (69) and (78):

$$\begin{aligned}F_1(0,y,t) &= a^2\int_0^t\int_0^b F_2(0,\eta,\tau)\Gamma_1(0,y,0,\eta,t-\tau)\,d\eta d\tau \\ &+ C_1(y,t),\end{aligned} \tag{80}$$

where

$$\Gamma_1(x,y,\xi,\eta,t) = \left(\frac{\partial}{\partial x}-\beta_0\right)G_1(x,y,\xi,\eta,t),$$

$$\begin{aligned}C_1(y,t) &= \frac{1}{\beta_0}\int_{-l_1}^0\int_0^b U_1^0(\xi,\eta)\Gamma_1(0,y,\xi,\eta,t)\,d\xi d\eta \\ &+ a^2\int_0^t\int_0^b \Gamma_1(0,y,-l_1,\eta,t-\tau)\,d\eta d\tau \\ &+ a^2\int_0^t\int_{-l_1}^0 \Gamma_1(0,y,\xi,b,t-\tau)\,d\eta d\tau.\end{aligned}$$

Next,we find equation for $F(\delta,y,t)$ by using (74) and (72):

$$\begin{aligned}F(\delta,y,t) &= -a^2\frac{\beta_0}{\beta}\int_0^t\int_0^1 F_1(0,\eta,\tau)\Gamma_0(\delta,y,0,\eta,t-\tau)\,d\eta d\tau \\ & a^2\int_0^t\int_0^b F_0(\delta,\eta,\tau)\Gamma_0(\delta,y,\delta,\eta,t-\tau)\,d\eta d\tau + C(y,t)\end{aligned} \tag{81}$$

where

$$\Gamma_0(x,y,\xi,\eta,t) = \left(\frac{\partial}{\partial x}-\beta\right)G_0(x,y,\xi,\eta,t),$$

$$C(y,t) = \frac{1}{\beta}\int_0^\delta\int_0^1 U_0^0(\xi,\eta)\Gamma_0(\delta,y,\xi,\eta,t)\,d\eta d\xi.$$

Finally, using (77) and (72) we get equation for $F_2(0,y,t)$:

$$F_2(0,y,t) = -a^2\int_0^t\int_0^1 F_1(0,\eta,\tau)\Gamma_2(0,y,0,\eta,t-\tau)d\eta d\tau$$
$$+a^2\frac{\beta}{\beta_0}\int_0^t\int_0^b F_0(\delta,\eta,\tau)\Gamma_2(0,y,\delta,\eta,t-\tau)d\eta d\tau \quad (82)$$
$$+C_2(y,t),$$

where

$$\Gamma_2(x,y,\xi,\eta,t) = \left(\frac{\partial}{\partial x}+\beta_0\right)G_0(x,y,\xi,\eta,t),$$
$$C_2(y,t) = \frac{1}{\beta_0}\int_0^\delta\int_0^1 U_0^0(\xi,\eta)\Gamma_2(0,y,\xi,\eta,t)d\xi d\eta.$$

When a system of Fredholm integral equations of the second kind (58) – (61) is solved, we obtain the temperatures fields in the wall (72), left fin (78) and right fin (75).

## VI. Solution for the system of integral equations

This section provides the method that can be used to solve the system of integral equations (79)-(82). Let us denote unknown functions $F_0(\delta,y,t)$, $F_1(0,y,t)$, $F(\delta,y,t)$, $F_2(\delta,y,t)$ by the functions $y_1,y_2,y_3,y_4$:

$$y_1(y,t) = F_0(\delta,y,t),$$
$$y_2(y,t) = F_1(0,y,t),$$
$$y_3(y,t) = F(\delta,y,t),$$
$$y_4(y,t) = F_2(0,y,t).$$

