Analytical solution of eddy current problems for multilayer medium with varying electrical conductivity and magnetic permeability

Valentina Koliskina, and Inta Volodko

***Abstract*—**Analytical solution of eddy current testing problem for the case of a planar multilayer medium is constructed in the present paper. It is assumed that the electrical conductivity and magnetic permeability of each layer are exponential functions of the vertical coordinate. The problem is solved by the method of the Hankel integral transform. The resulting system of ordinary differential equations is solved analytically using the modified Bessel functions of the first and second kind. Two particular cases are considered in detail: (a) a single-turn coil above a conducting half-space with varying electrical conductivity and magnetic permeability and (b) a single-turn coil above a two-layer medium where the properties of the upper layer exponentially vary with depth while the properties of the lower half-space are assumed to be constant. Results of numerical calculations of the change in impedance are presented.

***Keywords*—**eddy current testing, multilayer medium, electrical conductivity, magnetic permeability.

I. Introduction

MATHEMATICAL models of eddy current testing problems for planar multilayer medium with constant electrical and magnetic properties are well-developed in the literature [1]-[4]. Analytical solutions of direct problems for multilayer medium are usually constructed by means of integral transforms (such as Hankel or Fourier integral transforms). The resulting system of ordinary differential equations in the transformed space is then easily solved analytically if the properties of each layer of the conducting medium are constant.

Alternative approach is developed in [5] where the magnetic field is assumed to be zero at a sufficiently large distance from the axis of a coil. Such a method is known as the TREE (TRuncated Eigenfuction Expansion) method in the literature [4]. In this case analytical solutions to eddy current problems can be constructed in the form of series expansions.

Since the range of applications of eddy current method is quite wide it is not surprising to know that in some cases the electrical conductivity and magnetic permeability of a conducting medium can vary with respect to spatial coordinates. It is shown in [6] and [7] that special type of treatment of ferromagnetic metals (such as surface hardening) can lead to the presence of a surface layer with reduced magnetic permeability which varies exponentially with respect to the vertical coordinate. In order to optimize the performance of gas turbines it is necessary to increase firing temperatures. As a result, blades are usually protected from the exposure to high temperatures by special layers containing aluminium and chrome in special proportions [8]. The depletion of aluminimum in this case leads to the variation of electrical conductivity with respect to the vertical coordinate.

Hence, in order to adequately describe eddy current problem in the above mentioned cases it is necessary to develop mathematical models which take into account variability of the electrical conductivity and/or magnetic permeability with respect to one geometrical coordinate (the vertical coordinate in the case of a planar multilayer medium). Two approaches are usually used in such cases. One method is based on the assumption that the variation of electrical and/or magnetic properties of the medium can be represented by piecewise constant functions. In other words, a conducting layer with varying properties is divided into a large number of relatively thin sub-layers where the electrical conductivity and/or magnetic permeability of each sub-layer are constant. This approach is used in [8], [9] for a rectangular coordinate system, in [10] for a cylindrical coordinate system and in [11] for a spherical coordinate system. Note that up to 50 layers are used in [9] and up to 20 layers in [8]. In addition, it is estimated in [8] that the use of many layers affects computational efficiency by increasing computational time.

Alternative approach is based on the assumption that simple model profiles (for example, in the form of exponential or power functions) can be used in order to represent variability of the properties of the conducting medium in one spatial direction (see, for example, [3], [12]-[16]).

In the present paper we construct analytical solutions for the case where a single-turn coil is located above a multilayer conducting medium. It is assumed that the electrical conductivity and magnetic permeability of each layer can vary with respect to the vertical coordinate. In particular, the electrical conductivity and magnetic permeability in each

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layer are assumed to be exponential functions of the vertical coordinate. Some particular cases of the suggested solutions are considered in detail. Results of numerical calculations are presented.

II. A COIL ABOVE A MULTILAYER MEDIUM WITH VARYING PROPERTIES

Consider an air core coil located above a multilayer medium (see Fig. 1). The outer and inner radii of the coil are r_o and r_i, respectively. The height of the coil is $h_2 - h_1$, where h_1 is the lift-off. The coil is located in free space (region R_1). The thickness of each conducting layer R_i is denoted by d_i, $i = 2,3,...,n-1$. The bottom layer R_n is assumed to be infinite in the vertical direction.

