Coordination of multi-agents with a revenue-cost-sharing mechanism: A cooperative game theory approach

Athanasios C. Karmperis, Konstantinos Aravossis, Anastasios Sotirchos, and Ilias P. Tatsiopoulos

Abstract— In this paper we focus on the coordination of multi-agents through a revenue-cost-sharing mechanism. We consider a grand-coalition consisting of finite agents, who undertake part of the costs individually, while the remaining costs C and the total revenues R are shared between them with a revenue-cost-sharing contract. We introduce a novel approach in the form of a cooperative game for a finite set of agents N and we estimate the finite set of possible solutions. Specifically, each of these solutions can be used for the coordination of the multi-agents, as it allocates the grand-coalition's profits and risks equally among them. A computation algorithm is developed and illustrated in a numerical example for the coordination of a grand-coalition with nine individual agents.

Keywords— coalitions, cooperative game, multi-agents, revenue-cost-sharing mechanism

I. Introduction

GLOBALLY, the development and exploitation of new products is implemented through contractual agreements, where at least two individual agents cooperate by forming a grand-coalition. The grand-coalition's coordination can be achieved under a centralized or decentralized scheme, with one or several decision makers respectively. However, the decentralized scheme is mostly used over the last decades, in which the contractual agreements may include a revenue-sharing or cost-sharing mechanism among the cooperative parties and each agent act in such a way that is optimal for the achievement of mutual targets. Specifically, these mechanisms are developed in order to coordinate all grand-coalition's members, so as both their individual objectives as well as the coalition-wide performance can be optimized [1] – [2]. In cases where the system's profits are equal to the relative profits arising through the centralized scheme, then the grand-coalition is perfectly coordinated [3]. However, taking into consideration that the coordination mechanism should be accepted by all the grand-coalition's members, it has to be fair allocating equally the total profits and risks among all agents.

Generally, in such multi-person situations, where individual decision makers examine to form a grand-coalition with a revenue-cost-sharing mechanism and the total outcome is influenced by each agent's outcome, game theory can be effectively applied [4]. Herein, we focus on a system consisting of multi-agents, who agree to undertake part of the grand-coalition's cost individually, while the remaining costs and the total revenues should be shared properly, in order to get all agents equal profits under an equal risk allocation scheme. We develop a basic model and we use cooperative game theory, in order to estimate the number of possible solutions and to introduce a computation algorithm that can be used by the individual decision makers. The rest of this paper, which is closely related to the multi-agent coordination and the quantitative risk analysis, is organized as follows. The review of the literature is presented in section II and the basic model is developed in section III. In section IV we present a computation algorithm, which is illustrated in a numerical example in section V, while useful conclusions and the future research issues are also discussed.

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II. Literature Review

A. Revenue Sharing and Cost Sharing Contracts

The design of revenue-sharing contracts as well as the negotiation process between the cooperative parties is presented in [5], while [6] examine the revenue-sharing in a supply network formation and propose some feasible allocation rules that ensure the positive profit for the networks' enterprises. A profit-sharing model for the coordination of a decentralized supply chain is developed in [7], [8] demonstrates that a system can be coordinated with a properly designed profit-sharing contract and [9] presents a profit-sharing and transfer pricing framework for the network companies. Furthermore, in [10] is analyzed the coordination that is achieved through a revenue-sharing contract, while in [11] is proposed a revenue-sharing contract that guarantees a win-win outcome and suggest that the revenue-sharing ratio can be settled through negotiation. Moreover, [2] indicate that the system's coordination can always be achieved through properly designed buy-back and lost-sales cost-sharing contracts, while in [12] is presented a method for the funding sharing between two cooperative parties and in [13] is developed a formula that increases the financial sustainability of partnerships in Greece.

However, taking into consideration that the coordination mechanisms should use risk as driver, as mentioned in [14], it is excluded the correlation between the revenue/cost-sharing contracts with the system's risks, which are shared between the cooperative parties. Generally, according to [15], the firms collaborate in order to have an efficient risk-sharing, as approximately 96% of the US ventures include the risk-sharing between partners, while [16] mention that the risk-sharing should be preferred and [17] suggest that the shared profit among agents should be proportional to their investment and risk taking. Moreover, there are several methods proposed in the literature for the quantitative risk analysis. In a decentralized case, where the decision makers want to estimate the expected profits, one of the mostly used methods is the Monte Carlo simulation that takes into account the impact of the system's variables [18] – [19], and defines the possible magnitude of the profits, graphically expressed as the cumulative probability distribution function [20] – [21].

B. Cooperative Game Theory for the System Coordination

The applications of cooperative game theory to the multi-agent systems, focusing on the profit allocation and stability, are surveyed in [4] and also examined in [22] – [23] – [24]. Generally, game theory is applied in a finite set of agents $N = \{1,2,3,\dots,n\}$, namely grand-coalition. Moreover, any subset in which this set can be divided is usually called a coalition [25], and any coalition with just one agent is called a singleton coalition [26]. A cooperative game is a pair (N, u) where u is the characteristic function representing the collective payoff for a set of agents that form a coalition [27]. The game's solution is a vector $x \in \mathbb{R}^N$ representing the allocation of the total profit to each agent. A formal solution for the cooperative bargaining process was first introduced by Nash [28], namely Nash-bargaining solution, which consists of an axiomatic derivation of the solution for a bargaining game between two agents, who have perfect information [29] and examine to cooperate and share the profits. The solution satisfies a set of axioms that is symmetry, Pareto-optimality and feasibility, i.e. identical agents receive identical profit allocations, any change to a different allocation that makes at least one agent better off will make at least one of the other agents worse off, and the sums of the agents' allocations do not exceed the total pie. Additionally, the solution is preserved under linear transformations and is independent of irrelevant alternatives.

However, especially in the decentralized systems, where there are individual agents who cooperate by undertaking different tasks, main challenge is to estimate a fair solution, in order to allocate the grand-coalition's profits and risks equally among the cooperative parties.

III. The Basic Model

We focus on a decentralized system with a finite set of agents $N = \{1,2,3,\dots,n\}$ that is the grand-coalition. These agents agree to cooperate by undertaking part of the grand-coalition's cost individually, i.e. the costs c_1, c_2, c_3, …., and c_n, are undertaken by agents 1,2,3,…., and n, respectively. Furthermore, the grand-coalition's remaining costs C and revenues R are shared between all agents, through a revenue-cost-sharing mechanism. Let P_i denote the profit allocated to each agent. A complete list of the notations used in this paper is presented in Table 1.

TABLE I
LIST OF NOTATIONS

Symbol	Description	
N	Finite set of agents (grand-coalition)	$N = \{1,2,3,\dots,n\}$
c_i	Cost undertaken by agent i	$i \in N$, i.e. $i = 1,2,\dots,n$
R	Grand-coalition's revenues	(to be divided in all agents)
C	Grand-coalition's remaining cost	(to be divided in all agents)
R_i	Revenue-sharing ratio of agent i	$\mathrm{R}_i \in (0,1)$, $\sum_{i=1}^{n} \mathrm{R}_i = 1$
C_i	Cost-sharing ratio of agent i	$\sum_{i=1}^{n} \mathrm{C}_i = 1$
P_i	Profit allocated to agent i	$\mathrm{P}_i = R\mathrm{R}_i - C\mathrm{C}_i - c_i$, $\forall i \in \mathrm{N}$
Π	Probability density functions: R, C	$\Pi_R(\mu_R,\sigma_R^2)$, $\Pi_C(\mu_C,\sigma_C^2)$
P_i	Profit's probability distribution function	$P_i = (\Pi_R \mathrm{R}_i, \Pi_C \mathrm{C}_i, c_i)$ $\forall i \in N$
μ_i	Expected profit	$\mu_i = \mu_R \mathrm{R}_i - \mu_C \mathrm{C}_i - c_i$, $\forall i \in N$
Confidence Intervals	probability : $(\mu_i - \sigma_i \le \mathrm{P}_i \le \mu_i + \sigma_i) \approx 0.6827$ probability : $(\mu_i - 2\sigma_i \le \mathrm{P}_i \le \mu_i + 2\sigma_i) \approx 0.9545$ probability : $(\mu_i - 3\sigma_i \le \mathrm{P}_i \le \mu_i + 3\sigma_i) \approx 0.9973$	
Equal profit and risk allocation	$P_1(\mu_1,\sigma_1^2) \equiv P_2(\mu_2,\sigma_2^2) \equiv \dots \equiv P_n(\mu_n,\sigma_n^2) \Leftrightarrow$ $\mu_1 = \mu_2 = \dots = \mu_n$, $\sigma_1 = \sigma_2 = \dots = \sigma_n$	
$s(n)$	Number of possible solutions	$j = 1,2,3,\dots,s(n)$
S	Finite set of possible solutions	$S = \{1,2,3,\dots,s(n)\}$
r, c $\in \mathbb{R}^N$	Solutions for equal profit, risk allocation (pairs of vectors)	$\mathrm{r}^j = (\mathrm{R}_1^j, \mathrm{R}_2^j, \dots, \mathrm{R}_n^j)$, $\mathrm{c}^j = (\mathrm{C}_1^j, \mathrm{C}_2^j, \dots, \mathrm{C}_n^j)$, $\forall j \in S$