Let us denote the kernels f integral equations in such way:

$$K_{1,3}(y,\eta,t-\tau) = -\Gamma(\delta,y,\delta,\eta,t-\tau),$$
$$K_{2,4}(y,\eta,t-\tau) = \Gamma_1(0,y,0_1,\eta,t-\tau),$$
$$K_{3,1}(y,\eta,t-\tau) = \Gamma_0(\delta,y,\delta,\eta,t-\tau),$$
$$K_{3,2}(y,\eta,t-\tau) = -\frac{\beta_0}{\beta}\Gamma_0(\delta,y,0,\eta,t-\tau),$$
$$K_{4,1}(y,\eta,t-\tau) = \frac{\beta_0}{\beta}\Gamma_2(0,y,\delta,\eta,t-\tau),$$
$$K_{4,2}(y,\eta,t-\tau) = -\Gamma_2(0,y,0,\eta,t-\tau).$$

Thus we can rewrite the system of integral equation in such way:

$$y_1(y,t) - a^2\int_0^t\int_0^b y_3(\eta,\tau)K_{1,3}(y,\eta,t-\tau)d\eta d\tau = C_0(y,t), \quad (83)$$
$$y_2(y,t) - a^2\int_0^t\int_0^b y_4(\eta,\tau)K_{2,4}(y,\eta,t-\tau)d\eta d\tau = C_1(y,t), \quad (84)$$
$$y_3(y,t) - a^2\int_0^t\int_0^b y_1(\eta,\tau)K_{3,1}(y,\eta,t-\tau)d\eta d\tau$$
$$-a^2\int_0^t\int_0^1 y_2(\eta,\tau)K_{3,2}(y,\eta,t-\tau)d\eta d\tau = C(y,t), \quad (85)$$
$$y_4(y,t) - a^2\int_0^t\int_0^b y_1(\eta,\tau)K_{4,1}(y,\eta,t-\tau)d\eta d\tau$$
$$-a^2\int_0^t\int_0^1 y_2(\eta,\tau)K_{4,2}(y,\eta,t-\tau)d\eta d\tau \quad (86)$$
$$= C_2(y,t).$$

On the assumption that the kernels $K_{4,2}(y,\eta,t-\tau), K_{3,2}(y,\eta,t-\tau)$ are continuous or square integrable on the square $[0\le y\le 1, 0\le\eta\le 1]$, other kernels $K_{1,3}(y,\eta,t-\tau), K_{2,4}(y,\eta,t-\tau), K_{3,1}(y,\eta,t-\tau), K_{4,1}(y,\eta,t-\tau)$ are continuous or square integrable on the square $[0\le y\le b, 0\le\eta\le b,]$ and the right-hand sides $C(y,t), C_0(y,t)C_1(y,t)C_2(y,t)$ are continuous or square integrable on $[0\le y\le 1]$ the theory for Fredholm equations of the second kind can be completely extended to such systems. Thus system of integral equations (83)-(86) can be solved by means of the method of successive approximations. To this end, one should use the recurrent formula:

$$y_1^n(y,t) = C_0(y,t) + a^2\int_0^t\int_0^b y_3^{n-1}(\eta,\tau)K_{1,3}(y,\eta,t-\tau)d\eta d\tau,$$
$$y_2^n(y,t) = C_1(y,t) + a^2\int_0^t\int_0^b y_4^{n-1}(\eta,\tau)K_{2,4}(y,\eta,t-\tau)d\eta d\tau,$$
$$y_3^n(y,t) = C(y,t) + a^2\int_0^t\int_0^1 y_2^{n-1}(\eta,\tau)K_{3,2}(y,\eta,t-\tau)d\eta d\tau$$
$$+a^2\int_0^t\int_0^b y_1^{n-1}(\eta,\tau)K_{3,1}(y,\eta,t-\tau)d\eta d\tau,$$
$$y_4^n(y,t) = C_2(y,t)$$
$$+a^2\int_0^t\int_0^1 y_2^{n-1}(\eta,\tau)K_{4,2}(y,\eta,t-\tau)d\eta d\tau$$
$$+a^2\int_0^t\int_0^b y_1^{n-1}(\eta,\tau)K_{4,1}(y,\eta,t-\tau)d\eta d\tau,$$
$$n = 1,2,\ldots$$

with the zeroth approximations:

$$y_1^0(y,t) = C_0(y,t),$$
$$y_2^0(y,t) = C_1(y,t),$$
$$y_3^0(y,t) = C(y,t),$$
$$y_4^0(y,t) = C_2(y,t).$$

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