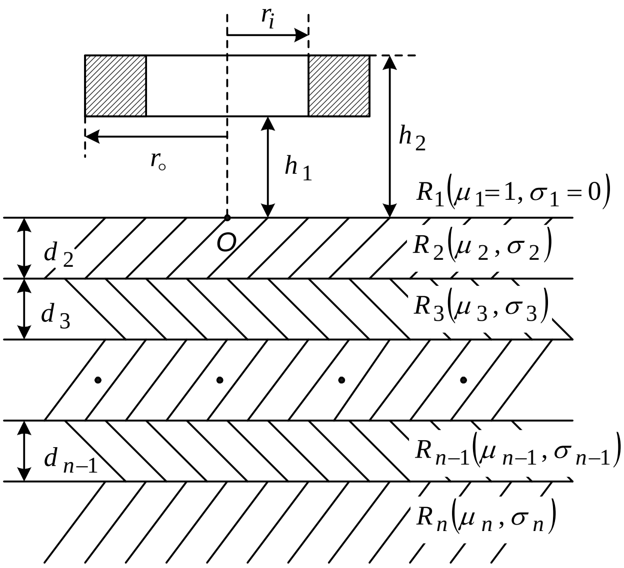

Fig. 1. An air core coil above a multilayer medium.

Each conducting layer $R_i, i = 2,3,...,n$ is characterized by the two parameters: the electrical conductivity σ_i and magnetic permeability μ_i.

We consider a system of cylindrical polar coordinates (r, φ, z) centered at O, the z-axis is directed upwards. The coil is carrying alternating current of the form

$$\vec{I}^{\,e} = I(r,h)e^{j\omega t}\vec{e}_\varphi, \quad (1)$$

where $r_i \le r \le r_o, h_1 \le h \le h_2$, ω is the frequency and $\vec{e}_\varphi$ is the unit vector in the $\varphi-$direction. In this case it is natural to assume that the vector potential $\vec{A}_i$ in each region $R_i, i = 1,2,...,n$ has only one non-zero component of the form

$$\vec{A}_i = A_i(r,z)e^{j\omega t}\vec{e}_\varphi. \quad (2)$$

We assume that the electrical conductivity σ_i and magnetic permeability μ_i in each region R_i depends on the vertical coordinate. More precisely, σ_i and μ_i are modeled by the following relations

$$\mu_i = \mu_0 \mu_i^m \exp(\alpha_i z), \quad \sigma_i = \sigma_i^m \exp(\beta_i z), \quad i = 2,3,..n, \quad (3)$$

where μ_0 is the magnetic constant and $\alpha_i, \beta_i, \mu_i^m, \sigma_i^m$ are constants.

III. A SINGLE-TURN COIL ABOVE A MULTILAYER MEDIUM

The first step in solving the problem is to find the solution for the case where a single-turn coil of radius r_c is located at distance h above a conducting multilayer medium (see Fig. 2).

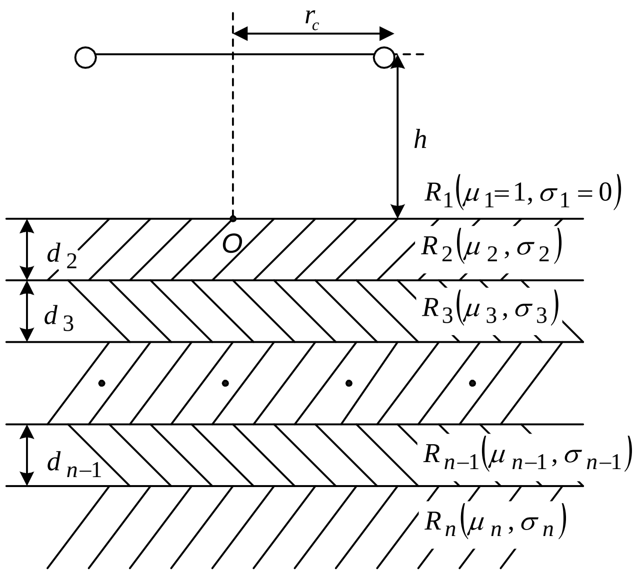

Fig. 2. A single-turn coil above a multilayer medium.