Obviously, the revenue-cost-sharing mechanism has to be feasible and individually rational, i.e. the sum of the agents' allocations does not exceed the total pie and each agent gets at least as much as what it could obtain through the non-cooperative option:

$$\mathrm{R}_i \in (0,1) \text{ and } \sum_{i=1}^{n} \mathrm{R}_i = \sum_{i=1}^{n} \mathrm{C}_i = 1 \tag{1}$$

$$\mathrm{P}_i = R\,(\mathrm{R}_i) - C\,(\mathrm{C}_i) - c_i > 0, \quad \forall i \in N \tag{2}$$

We assume that there is full information among agents and we examine the case where the grand-coalition's profits should be shared equally and be proportional to each agent's investment and risk taking [17]. Generally, if there is no probability distribution assigned to the revenues R and costs C to be shared, yields the following proposition:

Proposition 1. *For a finite grand-coalition N, with i=1,2,...,n agents, there are infinite revenue-cost-sharing ratios* $(\mathrm{R}_i),(\mathrm{C}_i)$ *which define the equal profit allocation among all agents.*

Proof of Proposition 1. The profit for each agent i = 1,2,3,…,n who is member in a grand-coalition N is given:

$$\mathrm{P}_i = R\mathrm{R}_i - C\mathrm{C}_i - c_i \tag{3}$$

Obviously, there are infinite revenue-cost-sharing ratios $(\mathrm{R}_i),(\mathrm{C}_i)$, because even in the simplest case with two agents (n = 2), the grand-coalition profit P_N is calculated with (4):

$$\mathrm{P}_N = R - C - c_1 - c_2 \tag{4}$$

In order to estimate the agents' revenue-cost-sharing ratios: $(\mathrm{R}_1,\mathrm{R}_2),(\mathrm{C}_1,\mathrm{C}_2)$, with which the grand-coalition profits are allocated equally among them, we have a system of four equations with four unknowns $(\mathrm{R}_1,\mathrm{R}_2),(\mathrm{C}_1,\mathrm{C}_2)$, as follows:

$$\mathrm{P}_1 = R\mathrm{R}_1 - C\mathrm{C}_1 - c_1 = \frac{\mathrm{P}_N}{2} \tag{5}$$

$$\mathrm{P}_2 = R\mathrm{R}_2 - C\mathrm{C}_2 - c_2 = \frac{\mathrm{P}_N}{2} \tag{6}$$

$$\mathrm{R}_1 + \mathrm{R}_2 = \mathrm{C}_1 + \mathrm{C}_2 = 1 \tag{7}$$

Summarizing (5) with (6) we get:

$$R(\mathrm{R}_1 + \mathrm{R}_2) - C(\mathrm{C}_1 + \mathrm{C}_2) - (c_1 + c_2) \overset{(4)}{\underset{(7)}{=}} \mathrm{P}_N \Leftrightarrow 0 = 0$$

Moreover, we mention that in all other cases, where $n > 2$, we have to calculate $2n$ unknowns, while the available equations will be: $n + 2 < 2\,n$.

However, in order to define the system's risks and to allocate them with fairness among agents, the costs and revenues to be shared are normally distributed with specific mean value μ and variance σ^2. That is, a different probability distribution function Π is assigned in the grand-coalition's revenues: Π_R (μ_R, σ_R^2) and the shared costs: Π_C (μ_C, σ_C^2). According to these distributions, we get the following proposition.

Proposition 2. *The grand-coalition's profits and risks are allocated equally when all agents' profits get equal probability distribution functions, satisfying* (8):

$$\begin{aligned} &\mu_1 = \mu_2 = \mu_3 = ... = \mu_n \\ &\sigma_1 = \sigma_2 = \sigma_3 = ... = \sigma_n \end{aligned} \tag{8}$$

Proof of Proposition 2. Taking into consideration that both the grand-coalition's revenues R and shared costs C are normally distributed, the profits P_1, P_2, P_3 ,…., and P_n , which are allocated to agents 1,2,3,…., and n, respectively, follow normal probability distribution functions:

$$P_1(\mu_1,\sigma_1^2) = \left(1/\sqrt{2\pi}\,\sigma_1\right)e^{-\frac{(P_1-\mu_1)^2}{2\sigma_1^2}},$$

$$P_2(\mu_2,\sigma_2^2) = \left(1/\sqrt{2\pi}\sigma_2\right)e^{-\frac{(P_2-\mu_2)^2}{2\sigma_2^2}},........,$$

$$P_n(\mu_n,\sigma_n^2) = \left(1/\sqrt{2\pi}\,\sigma_n\right)e^{-\frac{(P_n-\mu_n)^2}{2\sigma_n^2}}$$ for agent i = 1,2,.., and n, respectively.

From these functions, the profits and risks are allocated equally when: $P_1(\mu_1,\sigma_1^2) \equiv P_2(\mu_2,\sigma_2^2) \equiv ... \equiv P_n(\mu_n,\sigma_n^2) \Leftrightarrow$ $\mu_1 = \mu_2 = ... = \mu_n$ and $\sigma_1 = \sigma_2 = ... = \sigma_n$

IV. Equal Profit and Risk Allocation among All Agents

In this section we use insights from the cooperative game theory, in order to estimate the possible solutions for the agents' revenue-sharing and cost-sharing ratios, which define the equal profit and risk allocation among them. Initially, we examine the case with 2 agents and further the cases where the grand-coalition consists of $n > 2$ agents.

A. Grand-Coalition with 2 Agents

In cases where the grand-coalition consists of two agents, i.e. N = {1,2}, the system's profits and risks are allocated equally when (8) is satisfied: $P_1(\mu_1,\sigma_1^2) \equiv P_2(\mu_2,\sigma_2^2) \Leftrightarrow$ $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$. Moreover, we derive Theorem 1.

Theorem 1. *There is a unique solution:* $(\mathrm{R}_1^*,\mathrm{R}_2^*),(\mathrm{C}_1^*,\mathrm{C}_2^*)$, *with which the system's profits and risks are allocated equally among two* agents.

Proof of Theorem 1. The probability distribution functions of the agents' 1 and 2 profits are given by: $P_1 = \Pi_R\mathrm{R}_1 - \Pi_C\mathrm{C}_1 - c_1$ and $P_2 = \Pi_R\mathrm{R}_2 - \Pi_C\mathrm{C}_2 - c_2$.

According to Proposition 2, the mean values and the profit values in the same confidence interval, e.g. the $(\mu \pm 2\sigma)$ should be equal.

Due to the fact that R_1 + R_2 = 1 and C_1 + C_2 = 1 from (1), where $\mathrm{R}_1,\mathrm{R}_2 \in (0,1)$, we define that $\mathrm{C}_1,\mathrm{C}_2$ can take negative or higher than 1 values, with respect to (1) and (2), so as to ensure that there is at least one solution $(\mathrm{R}_1^*,\mathrm{R}_2^*),(\mathrm{C}_1^*,\mathrm{C}_2^*)$, with which $\mu_1 = \mu_2, \sigma_1 = \sigma_2$. Furthermore, in order to prove the uniqueness of this solution, we suppose that there is another solution, denoted by $\left(R_1^{**}, R_2^{**}\right), \left(C_1^{**}, C_2^{**}\right)$, which also satisfies (8). Particularly, at least one of $R_1^* \neq R_1^{**}$, or $R_2^* \neq R_2^{**}$, or $C_1^* \neq C_1^{**}$, or $C_2^* \neq C_2^{**}$, while in both cases there is:

$$R_1^* + R_2^* = R_1^{**} + R_2^{**} = C_1^* + C_2^* = C_1^{**} + C_2^{**} = 1 \tag{9}$$

We consider that both $\left(R_1^*, R_2^*\right), \left(C_1^*, C_2^*\right)$ and $\left(R_1^{**}, R_2^{**}\right), \left(C_1^{**}, C_2^{**}\right)$ solutions satisfy (10), (12) and (11), (13) respectively:

$$\mu_1 = \mu_2 \Leftrightarrow \mu_R R_1^* - \mu_C C_1^* - c_1 = \mu_R R_2^* - \mu_C C_2^* - c_2 \tag{10}$$

$$\mu_1 = \mu_2 \Leftrightarrow \mu_R R_1^{**} - \mu_C C_1^{**} - c_1 = \mu_R R_2^{**} - \mu_C C_2^{**} - c_2 \tag{11}$$

$$\mu_1 \pm 2\sigma_1 = \mu_2 \pm 2\sigma_2 \Leftrightarrow \Pi_R R_1^* - \Pi_C C_1^* - c_1 = \Pi_R R_2^* - \Pi_C C_2^* - c_2 \tag{12}$$

$$\begin{gathered} \mu_1 \pm 2\sigma_1 = \mu_2 \pm 2\sigma_2 \Leftrightarrow \\ \Pi_R R_1^{**} - \Pi_C C_1^{**} - c_1 = \Pi_R R_2^{**} - \Pi_C C_2^{**} - c_2 \end{gathered} \tag{13}$$