Using the Maxwell's equations we obtain the following system of equations for the amplitudes $A_i(r,z)$ of the vector potential in each region $R_i, i = 1,2,...,n$ (see, for example, [3]):

$$\frac{\partial^2 A_1}{\partial r^2}+\frac{1}{r}\frac{\partial A_1}{\partial r}-\frac{A_1}{r^2}+\frac{\partial^2 A_1}{\partial z^2} = -\mu_0 I\delta(r-r_c)\delta(z-h), \quad (4)$$

$$\frac{\partial^2 A_i}{\partial r^2}+\frac{1}{r}\frac{\partial A_i}{\partial r}-\frac{A_i}{r^2}+\frac{\partial^2 A_i}{\partial z^2} -\frac{1}{\mu_i(z)}\frac{d\mu_i(z)}{dz}\frac{\partial A_i}{\partial z}-j\sigma_i(z)\mu_i(z)A_i=0, \quad (5)$$
$$i=2,3,...,n,$$

where $\delta(x)$ is the Dirac delta-function.
Using (3) we rewrite equations (5) in the form

$$\frac{\partial^2 A_i}{\partial r^2}+\frac{1}{r}\frac{\partial A_i}{\partial r}-\frac{A_i}{r^2}+\frac{\partial^2 A_i}{\partial z^2} -\beta_i\frac{\partial A_i}{\partial z}-j\mu_0\sigma_i^m\mu_i^m\exp[(\alpha_i+\beta_i)z]A_i=0, \quad (6)$$
$$i=2,3,...,n.$$

The boundary conditions are

$$A_1|_{z=0}=A_2|_{z=0}, \quad \frac{\partial A_1}{\partial z}|_{z=0}=\frac{1}{\hat{\mu}_1}\frac{\partial A_2}{\partial z}|_{z=0}, \quad (7)$$

$$A_i|_{z=-\hat{d}_i}=A_{i+1}|_{z=-\hat{d}_i},$$
$$\frac{1}{\hat{\mu}_i}\frac{\partial A_i}{\partial z}|_{z=-\hat{d}_i}=\frac{1}{\hat{\mu}_{i+1}}\frac{\partial A_{i+1}}{\partial z}|_{z=-\hat{d}_i}, \quad i=2,3,..,n-1, \quad (8)$$

where

$$\hat{d}_i=\sum_{k=2}^{i}d_i, \quad \hat{\mu}_i=\mu_i^m\exp(-\alpha_i\hat{d}_i), i=2,3,...n-1,$$
$$\hat{\mu}_1=\mu_1^m.$$

The following conditions hold at infinity:

$$A_i, \frac{\partial A_i}{\partial r}\to 0 \text{ as } r\to\infty, \ i=1,2,...,n, \quad (9)$$

$$A_1\to 0 \text{ as } z\to+\infty, \ A_n\to 0 \text{ as } z\to-\infty. \quad (10)$$

Applying the Hankel integral transform of the form

$$\tilde{A}_i(\lambda,z)=\int_0^\infty A_i(r,z)rJ_1(\lambda r)dr, \quad i=1,2,..,n \quad (11)$$

to problem (4), (6)-(10) we obtain

$$\frac{d^2\tilde{A}_1}{dz^2}-\lambda^2\tilde{A}_1=-\mu_0 Ir_c J_1(\lambda r_c)\delta(z-h), \quad (12)$$

$$\frac{d^2\tilde{A}_i}{dz^2}-\lambda^2\tilde{A}_i\beta_i\frac{d\tilde{A}_i}{dz} -j\omega\mu_0\mu_i^m\sigma_i^m\exp[(\alpha_i+\beta_i)z]\tilde{A}_i=0, \quad (13)$$
$$i=2,3,..,n,$$

$$\tilde{A}_1|_{z=0}=\tilde{A}_2|_{z=0}, \quad \frac{d\tilde{A}_1}{dz}|_{z=0}=\frac{1}{\hat{\mu}_1}\frac{d\tilde{A}_2}{dz}|_{z=0}, \quad (14)$$

$$\tilde{A}_i|_{z=-\hat{d}_i}=\tilde{A}_{i+1}|_{z=-\hat{d}_i},$$
$$\frac{1}{\hat{\mu}_i}\frac{d\tilde{A}_i}{dz}|_{z=-\hat{d}_i}=\frac{1}{\hat{\mu}_{i+1}}\frac{d\tilde{A}_{i+1}}{dz}|_{z=-\hat{d}_i}, \quad i=2,3,..,n-1, \quad (15)$$

$$\tilde{A}_1\to 0 \text{ as } z\to+\infty, \ \tilde{A}_n\to 0 \text{ as } z\to-\infty. \quad (16)$$