From (10) minus (11) we get:

$$\begin{gathered} (10) - (11) \Leftrightarrow \\ \mu_R\left(R_1^* - R_2^* - R_1^{**} + R_2^{**}\right) - \mu_C\left(C_1^* - C_2^* + C_2^{**} - C_1^{**}\right) = 0 \end{gathered} \tag{14}$$

and from (12) minus (13):

$$\begin{gathered} (12) - (13) \Leftrightarrow \\ \Pi_R\left(R_1^* - R_2^* - R_1^{**} + R_2^{**}\right) - \Pi_C\left(C_1^* - C_2^* + C_2^{**} - C_1^{**}\right) = 0 \end{gathered} \tag{15}$$

There is $\mu_R > \mu_C, \Pi_R > \Pi_C$ and thus both the parentheses in (14) as well as in (15) equal zero:

$$\begin{aligned} &R_1^* - R_2^* - R_1^{**} + R_2^{**} = 0 \overset{(9)}{\Leftrightarrow} 1 - R_2^* - R_2^* - 1 + R_2^{**} + R_2^{**} = 0 \\ &C_1^* - C_2^* - C_1^{**} + C_2^{**} = 0 \quad\Leftrightarrow\quad 1 - C_2^* - C_2^* - 1 + C_2^{**} + C_2^{**} = 0 \end{aligned} \Leftrightarrow$$

$$\begin{aligned} &R_2^{**} = R_2^* \overset{(9)}{\Leftrightarrow} R_1^{**} = R_1^* \\ &C_2^{**} = C_2^* \Leftrightarrow C_1^{**} = C_1^* \end{aligned}$$

Therefore, the second solution is equal to the first: $R_1^* = R_1^{**}, R_2^* = R_2^{**}, C_1^* = C_1^{**}, C_2^* = C_2^{**}$ and there is a unique solution: $\left(R_1^*, R_2^*\right), \left(C_1^*, C_2^*\right)$ for the equal profit and risk allocation among agents 1 and 2.

B. Grand-Coalition with n > 2 Agents

In order to find the solution/s that allocates the grand-coalition's profits and risks equally among all agents, we use a cooperative game theory approach. Specifically, the grand-coalition is divided in two coalitions, namely: $N_A = \{1,2,\ldots,h\}$ $N_B = \{h+1,h+2,\ldots,n\}$, with: $1 \le h < n$. Due to (2), there is no coalition that can be profitably blocked by any coalition of agents. Hence, there is no constraint considered for the division of the agents, i.e. any agent can be placed either in the N_A or in the N_B coalition. However, there is: $R_A + R_B = C_A + C_B = 1$.

Further, we derive Theorems 2 and 3.

Theorem 2. *For each pair of non-empty coalitions N_A, N_B, that the grand-coalition N can be divided, there is a unique solution: $\left(R_A^*, R_B^*\right), \left(C_A^*, C_B^*\right)$, with which the system's profits and risks are allocated equally among all agents.*

Proof of Theorem 2. In order to demonstrate that the grand-coalition's profits and risks are allocated equally, the mean values and the profit values in the confidence intervals: $(\mu \pm \sigma)$, $(\mu \pm 2\sigma)$ and $(\mu \pm 3\sigma)$ should be equal for all agents. We mention that these confidence intervals include the profit values for each agent with 68.27%, 95.45% and 99.73% probability, respectively. However, the probability distribution functions of the N_A, N_B coalitions, are given by:

$$P_A = \Pi_R R_A - \Pi_C C_A - \sum_{a=1}^{h} c_a \tag{16}$$

$$P_B = \Pi_R R_B - \Pi_C C_B - \sum_{b=h+1}^{n} c_b \tag{17}$$

According to Proposition 2, the mean values and the profit values in the same confidence interval, e.g. the $(\mu \pm \sigma)$, or the $(\mu \pm 2\sigma)$, or the $(\mu \pm 3\sigma)$, should be equal. Similarly with the proof of Theorem 1, we suppose that there are two solutions: $\left(R_A^*, R_B^*\right), \left(C_A^*, C_B^*\right)$ and $\left(R_A^{**}, R_B^{**}\right), \left(C_A^{**}, C_B^{**}\right)$, which satisfy the following equations (18) and (19):

$$\begin{gathered} \mu_A = h(\mu_i),\ \mu_B = (n-h)(\mu_i) \Leftrightarrow \mu_A = \frac{h}{n-h}\mu_B \Leftrightarrow \\ (\mu_R R_A^* - \mu_C C_A^* - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\mu_R R_B^* - \mu_C C_B^* - \sum_{b=h+1}^{n} c_b) = \\ (\mu_R R_A^{**} - \mu_C C_A^{**} - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\mu_R R_B^{**} - \mu_C C_B^{**} - \sum_{b=h+1}^{n} c_b) = 0 \end{gathered} \tag{18}$$

$$\begin{gathered} (\Pi_R R_A^* - \Pi_C C_A^* - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\Pi_R R_B^* - \Pi_C C_B^* - \sum_{b=h+1}^{n} c_b) = \\ (\Pi_R R_A^{**} - \Pi_C C_A^{**} - \sum_{a=1}^{h} c_a) - \frac{h}{n-h}(\Pi_R R_B^{**} - \Pi_C C_B^{**} - \sum_{b=h+1}^{n} c_b) = 0 \end{gathered} \tag{19}$$

Following the same concept with the proof of Theorem 1, we solve (18), (19) and we get: $R_A^* = R_A^{**}, R_B^* = R_B^{**}$, $C_A^* = C_A^{**}, C_B^* = C_B^{**}$. Hence, there is a unique solution: $\left(R_A^*, R_B^*\right), \left(C_A^*, C_B^*\right)$ for each pair of non-empty coalitions N_A, N_B, that the grand-coalition N can be divided, with which the system's profits and risks are allocated equally among all agents.

Theorem 3. *The number of possible solutions* $s(n)$, *with which the system's profits and risks are allocated equally among all agents, is equal to the combinations of agents divided in pairs of non-empty coalitions iteratively, until all are divided in singleton coalitions:* $\{\{1\},\{2\},\ldots,\{n\}\}$. *Specifically, the number of possible solutions:* $s(n)$, $\forall n \geq 2$ *, is given from* (20) *and* (21), *whether n is odd or even number respectively:*

$$s(n) = \frac{n!}{(n-1)!}s(n-1) + \sum_{k=2}^{\frac{n-1}{2}} \frac{n!}{(n-k)!(k)!}s(n-k)s(k) \tag{20}$$

$$s(n) = \frac{n!}{(n-1)!}s(n-1) + \sum_{k=2}^{\frac{n}{2}-1} \frac{n!}{(n-k)!(k)!}s(n-k)s(k) + \frac{n!}{(\frac{n}{2}!)^2}(s(\frac{n}{2}))^2(\frac{1}{2}) \tag{21}$$

Proof of Theorem 3. According to Theorem 2, for each pair of non-empty coalitions N_A, N_B, that the grand-coalition N can be divided, there is a unique solution: $\left(R_A^*, R_B^*\right),\left(C_A^*, C_B^*\right)$. However, if we consider the further division of the N_A coalition that includes h agents in another pair of coalitions, namely N_{AA} and N_{AB}, where N_{AA} consists of t agents and N_{AB} consists of h-t agents, then according to Theorem 2, there is a unique solution: $\left(R_{AA}^*, R_{AB}^*\right),\left(C_{AA}^*, C_{AB}^*\right)$. Moreover, the rest agents n-h of the N_B coalition can also be divided further in two other coalitions, namely N_{BA}, N_{BB}, and with the same concept there is another unique solution: $\left(R_{BA}^*, R_{BB}^*\right),\left(C_{BA}^*, C_{BB}^*\right)$. We consider the iterative divisions in pairs of coalitions, until all the n agents of the grand-coalition are divided in singleton coalitions: $\{\{1\},\{2\},\ldots,\{n\}\}$. Taking into account that for each coalition considered there is a unique solution, we conclude that there is a unique solution for each agent who is included in specific coalitions. This is calculated when all the solutions of the coalitions including him are multiplied. For instance, the solution for agent i, who is included in N_B, N_{BA}, N_{BAA}, N_{BAAB} coalitions, is given: $R_i = (R_B)(R_{BA})(R_{BAA})(R_{BAAB})$, and $C_i = (C_B)(C_{BA})(C_{BAA})(C_{BAAB})$.