In order to solve equation (12) we consider the following two sub-regions of $R_1: 0<z<h$ and $z>h$. The solutions in these regions are denoted by $\tilde{A}_{10}$ and $\tilde{A}_{11}$, respectively. Hence,

$$\frac{d^2\tilde{A}_{10}}{dz^2}-\lambda^2\tilde{A}_{10}=0, \quad 0<z<h, \quad (17)$$
$$\frac{d^2\tilde{A}_{11}}{dz^2}-\lambda^2\tilde{A}_{11}=0, \quad z>h. \quad (18)$$

The general solution to (17) can be written in the form

$$\tilde{A}_{10}=C_1e^{\lambda z}+C_2e^{-\lambda z}. \quad (19)$$

The bounded solution to (18) is

$$\tilde{A}_{11}=C_3e^{-\lambda z}. \quad (20)$$

The solution to (13) can be expressed in terms of modified Bessel functions (see [17], formula 2.1.3.10, page 247):

$$\tilde{A}_i(\lambda,z)=C_{4i}e^{\beta_i z/2}I_{\nu_i}\left(c_ie^{(\alpha_i+\beta_i)z/2}\right) +C_{5i}e^{\beta_i z/2}K_{\nu_i}\left(c_ie^{(\alpha_i+\beta_i)z/2}\right), i=2,3,..,n-1 \quad (21)$$

where

$$c_i = \frac{2\sqrt{j\omega\mu_0\mu_i^m\sigma_i^m}}{\alpha_i + \beta_i}, \quad \nu_i = \frac{\sqrt{\beta_i^2 + 4\lambda^2}}{\alpha_i + \beta_i}, \tag{22}$$

and I_{ν_i}, K_{ν_i} are the modified Bessel functions of the first and second kind, respectively.

The bounded solution to (13) in region R_n is

$$\tilde{A}_n(\lambda, z) = C_{4n} e^{\beta_i z/2} I_{\nu_i}\left(c_i e^{(\alpha_i+\beta_i)z/2}\right). \tag{23}$$

Continuity of the vector potential at $z = h$ gives

$$C_1 e^{\lambda h} + C_2 e^{-\lambda h} = C_3 e^{-\lambda h}. \tag{24}$$

Integrating (12) with respect to z from $h-\varepsilon$ to $h+\varepsilon$, considering the limit as $\varepsilon \to +0$ and using continuity of the function $\tilde{A}_1$ at $z = h$ we obtain

$$-C_3\lambda e^{-\lambda h} - C_1\lambda e^{\lambda h} + C_2\lambda e^{-\lambda h} = -\mu_0 I r_c J_1(\lambda r_c). \tag{25}$$

Arbitrary constants $C_1, C_2, C_3, C_{4i}, C_{5i}\,(i = 2,3,..,n-1)$ and C_{4n} can be obtained from (14), (15), (24) and (25). The solution in each region $R_i\,(i = 1,2,...,n)$ is then found by means of the inverse Hankel transform of the form

$$A_i(r,z) = \int_0^\infty \tilde{A}_i(\lambda, z)\lambda J_1(\lambda r)d\lambda, \quad i = 1,2,...,n. \tag{26}$$

In the next section we consider one particular case – a single-turn coil above a conducting half-space with varying electrical and magnetic properties.

IV. A SINGLE-TURN COIL ABOVE A CONDUCTING HALF-SPACE WITH VARYING PROPERTIES

Consider a single-turn coil of radius r_c located at a distance h above a conducting half-space [18] (see Fig. 2 in the limit as $d_2 \to \infty$).

The solution in region R_1 is given by (19) and (20). The solution in the conducting half-space where the electrical conductivity and magnetic permeability are given by (3) is represented in the form (23). Using the boundary conditions we determine all unknown constants C_1, C_2, C_3, C_{4n}. In particular, the value of C_2 is

$$C_2 = \frac{\mu_0 I r_c J_1(\lambda r_c)e^{-\lambda h}[(2\lambda\mu_m - \beta)I_\nu(c) - c(\alpha+\beta)I'_\nu(c)]}{2\lambda[(2\lambda\mu_m + \beta)I_\nu(c) + c(\alpha+\beta)I'_\nu(c)]}, \tag{27}$$

where for simplicity the subscript i in α_i, β_i, c_i and ν_i is omitted.