Obviously, each time we consider the division of a set N in a pair of coalitions with h and n-h agents respectively, where $1 \leq h < n$, there are alternative possible combinations of the agents in the coalitions and each combination results in a unique system's solution: $\left(R_1^*, R_2^*, \ldots, R_n^*\right),\left(C_1^*, C_2^*, \ldots, C_n^*\right)$. Thus, the number of the system's possible solutions, with which the grand-coalition's profits and risks are allocated equally among agents, is equal to the number of possible combinations of the agents in the coalitions, until all agents are divided in a singleton coalition. Let $s(n)$ denote the number of solutions for a grand-coalition with $n \geq 2$ agents. We mention that $s(n)$ is equal to the sum of possible combinations of the n agents in coalitions, which is multiplied with the number of possible solutions for the specific coalitions. That is, $s(n)$ increases with the number of agents, as for a grand-coalition with 3 agents, i.e. $N = \{1,2,3\}$, there are 3 combinations in a 2-agent coalition:

1) $\{\{1,2\}=\{2,1\},\{3\}\}$ that gives a unique solution: $\left(R_1^{*1}, R_2^{*1}, R_3^{*1}\right),\left(C_1^{*1}, C_2^{*1}, C_3^{*1}\right)$
2) $\{\{1,3\}=\{3,1\},\{2\}\}$ that gives another solution: $\left(R_1^{*2}, R_2^{*2}, R_3^{*2}\right),\left(C_1^{*2}, C_2^{*2}, C_3^{*2}\right)$
3) $\{\{2,3\}=\{3,2\},\{1\}\}$ that gives another solution: $\left(R_1^{*3}, R_2^{*3}, R_3^{*3}\right),\left(C_1^{*3}, C_2^{*3}, C_3^{*3}\right)$

Each of these solutions allocates the grand-coalition's profits and risks equally among all agents, i.e. the $s(3)=3$. Particularly, the mean values are equal for all agents in all solutions: $\mu_i^j = \mu_N / n$, $\forall i = 1,2,3$, $j = 1,2,3$ while the standard deviations are also equal for all agents in all solutions: σ_i^j , $\forall j = 1,2,3$, $i = 1,2,3$ even though the revenue-sharing and cost-sharing ratios are different between these three solutions.

Moreover, for a grand-coalition with 4 agents, i.e. $N = \{1,2,3,4\}$, there are $\frac{4!}{3!(4-3)!} = 4$ combinations of the 4 agents in a 3-agent coalition: $\{\{1,2,3\},\{4\}\}$, $\{\{1,2,4\},\{3\}\}$, $\{\{1,3,4\},\{2\}\}$ and $\{\{2,3,4\},\{1\}\}$, where each one has 3 possible solutions, while there are also three combinations of the 4 agents in a 2-agent coalition respectively: $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$ and each one has 1 solution, thus: $s(4) = 4s(3) + 3s(2) \Leftrightarrow s(4) = 4(3) + 3(1) = 15$. Furthermore, the number of possible solutions, with which the system's profits and risks are allocated equally among all agents, is calculated with (20) and (21), whether n is odd or even number respectively.

C. *Computation of Possible Solutions*

Taking into consideration that the grand-coalition is a finite set of agents and the number of possible solutions is calculated with (20) and (21), whether n is odd or even number respectively, we conclude that there are finite possible solutions that define the equal profit and risk allocation among agents. However, the number of possible solutions is rapidly increased with the number of agents. Particularly, from the Proof of Theorem 1 we get $s(2)=1$ and from the Proof of Theorem 3, we get $s(3)=3$ and $s(4)=15$ and thus, for $n=5,6,\ldots,10$ we compute:

$$s(5) = \frac{5!}{4!}s(4) + \frac{5!}{3!2!}s(3)s(2) = 5(15) + 10(3)(1) = 105$$

$$s(6) = \frac{6!}{5!}s(5) + \frac{6!}{4!2!}s(4)s(2) + \frac{6!}{(3!)^2}(s(3))^2(\frac{1}{2}) = 945$$

$$s(7) = \frac{7!}{6!}s(6) + \frac{7!}{5!2!}s(5)s(2) + \frac{7!}{4!3!}s(4)s(3) = 10{,}395$$

$$s(8)=\frac{8!}{7!}s(7)+\frac{8!}{6!2!}s(6)\mathrm{s}(2)+\frac{8!}{5!3!}s(5)s(3)+\frac{8!}{(4!)^2}(s(4))^2(\frac{1}{2})=$$
135,135

$$s(9)=\frac{9!}{8!}s(8)+\frac{9!}{7!2!}s(7)\mathrm{s}(2)+\frac{9!}{6!3!}s(6)s(3)+\frac{9!}{5!4!}s(5)s(4)=$$
2,027,025

$$s(10)=\frac{10!}{9!}s(9)+\frac{10!}{8!2!}s(8)\mathrm{s}(2)+\frac{10!}{7!3!}s(7)s(3)+\frac{10!}{6!4!}s(6)s(4)+$$
$$\frac{10!}{(5!)^2}(s(5))^2(\frac{1}{2})=34{,}429{,}425$$

That is, in Fig. 1 we introduce a code that can be used in the Wolfram Research, Inc., Mathematica, Version 7.0, Champaign, IL (2008), for the calculation of the precise number of possible solutions.

```
In[1]:= s[n_] := Piecewise[
      {{n!/(n - 1)! s[n - 1] + Sum[(n!/((n - k)! k!) s[n - k] s[k]), {k, 2, (n - 1)/2}],
        (n - 1)/2 ∈ Integers && n ≥ 3},
       {n!/(n - 1)! s[n - 1] + Sum[(n!/((n - k)! k!) s[n - k] s[k]), {k, 2, n/2 - 1}] +
        n!/((n/2)!)^2 s[n/2]^2 1/2, n/2 ∈ Integers && n ≥ 4}}
      , 1]
     s[2]
Out[2]= 1
```

Fig. 1: Calculations of possible solutions with the Wolfram Mathematica, (2008).

Moreover, in Table II we illustrate the results arising for n = 2,3,4,…., and 25 and we derive the following Proposition 3.

Proposition 3. *The revenue-cost-sharing ratios for a finite grand-coalition with n agents, can be computed through the random division of the agents in pairs of coalitions iteratively for n -1 times (until all agents are divided in a singleton coalition), and the calculation of the unique revenue-cost-sharing solution for each coalition.*

Proof of Proposition 3. According to the proof of Theorem 2, there is a unique $\left(\mathrm{R}_A^*,\mathrm{R}_B^*\right)\left(\mathrm{C}_A^*,\mathrm{C}_B^*\right)$ solution for each pair of coalitions $N_A=\{1,2,..,h\}$, $N_B=\{h+1,h+2,..,n\}$, in which a grand-coalition N can be divided. We consider the division of the N in pairs of coalitions iteratively, where the first coalition in each pair consists of a singleton coalition, i.e. $h=1$. That is, the calculations in the first division, give: $\mathrm{R}_A^*=\mathrm{R}_1^*,\mathrm{C}_A^*=\mathrm{C}_1^*$, and $\mathrm{R}_B^*=\mathrm{R}_{n\text{-}1}^*,\mathrm{C}_B^*=\mathrm{C}_{n\text{-}1}^*$. In the second division, the coalition with the n-1agents is divided again, with agent 2 in the first and the n-2 agents in the second coalition respectively, in the third division the two coalitions consist of agent 3 and the n-3 agents respectively, etc. Specifically, the set N is divided in pairs of coalitions for n-1 times, until all agents are divided in a singleton coalition. However, the number of repeated divisions in pair of coalitions is the same: n-1 times, for all the possible combinations, while all the possible solutions $s(n)$ result in the same solution: $P_1=P_2=P_3=...=P_n$. Hence, we conclude that the revenue-cost-sharing solution for a finite grand-coalition N, can be estimated through the random division of the N's agents in pairs of coalitions iteratively for n-1 times and the calculation of the agents' revenue-cost-sharing ratios: $\left(\mathrm{R}_i\right),\left(\mathrm{C}_i\right)\ \forall i\in N,\ \ i=1,2,3,...,\mathrm{n}$.

TABLE II
POSSIBLE SOLUTIONS $s(n)$ FOR THE GRAND-COALITION $N=\{1,2,3,...n\}$

n	**s(*n*)**
n = 2	1
n = 3	3
n = 4	15
n = 5	105
n = 6	945
n = 7	10,395
n = 8	135,135
n = 9	2,027,025
n = 10	34,429,425
n = 11	654,729,075
n = 12	13,749,310,575
n = 13	31,623,414,225
n = 14	7,905,853,580,625
n = 15	213,458,046,676,875
n = 16	6,190,283,353,629,375
n = 17	191,898,783,962,510,625
n = 18	6,332,659,870,762,850,625
n = 19	221,643,095,476,699,771,875
n = 20	8,200,794,532,637,891,559,375
n = 21	319,830,986,772,877,770,815,625
n = 22	13,113,070,457,687,988,603,440,625
n = 23	563,862,029,680,583,509,947,946,875
n = 24	25,373,791,335,626,257,947,657,609,375
n = 25	1,192,568,192,774,434,123,539,907,640,625

V. COMPUTATION ALGORITHM

In this section we introduce a novel algorithm that can be used for the coordination of multi-agents. Particularly, the algorithm computes a specific solution that is fair, as the grand-coalition's profits and risks are allocated equally among all agents through a revenue-cost-sharing mechanism. The computation algorithm includes seven basic steps, as follows:

Step 1: Estimate the cost that is individually undertaken by each agent: c_1, c_2, c_2 ,..., c_n and assign a normal probability density function to the revenues R and costs C, which will be shared in all agents: $\Pi_R(\mu_R,\sigma_R{}^2)$, $\Pi_C(\mu_C,\sigma_C{}^2)$. Select a specific confidence interval: $(\mu\pm\sigma)$, or $(\mu\pm 2\sigma)$, or $(\mu\pm 3\sigma)$.

Step 2: Divide randomly the grand-coalition $N=\{1,2,3,\ldots,n\}$ in a pair of non-empty coalitions: $N_{\rm A}=\{1,2,..,h\}$, $N_{\rm B}=\{h+1,h+2,..,n\}$, with $1\le h\prec n$.

Step 3: Define the equations for the calculations of the profits:

$$\mathrm{P_A}=\Pi_R\mathrm{R_A}-\Pi_C\mathrm{C_A}-\sum_{a=1}^{h}c_a \tag{16}$$

$$\mathrm{P_B}=\Pi_R\mathrm{R_B}-\Pi_C\mathrm{C_B}-\sum_{b=h+1}^{n}c_b \tag{17}$$

Step 4: Develop a Monte Carlo simulation model, in which the Π_R,Π_C are defined as inputs and the $\mathrm{P_A},\mathrm{P_B}$ as outputs.

Step 5: Use equation (1) and examine alternative values of $\mathrm{R_A}\in(0,1)$. Particularly, for each value of $\mathrm{R_A}$, the $\mathrm{R_B}$ and the $\mathrm{C_A}$ as well as the $\mathrm{C_B}$ (which can be negative or higher than 1), take specific values, in order to satisfy: $\frac{\mu_{\rm A}}{\mu_{\rm B}}=\frac{h}{n\text{-}h}=\frac{\mu_R\mathrm{R^*_A}\text{-}\mu_C\mathrm{C^*_A}\text{-}\sum_{a=1}^{h}c_a}{\mu_R\mathrm{R^*_B}\text{-}\mu_C\mathrm{C^*_B}\text{-}\sum_{b=h+1}^{n}c_b}$. For each scenario $\mathrm{R_A},\mathrm{C_A},\mathrm{R_B},\mathrm{C_B}$ run the Monte Carlo simulation, analyze the probability distribution functions $P_{\rm A}$, $P_{\rm B}$ and calculate the coalitions' profits in the selected confidence interval: $(\mu_{\rm A}\pm\sigma_{\rm A})$ and $(\mu_{\rm B}\pm\sigma_{\rm B})$, or $(\mu_{\rm A}\pm 2\sigma_{\rm A})$ and $(\mu_{\rm B}\pm 2\sigma_{\rm B})$ or $(\mu_{\rm A}\pm 3\sigma_{\rm A})$ and $(\mu_{\rm B}\pm 3\sigma_{\rm B})$ Following the trial and error method, select the scenario $\mathrm{R_A},\mathrm{C_A},\mathrm{R_B},\mathrm{C_B}$ that satisfies the proportionality presented in equation (22):

$$\frac{h}{n\text{-}h}-\frac{\mu_{\rm A}}{\mu_{\rm B}}=0=\frac{h}{n\text{-}h}-\frac{(\mu_{\rm A}+2\sigma_{\rm A})-(\mu_{\rm A}-2\sigma_{\rm A})}{(\mu_{\rm B}+2\sigma_{\rm B})-(\mu_{\rm B}-2\sigma_{\rm B})}\Leftrightarrow$$

$$\frac{\mu_{\rm A}}{\mu_{\rm B}}=\frac{2\sigma_{\rm A}+2\sigma_{\rm A}}{2\sigma_{\rm B}+2\sigma_{\rm B}}\Leftrightarrow$$

$$\frac{\mu_{\rm A}}{\mu_{\rm B}}=\frac{\sigma_{\rm A}}{\sigma_{\rm B}} \tag{22}$$

This scenario is a solution: $(\mathrm{R^*_A},\mathrm{R^*_B}),(\mathrm{C^*_A},\mathrm{C^*_B})$, which is unique for the selected $N_{\rm A},N_{\rm B}$ coalitions.

Step 5: Use the $(\mathrm{R^*_A},\mathrm{R^*_B}),(\mathrm{C^*_A},\mathrm{C^*_B})$ solution and repeat the 2 to 4 Steps, i.e. divide randomly $N_{\rm A},N_{\rm B}$ in two other pairs of coalitions, namely $N_{\rm AA},N_{\rm AB}$ and $N_{\rm BA},N_{\rm BB}$ respectively. Specifically, the first pair of coalitions from $N_{\rm A}$ is the: $N_{\rm AA}=\{1,2,..,k\},N_{\rm AB}=\{k+1,k+2,..,h\}$, with $1\le k\prec h$ and the second pair of coalitions from $N_{\rm B}$ is the: $N_{\rm BA}=\{h+1,h+2,..,m\},N_{\rm BB}=\{m+1,m+2,..,n\}$, where the $h+1\le \mathrm{m}\prec n$. Further, calculate the unique scenarios: $(\mathrm{R^*_{AA}},\mathrm{R^*_{AB}}),(\mathrm{C^*_{AA}},\mathrm{C^*_{AB}})$ that satisfies $\frac{k}{h\text{ - }k}=\frac{\mu_{\rm AA}}{\mu_{\rm AB}}=\frac{\sigma_{\rm AA}}{\sigma_{\rm AB}}$, $(\mathrm{R^*_{BA}},\mathrm{R^*_{BB}}),(\mathrm{C^*_{BA}},\mathrm{C^*_{BB}})$ that satisfies $\frac{m\text{ - }h}{n\text{ - }m}=\frac{\mu_{\rm BA}}{\mu_{\rm BB}}=\frac{\sigma_{\rm BA}}{\sigma_{\rm BB}}$.

Repeat the above steps, until all agents are divided in a singleton coalition, i.e. the 2 to 4 steps should be followed for n-1 times.

Step 6: For each agent i = 1,2,3,…,n , multiply the revenue-cost-sharing ratios of the coalitions including him and get the system's solution: $(\mathrm{R^*_1},\mathrm{R^*_2},\mathrm{R^*_2},...,\mathrm{R^*_{\mathit{n}}})$ and $(\mathrm{C^*_1},\mathrm{C^*_2},\mathrm{C^*_3},\ldots,\mathrm{C^*_{\mathit{n}}})$.

For instance, the revenue-cost-sharing ratios $(\mathrm{R^*_5}),(\mathrm{C^*_5})$ for agent 5 who is included in the $N_{\rm A}$, $N_{\rm AA}$, $N_{\rm AAA}$ and $N_{\rm AAAB}$ coalitions, is calculated through:

$\mathrm{R^*_5}=(\mathrm{R^*_A})(\mathrm{R^*_{AA}})(\mathrm{R^*_{AAA}})(\mathrm{R^*_{AAAB}})$ and

$\mathrm{C^*_5}=(\mathrm{C_A})(\mathrm{C_{AA}})(\mathrm{C_{AAA}})(\mathrm{C_{AAAB}})$.

As can be seen, the grand-coalition's solution is a pair of vectors $\mathrm{r,c}\in\mathbb{R}^N$ that is:

$\mathrm{r}=(\mathrm{R^*_1},\mathrm{R^*_2},\mathrm{R^*_2},...,\mathrm{R^*_{\mathit{n}}})$, $\mathrm{c}=(\mathrm{C^*_1},\mathrm{C^*_2},\mathrm{C^*_2},...,\mathrm{C^*_{\mathit{n}}})$.

Step 7: Use: $\mathrm{R^*_1},\mathrm{R^*_2},\mathrm{R^*_2},...,\mathrm{R^*_{\mathit{n}}}$ and $\mathrm{C^*_1},\mathrm{C^*_2},\mathrm{C^*_3},\ldots,\mathrm{C^*_{\mathit{n}}}$ and run the Monte Carlo simulation model, with the agents' profits $\mathrm{P_1},\mathrm{P_2},\mathrm{P_3},....,\mathrm{P}_n$ as outputs, in order to verify that:

$$\mu_1=\mu_2=...=\mu_n,\quad \sigma_1=\sigma_2=...=\sigma_n\Leftrightarrow P_1\equiv P_2\equiv...\equiv P_n$$

VI. Numerical Example

A numerical example is presented here, in order to illustrate some significant features of the basic model as well as the application of the proposed computation algorithm.

A. Assumptions

We consider a grand-coalition with n=9 agents $N=\{1,2,3,4,5,6,7,8,9\}$, who examine to cooperate in a single project, by undertaking parts of the costs individually, while both remaining costs C and revenues R should be shared following a revenue-cost-sharing mechanism. Specifically, the costs c_1 = 900000, c_2 = 2550000, c_3 = 950000, c_4 = 880000, c_5 = 1200000, c_6 = 1050000, c_7 = 870000, c_8 = 1500000, and c_9 = 1200000, are undertaken by agent 1,2, 3, … , and 9, respectively. This system can be perfectly coordinated through the equal allocation of the grand-coalition's profits and risks to agents, i.e. all agents get equal profits with equal probability distributions. The shared revenues R and costs C are normally distributed: $\Pi_R(\mu_R=50000000,\sigma_R=4500000)$ and $\Pi_C(\mu_C=10000000,\sigma_C=4300000)$, respectively.