The induced vector potential in region R_1 is

$$\tilde{A}_1^{ind}(\lambda, z) = C_2 e^{-\lambda z}, \tag{28}$$

where C_2 is given by (27). Applying the inverse Hankel transform of the form (26) to (28) we obtain

$$A_1^{ind}(r,z) = \frac{\mu_0 I r_c}{2}\int_0^\infty F(\lambda) J_1(\lambda r_c) J_1(\lambda r) e^{-\lambda(z+h)} d\lambda, \tag{29}$$

where

$$F(\lambda) = \frac{(2\lambda\mu_m - \beta)I_\nu(c) - c(\alpha+\beta)I'_\nu(c)}{(2\lambda\mu_m - \beta)I_\nu(c) + c(\alpha+\beta)I'_\nu(c)}.$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I}\oint_L A_1^{ind}(r,z)dl, \tag{30}$$

where L is the contour of the coil. Substituting (29) into (30) we obtain

$$Z^{ind} = \pi\omega\mu_0 r_c Z, \tag{31}$$

where

$$Z = j\int_0^\infty \frac{(2s\mu_m - \tilde{\beta})I_\nu(\tilde{c}) - \tilde{c}(\tilde{\alpha}+\tilde{\beta})I'_\nu(\tilde{c})}{(2s\mu_m + \tilde{\beta})I_\nu(\tilde{c}) + \tilde{c}(\tilde{\alpha}+\tilde{\beta})I'_\nu(\tilde{c})} J_1^2(s) e^{-2\gamma s} ds. \tag{32}$$

The following dimensionless parameters are used in (32):

$$\tilde{c} = \frac{2\eta\sqrt{j}}{\tilde{\alpha}+\tilde{\beta}}, \ \nu = \frac{\sqrt{\tilde{\beta}^2 + 4s^2}}{\tilde{\alpha}+\tilde{\beta}}, \ \eta = r_c\sqrt{\omega\mu_0\sigma_2^m\mu_2^m},$$

$$\tilde{\alpha} = \alpha r_c, \ \tilde{\beta} = \beta r_c, \ \gamma = \frac{h}{r_c}. \tag{33}$$

Fig. 3 plots the change in impedance (Z) for three different values of $\tilde{\beta}$, namely, $\tilde{\beta}=5$, $\tilde{\beta}=2$ and $\tilde{\beta}=0$, respectively (from top to bottom). The other parameters of the problem are fixed at $\tilde{\alpha}=0, \gamma=0.05$ and $\mu_2^m=5.$ The calculated points in Fig. 3 correspond to different values of $\eta=1,2,...,10$ (from left to right). Computations are done with Mathematica.

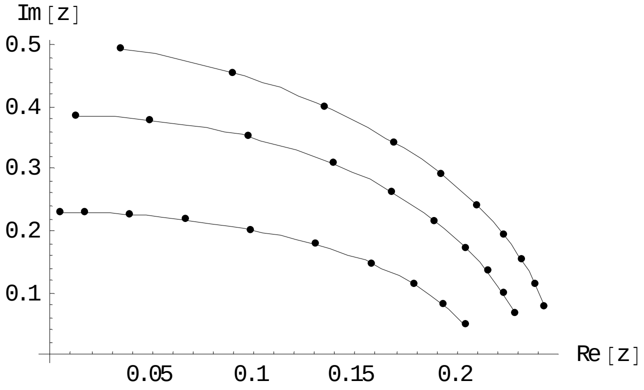

Fig. 3. The change in impedance (Z) for three different values of $\tilde{\beta}.$

It is seen from the graph that the modulus of Z increases as the parameter $\tilde{\beta}$ increases.