B. First Solution

Following the proposed algorithm, we divide the 9 agents in pairs of coalitions iteratively until all agents are divided in singleton coalitions, by following specific combinations, which are illustrated in Figure 2. According to equations (20) and (21) introduced in Theorem 3 as well as the calculations

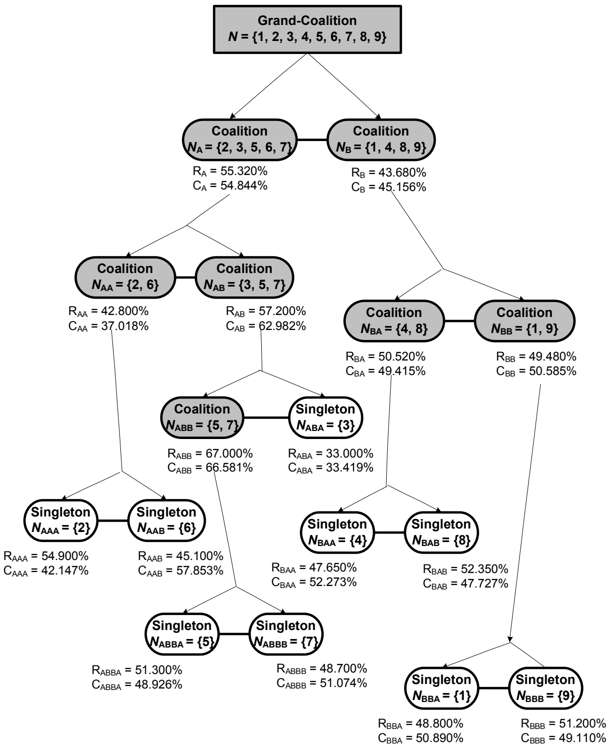

Fig. 2: first solution of the computation algorithm

presented in Table II, this set of combinations is one out of 2,027,025 possible sets that can be followed in this case (where n=9). Initially, following the 2nd to 4th Steps of the computation algorithm, we divide N in two coalitions: $N_A = \{2,3,5,6,7\}$, $N_B = \{1,4,8,9\}$ and we calculate the unique solution for the specific pair of coalitions:

TABLE III

FIRST SOLUTION OF THE REVENUE-COST-SHARING RATIOS $(R_i^1),(C_i^1)$

i	Calculations	R_i^{*1}	Calculations	C_i^{*1}
$i=1$	$R_B R_{BB} R_{BBA}$	10.54%	$C_B C_{BB} C_{BBA}$	11.62%
$i=2$	$R_A R_{AA} R_{AAA}$	13.23%	$C_A C_{AA} C_{AAA}$	8.55%
$i=3$	$R_A R_{AB} R_{ABA}$	10.63%	$C_A C_{AB} C_{ABA}$	11.54%
$i=4$	$R_B R_{BA} R_{BAA}$	10.51%	$C_B C_{BA} C_{BAA}$	11.66%
$i=5$	$R_A R_{AB} R_{ABB} R_{ABBA}$	11.07%	$C_A C_{AB} C_{ABB} C_{ABBA}$	11.25%
$i=6$	$R_A R_{AA} R_{AAB}$	10.87%	$C_A C_{AA} C_{AAB}$	11.74%
$i=7$	$R_A R_{AB} R_{ABB} R_{ABBB}$	10.51%	$C_A C_{AB} C_{ABB} C_{ABBB}$	11.74%
$i=8$	$R_B R_{BA} R_{BAB}$	11.55%	$C_B C_{BA} C_{BAB}$	10.64%
$i=9$	$R_B R_{BB} R_{BBB}$	11.06%	$C_B C_{BB} C_{BBB}$	11.21%
	$\sum_{i=1}^{9} R_i$	100.0%	$\sum_{i=1}^{9} C_i$	100.0%

$$\left(R_A^* = 55.320\%, R_B^* = 43.680\%\right), \left(C_A^* = 54.844\%, C_B^* = 45.156\%\right)$$

According to Proposition 3, we follow the basic Steps of the proposed algorithm (Steps 2 to 4), for n-1=8 times, in order to calculate the unique and optimum solutions for the selected set of combinations, which is illustrated in Figure 2.

Let: $\left(R_1^{*1}, R_2^{*1}, ..., R_9^{*1}\right), \left(C_1^{*1}, C_2^{*1}, ..., C_9^{*1}\right)$ denote the first revenue-cost-sharing solution, which is presented in Table III. As can be seen, the revenue-cost-sharing ratios for each agent are calculated with the multiplication of the revenue-cost-sharing ratios of the coalitions including him. For instance, the revenue-cost-sharing ratios for agent 7: $\left(R_7^{*1}\right), \left(C_7^{*1}\right)$, who is included in the N_A, N_{AB}, N_{ABB} and N_{ABBB} coalitions, is calculated through:

$$\begin{Bmatrix} R_7^{*1} = R_A R_{AB} R_{ABB} R_{ABBB} \\ C_7^{*1} = C_A C_{AB} C_{ABB} C_{ABBB} \end{Bmatrix} \Leftrightarrow$$

$$\begin{Bmatrix} R_7^{*1} = (55.320\%)(57.200\%)(67.000\%)(48.700\%), \\ C_7^{*1} = (54.844\%)(62.982\%)(66.581\%)(51.074\%) \end{Bmatrix} \Leftrightarrow$$

$$\left(R_7^{*1} = 10.5114\%\right), \left(C_7^{*1} = 11.7461\%\right)$$

Furthermore, according to Step 7 of the proposed algorithm, we verify the resulting revenue-cost-sharing ratios. Particularly, we use the $R_1^{*1}, R_2^{*1}, ..., R_9^{*1}$ and $C_1^{*1}, C_2^{*1}, ..., C_9^{*1}$ as well as the Π_R, Π_C and c_1, c_2, c_3, c_4, c_5, c_6, , c_8, and c_9 as inputs and the agents' profits: $P_1, P_2, P_3, ..., P_9$ as outputs in a Monte Carlo simulation model: $P_i = RR_i^1 - CC_i^1 - c_i$.

The simulation is performed with 5,000 runs calculating the probability distribution functions of the agents' profits, namely P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, and P_9, for the i = 1, 2, 3, 4, 5, 6, 7, 8, and 9 agent, respectively. These functions are illustrated together in Figure 3 and demonstrate that all agents get equal expected profits: $\mu_1 = \mu_2 = \mu_3 = ... = \mu_9 = 3{,}211{,}170$

Furthermore, we analyze the probability distribution functions in the same confidence intervals: $\mu_i \pm \sigma_i$, $\mu_i \pm 2\sigma_i$,

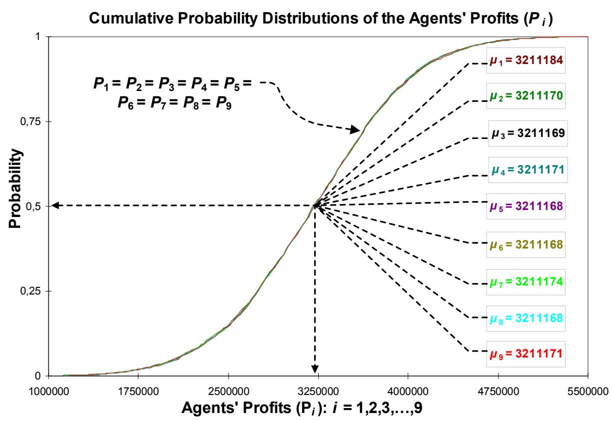

Fig. 3: cumulative probability distributions (first solution)

TABLE IV
FIRST SOLUTION: AGENTS' PROFITS IN THE CONFIDENCE INTERVALS

	$\mu_i \pm \sigma_i$	$\mu_i \pm 2\sigma_i$	$\mu_i \pm 3\sigma_i$
	Probability of Profit:		
Agent	$2{,}520{,}000 \le P_i \le 3{,}900{,}000$	$1{,}830{,}000 \le P_i \le 4{,}590{,}000$	$1{,}140{,}000 \le P_i \le 5{,}280{,}000$
$i = 1$	68.69%	95.69%	99.70%
$i = 2$	67.99%	95.46%	99.70%
$i = 3$	68.70%	95.70%	99.70%
$i = 4$	68.88%	95.71%	99.70%
$i = 5$	68.69%	95.53%	99.69%
$i = 6$	68.68%	95.14%	99.69%
$i = 7$	68.84%	95.69%	99.69%
$i = 8$	68.98%	95.64%	99.70%
$i = 9$	68.88%	95.62%	99.70%

and $\mu_i \pm 3\sigma_i$, as presented in Table IV. As can be seen, the probability for all agents' profits P_i (where i=1,2,3,..,9), to get values in the intervals: $[2520000,3900000]$, $[1830000,4590000]$, and $[1140000,5280000]$ is equal : 68.7%, 95.6%, and 99.7%, respectively, and thus, the first solution gives: $\sigma_1 = \sigma_2 = \sigma_3 = ... = \sigma_9$.
Hence, according to Proposition 2, all agents' profits have equal probability distributions: $P_1 \equiv P_2 \equiv P_1 \equiv ... \equiv P_9$ and the coordination of the grand-coalition $N = \{1,2,3,4,5,6,7,8,9\}$ can be achieved.

C. Second Solution

However, the first solution is one out of 2,027,025 solutions that can be calculated, according to the possible combinations for the divisions of the 9 agents in pairs of coalitions for n-1=8 times. For instance, if we use different combinations of the agents within the computation algorithm, as presented in Figure 4, we calculate a second revenue-cost-sharing solution: namely: $(R_1^{*2}, R_2^{*2},..., R_9^{*2})$ and $(C_1^{*2}, C_2^{*2},..., C_9^{*2})$. According to Theorem 3, we expect that the first and second solutions are different. This fact is verified through the results presented in Tables III and V, which present the agents' revenue-cost-sharing ratios in the first and second solution, respectively. As can be seen, these revenue-cost-sharing ratios are different for all agents in the two solutions:

$$\left(R_i^{*1}\right) \neq \left(R_i^{*2}\right), \text{ and } \left(C_i^{*1}\right) \neq \left(C_i^{*2}\right), \quad \forall i \in N, (i = 1,2,3,..., 9)$$

For instance, the revenue-cost-sharing ratios for agent 3, (i=3):

$R_3^{*1} = (10.63\%) \neq R_3^{*2} = (10.65\%)$, and

$C_3^{*1} = (11.54\%) \neq C_3^{*2} = (11.64\%)$

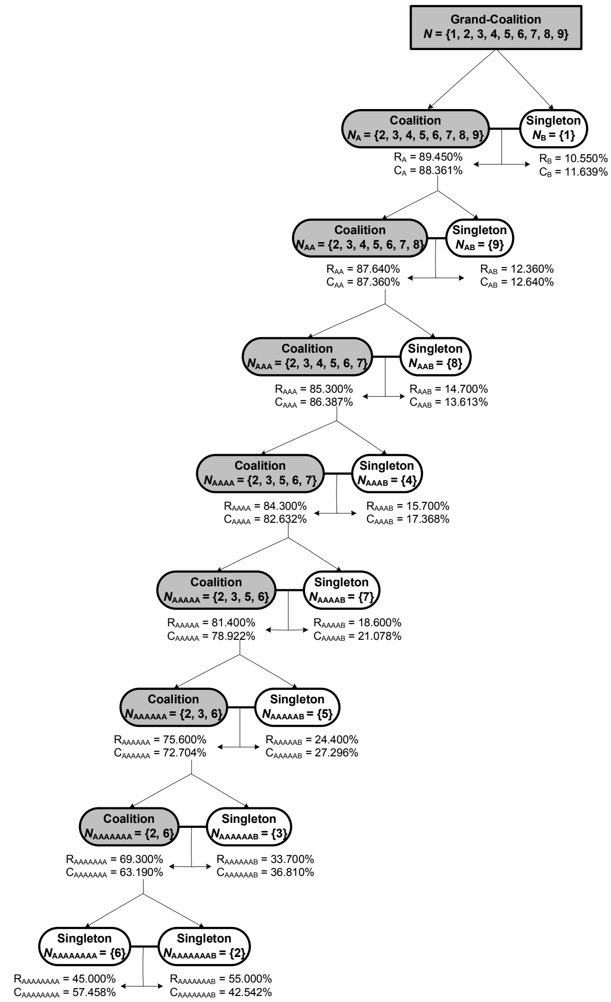

Fig. 4: second solution of the computation algorithm

Moreover, if we use the $R_1^{*2}, R_2^{*2},..., R_9^{*2}$, $C_1^{*2}, C_2^{*2},..., C_9^{*2}$, the Π_R, Π_C as well as the c_1, c_2, c_3, c_4, c_5, c_6, , c_8, and c_9 as

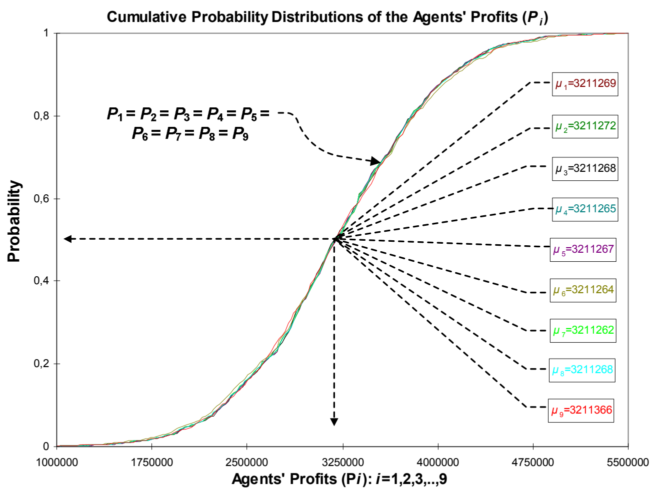

Fig. 5: cumulative probability distributions (second solution)

TABLE V

SECOND SOLUTION OF THE REVENUE-COST-SHARING RATIOS $\left(R_i^2\right),\left(C_i^2\right)$

i	Calculations	R_i^{*2}	Calculations	C_i^{*2}
$i = 1$	R_B	10.55%	C_B	11.64%
$i = 2$	$R_A R_{AA} R_{AAA}$ $R_{AAAA} R_{AAAAA}$ R_{AAAAAA} $R_{AAAAAAA}$ $R_{AAAAAAAB}$	13.22%	$C_A C_{AA} C_{AAA}$ $C_{AAAA} C_{AAAAA}$ C_{AAAAAA} $C_{AAAAAAA}$ $C_{AAAAAAAB}$	8.50%
$i = 3$	$R_A R_{AA} R_{AAA}$ R_{AAAA} R_{AAAAA} R_{AAAAAA} $R_{AAAAAAB}$	10.65%	$C_A C_{AA} C_{AAA}$ C_{AAAA} C_{AAAAA} C_{AAAAAA} $C_{AAAAAAB}$	11.64%
$i = 4$	$R_A R_{AA} R_{AAA}$ R_{AAAB}	10.50%	$C_A C_{AA} C_{AAA}$ C_{AAAB}	11.58%
$i = 5$	$R_A R_{AA} R_{AAA}$ R_{AAAA} R_{AAAAA} R_{AAAAAB}	11.20%	$C_A C_{AA} C_{AAA}$ C_{AAAA} C_{AAAAA} C_{AAAAAB}	11.87%
$i = 6$	$R_A R_{AA} R_{AAA}$ $R_{AAAA} R_{AAAAA}$ R_{AAAAAA} $R_{AAAAAAA}$ $R_{AAAAAAAA}$	10.82%	$C_A C_{AA} C_{AAA}$ $C_{AAAA} C_{AAAAA}$ C_{AAAAAA} $C_{AAAAAAA}$ $C_{AAAAAAAA}$	11.48%
$i = 7$	$R_A R_{AA} R_{AAA}$ R_{AAAA} R_{AAAAB}	10.49%	$C_A C_{AA} C_{AAA}$ C_{AAAA} C_{AAAAB}	11.61%
$i = 8$	$R_A R_{AA} R_{AAB}$	11.52%	$C_A C_{AA} C_{AAB}$	10.51%
$i = 9$	$R_A R_{AB}$	11.06%	$C_A C_{AB}$	11.17%
	$\sum_{i=1}^{9} R_i$	100.0%	$\sum_{i=1}^{9} C_i$	100.0%

inputs and the agents' profits: $P_1, P_2, ..., P_9$ as outputs in a Monte Carlo simulation model: $P_i = RR_i^2 - CC_i^2 - c_i$, we get the cumulative probability of the agents' profits as presented in Figure 5. As can be seen in Figures 3 and 5, as well as the results through the analysis in the confidence intervals presented in Tables IV and VI, all agents get equal expected profits and equal standard deviations (σ) in the first and second solutions:

$$\mu_1 = \mu_2 = \mu_3 = ... = \mu_9 = 3{,}211{,}170,\ \sigma_1 = \sigma_2 = \sigma_3 = ... = \sigma_9$$

D. Discussion

In the previous example, we have assumed that the shared revenues R and costs C are normally distributed: $\Pi_R(\mu_R = 50000000, \sigma_R = 4500000)$, $\Pi_C(\mu_C = 10000000, \sigma_C = 4300000)$.

TABLE VI

SECOND SOLUTION: AGENTS' PROFITS IN THE CONFIDENCE INTERVALS

	$\mu_i \pm \sigma_i$	$\mu_i \pm 2\sigma_i$	$\mu_i \pm 3\sigma_i$
	Probability of Profit:		
Agent	$2{,}520{,}000 \le P_i \le 3{,}900{,}000$	$1{,}830{,}000 \le P_i \le 4{,}590{,}000$	$1{,}140{,}000 \le P_i \le 5{,}280{,}000$
$i = 1$	69.99%	95.70%	99.90%
$i = 2$	69.39%	95.60%	99.83%
$i = 3$	69.81%	95.59%	99.81%
$i = 4$	69.69%	95.96%	99.87%
$i = 5$	68.53%	95.95%	99.93%
$i = 6$	68.03%	95.92%	99.80%
$i = 7$	68.45%	95.74%	99.88%
$i = 8$	69.91%	95.67%	99.87%
$i = 9$	69.86%	95.66%	99.79%

However, if we consider that the revenues and costs to be shared follow different normal probability distributions, e.g.