V. COIL OF FINITE DIMENSIONS

Using (29) one can compute the induced vector potential of a coil of finite dimensions shown in Fig. 1 (see [3], [18]). Assume that w is the number of turns in the coil. Consider two rings of the coil centered at the points (r_n, z_n) and (r_m, z_m), respectively. The induced vector potential on the ring centered at (r_m, z_m) due to eddy currents induced in the ring centered at (r_n, z_n) can be computed as follows

$$A_{mn}^{ind}=\frac{\mu_0 I r_n}{2}\frac{wdzdr}{(h_2-h_1)(r_o-r_i)} \times \int_0^\infty F(\lambda) J_1(\lambda r_n) J_1(\lambda r_m) e^{-\lambda(z_n+z_m)} d\lambda, \tag{34}$$

where $\frac{wdzdr}{(h_2-h_1)(r_o-r_i)}$ in the number of turns in the ring centered at (r_n, z_n). The induced vector potential in the ring centered at (r_m, z_m) due to eddy currents produced by the coil can be obtained as an integral of (34) with respect to r_n and z_n. The limits of integration are r_i, r_o and h_1, h_2, respectively. The result is

$$A_m^{ind}=-\frac{\mu_0 I w}{2(h_2-h_1)(r_o-r_i)}\int_0^\infty \frac{F(\lambda)}{\lambda} J_1(\lambda r_m) e^{-\lambda z_m} \times \left[\int_{r_i}^{r_o} r_n J_1(\lambda r_n) dr_n\right] \frac{e^{-\lambda h_1}}{\lambda}\left[e^{-\lambda(h_2-h_1)}-1\right] d\lambda. \tag{35}$$

Finally, integrating (35) with respect to r_m from r_i to r_o and with respect to z_m from h_1 to h_2 we obtain the vector potential in the coil induced by eddy currents in the conducting half-space in the form

$$A_{coil}^{ind}=\frac{\mu_0 I w^2}{2(h_2-h_1)^2(r_o-r_i)^2} \times \int_0^\infty \frac{F(\lambda)}{\lambda^6}\left(e^{-\lambda h_2}-e^{-\lambda h_1}\right)^2 \kappa(r_i, r_o, \lambda) d\lambda, \tag{36}$$

where $\kappa(r_i, r_o, \lambda)=\int_{\lambda r_i}^{\lambda r_o} \xi J_1(\xi) d\xi.$

Using (30) and (36) we obtain the induced change in impedance of the whole coil:

$$Z_{coil}^{ind}=\frac{j\omega\pi\mu_0 w^2}{2(h_2-h_1)^2(r_o-r_i)^2} \times \int_0^\infty \frac{F(\lambda)}{\lambda^6}\left(e^{-\lambda h_2}-e^{-\lambda h_1}\right)^2 \kappa(r_i, r_o, \lambda) d\lambda. \tag{37}$$

VI. A SINGLE-TURN COIL ABOVE A TWO-LAYER MEDIUM

In this section we consider another example of the general theory. Consider a single-turn coil of radius r_c located at a distance h from a two-layer medium (see Fig. 4).

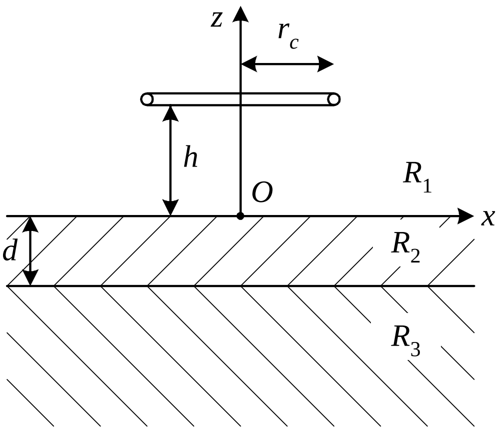

Fig. 4. A single-turn coil above a two-layer medium.

We assume that the electrical conductivity and magnetic permeability of region R_2 are given by (3) but the properties of the lower layer (region R_3) are constant. This model can be used to describe metal hardening [6], [7].

The functions A_1 and A_2 satisfy equations (4) and (6) while the function A_3 is the solution of the following equation

$$\frac{\partial^2 A_3}{\partial r^2}+\frac{1}{r}\frac{\partial A_3}{\partial r}-\frac{A_3}{r^2}+\frac{\partial^2 A_3}{\partial z^2}-j\sigma_3^m\mu_0\mu_3^m A_3=0, \quad (38)$$

where σ_3^m and μ_3^m are, respectively, the constant electrical conductivity and relative magnetic permeability of region R_3. The boundary conditions are

$$A_1\,|_{z=0}=A_2\,|_{z=0},\ \frac{\partial A_1}{\partial z}|_{z=0}=\frac{1}{\hat{\mu}_1}\frac{\partial A_2}{\partial z}|_{z=0}, \quad (39)$$

$$A_2\,|_{z=-d}=A_3\,|_{z=-d},\ \frac{1}{\hat{\mu}_2}\frac{\partial A_2}{\partial z}|_{z=-d}=\frac{1}{\mu_3^m}\frac{\partial A_3}{\partial z}|_{z=-d}. \quad (40)$$