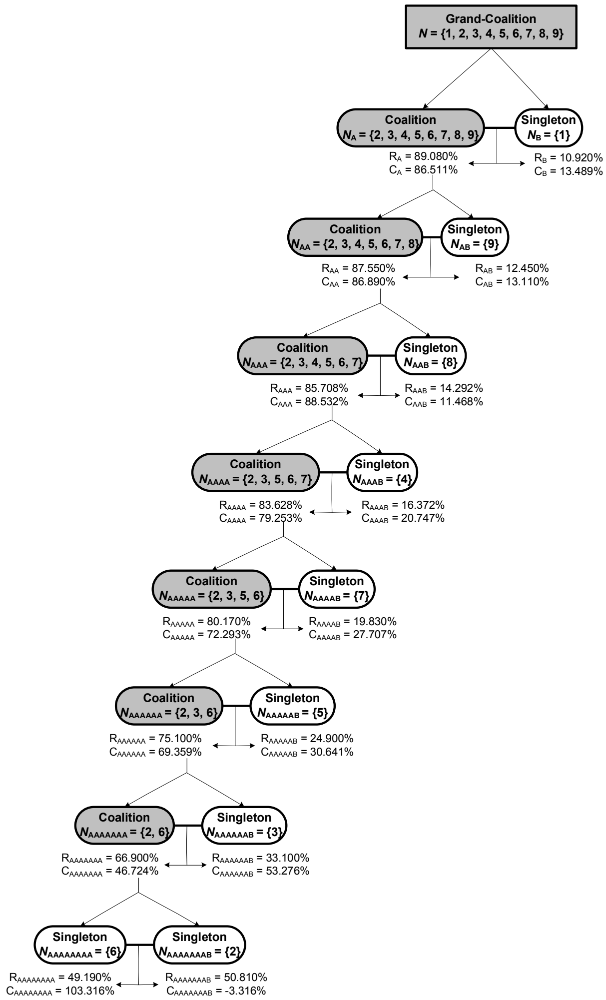

Fig. 6: solution of the computation algorithm

with the same mean values but with lower standard deviations, then the system's solution for the equal profit and risk allocation will be different. For instance, we assign initially: $\Pi_R(\mu_R = 50000000, \sigma_R = 2150000)$, and $\Pi_C(\mu_C = 10000000, \sigma_C = 430000)$. According to these probability distributions, we implement the proposed algorithm, by following the same set of combinations (for the divisions of the grand-coalition in pairs of coalitions) with the second solution in the previous example. The results are presented in Figure 6, while the agent profits' cumulative distribution functions are illustrated in Figure 7 and the analysis in the confidence intervals in Table VIII. These

TABLE VII

SOLUTION OF THE REVENUE-COST-SHARING RATIOS $(R_i^*),(C_i^*)$

i	Calculations	R_i^*	Calculations	C_i^*
$i=1$	R_B	10.92%	C_B	13.48%
$i=2$	$R_A R_{AA} R_{AAA}$ $R_{AAAA} R_{AAAAA}$ R_{AAAAAA} $R_{AAAAAAA}$ $R_{AAAAAAAB}$	11.44%	$C_A C_{AA} C_{AAA}$ $C_{AAAA} C_{AAAAA}$ C_{AAAAAA} $C_{AAAAAAA}$ $C_{AAAAAAAB}$	-0.40%
$i=3$	$R_A R_{AA} R_{AAA}$ R_{AAAA} R_{AAAAA} R_{AAAAAA} $R_{AAAAAAB}$	11.14%	$C_A C_{AA} C_{AAA}$ C_{AAAA} C_{AAAAA} C_{AAAAAA} $C_{AAAAAAB}$	14.08%
$i=4$	$R_A R_{AA} R_{AAA}$ R_{AAAB}	10.94%	$C_A C_{AA} C_{AAA}$ C_{AAAB}	13.80%
$i=5$	$R_A R_{AA} R_{AAA}$ R_{AAAA} R_{AAAAA} R_{AAAAAB}	11.15%	$C_A C_{AA} C_{AAA}$ C_{AAAA} C_{AAAAA} C_{AAAAAB}	11.68%
$i=6$	$R_A R_{AA} R_{AAA}$ $R_{AAAA} R_{AAAAA}$ R_{AAAAAA} $R_{AAAAAAA}$ $R_{AAAAAAAA}$	11.07%	$C_A C_{AA} C_{AAA}$ $C_{AAAA} C_{AAAAA}$ C_{AAAAAA} $C_{AAAAAAA}$ $C_{AAAAAAAA}$	12.76%
$i=7$	$R_A R_{AA} R_{AAA}$ R_{AAAA} R_{AAAAB}	11.08%	$C_A C_{AA} C_{AAA}$ C_{AAAA} C_{AAAAB}	14.61%
$i=8$	$R_A R_{AA} R_{AAB}$	11.14%	$C_A C_{AA} C_{AAB}$	8.62%
$i=9$	$R_A R_{AB}$	11.09%	$C_A C_{AB}$	11.34%
	$\sum_{i=1}^{9} R_i$	100.0%	$\sum_{i=1}^{9} C_i$	100.0%

results verify that the grand-coalitions profits and risks are allocated with fairness in all agents, as they all have the same probability for having lower or higher profits than the expected profit values. However, in this case (where the shared revenues and costs have lower standard deviations), the agents' profits

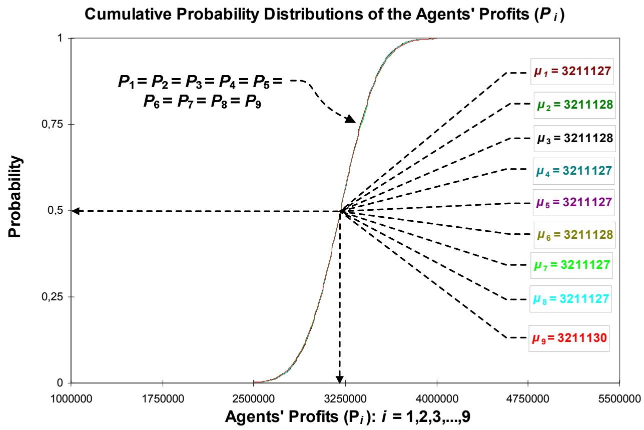

Fig. 7: cumulative probability distributions (second solution)

have lower standard deviations too. This fact is demonstrated through the analysis of the cumulative probability distribution functions in the same confidence intervals: $\mu_i \pm \sigma_i$, $\mu_i \pm 2\sigma_i$, and $\mu_i \pm 3\sigma_i$, as presented in Table VIII.

As can be seen, there is equal probability for all agent profits P_i: 68.7%, 95.6%, and 99.7%, to get values in the profit intervals: $[2967000, 3450000]$, $[2720000, 3700000]$, and $[2475000, 3945000]$ respectively.

Conclusively, the computation algorithm gives the desired results in both the first and second examples, i.e. it calculates the revenue-cost-sharing ratios for all agents, in order to allocate the grand-coalition's profits and risks equally among them. We mention that the specific combinations that were followed in the first example (in both solutions) as well as in the second example, were randomly selected.

TABLE VIII
SECOND SOLUTION: AGENTS' PROFITS IN THE CONFIDENCE INTERVALS

	$\mu_i \pm \sigma_i$	$\mu_i \pm 2\sigma_i$	$\mu_i \pm 3\sigma_i$
	Probability of Profit:		
Agent	2,967,000 $\leq P_i \leq$ 3,450,000	2,720,000 $\leq P_i \leq$ 3,700,000	2,475,000 $\leq P_i \leq$ 3,945,000
$i=1$	68.02%	95.69%	99.70%
$i=2$	67.64%	95.46%	99.70%
$i=3$	68.22%	95.70%	99.70%
$i=4$	67.94%	95.71%	99.70%
$i=5$	67.90%	95.53%	99.69%
$i=6$	68.00%	95.14%	99.69%
$i=7$	68.00%	95.69%	99.69%
$i=8$	68.98%	95.64%	99.70%
$i=9$	68.88%	95.62%	99.70%

Taking into consideration that the basic model developed here assumes that all players are identical (satisfying the symmetry axiom), the development of a model for the profit and risk allocation in situations with non-symmetric players, can be a subject for future research. Future papers can also be focused on the comparison between the solutions arising through the algorithm with the Nash-bargaining solution and the Shapley value, as well as on the application of the computation algorithm in different types of investments, e.g. the exploitation of a single product in a two-level supply chain, or a construction project, etc.

VII. CONCLUSIONS

In situations where individual agents examine to cooperate by forming a grand-coalition, which is the case in the decentralized systems, coordination can be achieved with revenue-cost-sharing mechanisms. Herein, we focus on cases where parts of the grand-coalition's costs are undertaken individually by the agents, while the remaining costs and the revenues should be shared among them, in order to allocate equally the grand-coalition's profits under an equal risk allocation scheme. A novel approach in the form of a cooperative game is used, in order to estimate the possible coalitions of agents and to compute the finite set of solutions. Each of these solutions is a pair of vectors r, c $\in \mathbb{R}^N$, with which the profits of all agents are normally distributed, having equal mean values and variances. Furthermore, we introduce a novel computation algorithm, which estimates the revenue-cost-sharing ratios for all agents, in order to allocate equally the grand-coalition profits and risks. The proposed algorithm includes the random division of the agents in pairs of coalitions until all agents are divided in singleton coalitions. Moreover, we present a numerical example, in order to highlight some significant features of the basic model, as well as the application of the novel approach.

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