Applying the Hankel transform (11) to (4) and (6) we obtain the solution (19)-(21) in regions R_1 and R_2. The solution to (38) in the transformed space which is bounded as $z\to-\infty$ is

$$\tilde{A}_3(\lambda,z)=C_6e^{qz}, \quad (41)$$

where $q=\sqrt{\lambda^2+j\omega\sigma_3^m\mu_0\mu_3^m}$. The constants $C_1,C_2,C_3,C_{42},C_{52}$ and C_6 can be found from (14), (15), (24), (25). In particular, the constant C_2 has the form

$$C_2=\frac{\mu_0 I r_c J_1(\lambda r_c)e^{-\lambda h}}{2\lambda}\cdot\frac{B}{D}, \quad (42)$$

where

$$B=-p[(\lambda\hat{\mu}_1-\beta/2)I_\nu(c)-c(\alpha+\beta)I_\nu^{'}(c)/2] + (\lambda\hat{\mu}_1-\beta/2)K_\nu(c)-c(\alpha+\beta)K_\nu^{'}(c)/2, \quad (43)$$

$$D=-p[(\lambda\hat{\mu}_1+\beta/2)I_\nu(c)+c(\alpha+\beta)I_\nu^{'}(c)/2] + (\lambda\hat{\mu}_1+\beta/2)K_\nu(c)+c(\alpha+\beta)K_\nu^{'}(c)/2, \quad (44)$$

and

$$p=\frac{E}{F}, \quad (45)$$

where

$$E=(\hat{\mu}_3\beta-2\hat{\mu}_2 q)K_\nu\left(ce^{-(\alpha+\beta)d/2}\right) + \hat{\mu}_3 c(\alpha+\beta)e^{-(\alpha+\beta)d/2}K_\nu^{'}\left(ce^{-(\alpha+\beta)d/2}\right),$$

$$F=(\hat{\mu}_3\beta-2\hat{\mu}_2 q)I_\nu\left(ce^{-(\alpha+\beta)d/2}\right) + \hat{\mu}_3 c(\alpha+\beta)e^{-(\alpha+\beta)d/2}I_\nu^{'}\left(ce^{-(\alpha+\beta)d/2}\right).$$

Here we used the notations

$$c=\frac{2\sqrt{j\omega\mu_0\mu_2^m\sigma_2^m}}{\alpha+\beta},\ \nu=\frac{\sqrt{\beta^2+4\lambda^2}}{\alpha+\beta},$$
$$\alpha=\alpha_2,\ \beta=\beta_2.$$

The induced vector potential (in the transformed space) is given by (28) and (42). Applying the inverse Hankel transform (26) to (28) we obtain the induced vector potential in the form

$$A_0^{ind}(r,z)=\frac{\mu_0 I r_c}{2}\int_0^\infty\frac{B}{D}J_1(\lambda r)J_1(\lambda r_c)e^{-\lambda(z+h)}d\lambda, \quad (46)$$

where B and D are given by (43) and (44), respectively. The change in impedance of the coil is computed using (30) and (46):

$$Z_{ind}=j\omega\mu_0\pi r_c Z, \quad (47)$$

where

$$Z = \int_0^\infty \frac{\tilde{B}}{\tilde{D}} J_1^2(s) e^{-\delta s} ds. \tag{48}$$

Here

$$\tilde{B} = -\tilde{p}[(s\hat{\mu}_1 - \eta/2) I_\nu(\tilde{c}) - \beta_1 \sqrt{j} I_\nu^{'}(\tilde{c})]$$
$$+ (s\hat{\mu}_1 - \eta/2) K_\nu(\tilde{c}) - \beta_1 \sqrt{j} K_\nu^{'}(\tilde{c}),$$
$$\tilde{D} = -\tilde{p}[(s\hat{\mu}_1 + \eta/2) I_\nu(\tilde{c}) + \beta_1 \sqrt{j} I_\nu^{'}(\tilde{c})]$$
$$+ (s\hat{\mu}_1 + \eta/2) K_\nu(\tilde{c}) + \beta_1 \sqrt{j} K_\nu^{'}(\tilde{c}),$$
$$\tilde{p} = \frac{\tilde{E}}{\tilde{F}},$$
$$\tilde{E} = (\hat{\mu}_3 \eta - 2\hat{\mu}_2 \sqrt{s^2 + j\beta_2^2}) K_\nu \left(\tilde{c} e^{-(\xi+\eta)\gamma/2}\right)$$
$$+ 2\hat{\mu}_3 \beta_1 \sqrt{j} e^{-(\xi+\eta)\gamma/2} K_\nu^{'} \left(\tilde{c} e^{-(\xi+\eta)\gamma/2}\right),$$
$$\tilde{D} = (\hat{\mu}_3 \eta - 2\hat{\mu}_2 \sqrt{s^2 + j\beta_2^2}) I_\nu \left(\tilde{c} e^{-(\xi+\eta)\gamma/2}\right)$$
$$+ 2\hat{\mu}_3 \beta_1 \sqrt{j} e^{-(\xi+\eta)\gamma/2} I_\nu^{'} \left(\tilde{c} e^{-(\xi+\eta)\gamma/2}\right),$$
$$\beta_1 = r_c \sqrt{\sigma_2^m \mu_0 \mu_2^m},$$
$$\gamma = \frac{d}{r_c},\ \delta = \frac{2h}{r_c},\ \xi = \alpha r_c,\ \eta = \beta r_c,\ \tilde{c} = \frac{2\beta_1 \sqrt{j}}{\xi + \eta},$$
$$\beta_2 = \beta_1 \theta,\ \nu = \frac{\sqrt{\eta^2 + 4s^2}}{\xi + \eta},\ \theta = \sqrt{\frac{\sigma_3 \mu_3}{\sigma_2^m \mu_2^m}}.$$

The change in impedance, Z, computed by means of (48) is shown in Fig. 5 for the following values of the parameters:

$$\gamma = 0.2,\ \theta = 1.5,\ \mu_m = 1,\ \mu_2 = 100,$$
$$\delta = 0.05,\ \eta = -4.$$

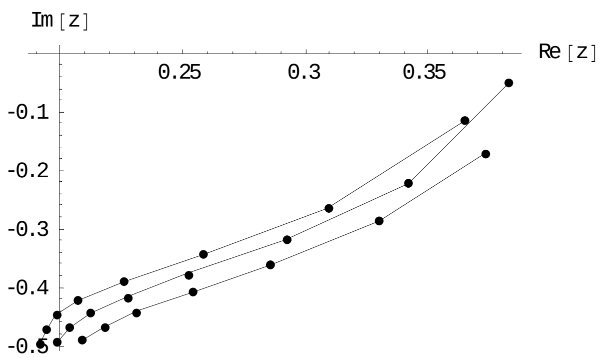

Fig. 5. The real and imaginary parts of Z for three values of $\xi: -4, -2$ and 0 (from top to bottom). Calculated points on each curve correspond to different values of β_1.

The solution for a single-turn coil can easily be generalized for the case of a coil of finite dimensions as it is done in the previous section.

Analytical solution can also be constructed for the case where a double conductor line is located above a multilayer medium with varying electrical and magnetic properties. Using the method of Fourier integral transform the system of ordinary differential equations in each layer can be solved analytically in terms of modified Bessel functions provided the electrical conductivity and magnetic permeability of each layer are exponential functions of the vertical coordinate. Examples for the case of a conducting half-space or a two-layer medium can be found in [20]-[22].

VII. Conclusions

Method of the Hankel integral transform is used in the present paper in order to construct analytical solution of eddy current problem where a coil with alternating current is located above a conducting multilayer medium. The electrical conductivity and magnetic permeability of each layer are exponential functions of the vertical coordinate. The resulting system of ordinary differential equations is solved analytically in terms of Bessel functions of the first and second kind. Two examples are considered in detail for the case of a conducting half-space and two-layer medium. The change in impedance is computed using Mathematica. The approach presented in the paper can also be used to construct analytical solution for the case where a double conductor line is located above a conducting medium with varying electrical conductivity and magnetic permeability.

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