Analysis of a Finite Difference Scheme for a Slow, 3-D Permeable Boundary, Navier-Stokes Flow

JOE HLOMUKA
Department of Mathematics and Applied Mathematics ,
University of Venda
P. Bag X5050,Thohoyandou, 0950
SOUTH AFRICA .
joe.hlomuka@univen.ac.za

Abstract:
We derive a finite difference scheme for a sufficiently slow permeable boundary Navier-Stokes flow, to justify the condition, $\rho\left(\mathbf{v}(x,t).\nabla\right)\mathbf{v}(x,t)\approx 0$, in its Navier-Stokes equation. The expression is the nonlinear part of the equation and features in the approximation of the flow's Reynold's number (see Remarks 5.2(3), on page 721 of [2]). In other words, we discretize the linearized , non-homogeneous Navier-Stokes problem, representing the 3-D slow flow of a fluid. A typical example of a slow Navier-Stokes fluid flow is ground water through an aquifer. It is to be noted that the condition of non-homogeneity stems from our further application of the Sauer-Maritz boundary permeation model (see Section 2,on page 718 of [2]). This model is also applied in [4], [5] and [6]. The homogeneous version of the problem is discussed by O.A. Ladyzhenskya in [7].The derived scheme will then be tested for convergence and error stability. The theoretic analysis of the problem, as discussed in [2], takes place in the Sobolev space, $L^2\left([0,T],H^2(\Omega)\right);T<\infty$; and therefore, the scheme will be derived in the same function space(similar to the scheme in [3]).

***Key Words*:** finite difference scheme; permeable boundary; slow Navier-Stokes flow.

***AMS Subject Codes*:** **65N06, 65N12,76D05, 76M20**

1 Introduction.

We seek a numerical approximation to the solution, $\mathbf{v}(x,t), p(x) \in H^2(\Omega)\times[0,T]$ whose existence and uniqueness was confirmed in[2], such that,

$$(1\text{a})\ \rho\ \partial_t \mathbf{v}(x,t) = \mu\Delta\mathbf{v}(x,t) - \nabla p(x,t) + \mathbf{f}(x,t);\ \mathbf{v}\in H^2(\Omega); x\in\Omega\subset R^3; \mathbf{f}\in L^2(\Omega), t>0,$$

subject to:

$$(1\text{b})\begin{cases} \nabla.\mathbf{v}(x,t)=0; \\ \gamma_0\mathbf{v}(y,t) = -\eta_v(y,t)\mathbf{n}(y);\ \ y\in\Gamma. \\ \sigma\partial_t\left[\gamma_0\mathbf{v}(\mathrm{y,t})\right] + \gamma_0 p(y,t) + 2\mu\kappa\eta_v(y,t) = p_0(\mathrm{t}) \ . \end{cases}$$

where,

$x := (x_1, x_2, x_3) \in \Omega;$

$y := (y_1, y_2) \in \partial\Omega = \Gamma;$

μ: fluid viscosity; assumed constant;

ρ : fluid density; assumed constant;

$\mathbf{v}(x,t)$: body fluid velocity field;

$\gamma_0\mathbf{v}(x,t)$: surface fluid velocity field;

$p(x,t)$: fluid pressure field;

$\gamma_0 p(x,t)$: surface fluid pressure field;

$\mathbf{f}(x,t)$: external force; assumed to be of potential type.s

2 The setting for the problem.

We fill up a container with a Navier-Stokes fluid. Inside this container we put a smaller container, with the same Navier-Stokes fluid and same height. The walls of the inner container are permeable; thus allowing the free flow of the fluid to and from the outer container. We denote the wall of the inner container by Γ and the wall of the outer container by Γ_0.

The body of fluid between Γ and Γ_0 is denoted by Ω. On the other hand the body of fluid beyond Γ, inside the inner container, is denoted by Ω_0. A fluid particle may accelerate from the position of rest from Ω_0, through Γ into Ω. Another particle could accelerate from the position of rest from Ω through Γ into Ω_0.

The motion of the fluid in regions Ω and Ω_0 is governed by the mathematical model 1(a), whilst the model 1(b)(the last two boundary equations) accounts for the fluid permeation through Γ. The scheme for the last boundary equation in 1(b) will be sought as its derivation in [4] has taken into consideration the Sauer-Maritz permeation model ($\gamma_0\mathbf{v}(y,t) = -\eta_v(y,t)\mathbf{n}(y)$) and the conservation laws.

(see also [2],[4],[5] and [6])

Remarks 2.1

In the following remarks we assume,

$\Delta t = t_j - t_{j-1}; t_j, t_{j-1} \in [0,T]; \Delta x = x_{ki} - x_{k(i-1)}, x_{ki}, x_{k(i-1)} \in \Omega; k = 1,2,3$

(1) The left-hand side time derivative scheme will generate the error term of the form $\frac{1}{2}\Delta t \frac{\partial^2 v^{(k)}\left(x^{(k)},\theta_k\right)}{\partial t^2}; x^{(k)} \in \Omega; \theta_k \in \left(t_{j-1}, t_j\right) \subset [0,T]$. Note that $x^{(1)}, x^{(2)}$ and $x^{(3)}$ represent the directions along the X_1, X_2 and X_3, axes respectively.

(2) The right-hand velocity scheme will generate the error term of the form $\frac{1}{12}[\Delta x]^2 \frac{\partial^4 v^{(k)}(\xi_k, t)}{\partial x_k^4}$. However, this error term vanishes in $H^2(\Omega)$.

(3) The pressure scheme generates the error term of the form $\frac{1}{2}\Delta x \frac{\partial^2 p^{(k)}(\xi_k)}{\partial x_k^2}$, which does not vanish in $H^2(\Omega)$.

(4) In the derivation of the following scheme, we will ignore the error terms as we use $v_{i,j}^{(1)}$; $v_{i,j}^{(2)}$ and $v_{i,j}^{(3)}$ to approximate $v\left(x_{1i}, t_j\right)$; $v\left(x_{2i}, t_j\right)$ and $v\left(x_{3i}, t_j\right)$, respectively. Similarly, we use $p_i^{(1)}$; $p_i^{(2)}$ and $p_i^{(3)}$ to approximate $p\left(x_{1i}\right)$; $p\left(x_{2i}\right)$ and $p\left(x_{3i}\right)$ respectively. (See (4) and (5) on page 510 of [7] and 12.2 and 12.3 in [2])

(5) See page 2 of [4] for the existence of a bijection between $v\left(x_{1i}, t_j\right)$ and $\eta_v\left(y_{1i}, t_j\right)$ on one hand; $p\left(x_{1i}\right)$ and $\gamma_0 p\left(y_{1i}\right)$ on the other hand.

3 The scheme for the region ,Ω between Γ and Γ_0

Taking a square grid, $i, j = 1,2,3,\ldots\ldots, N,$ for each axis plane, we develop the following scheme; assuming the mesh information under Remarks 2.1:

Along the X_1-axis, we have:

$$(1)\rho\left[\frac{v_{i,j+1}^{(1)} - v_{i,j}^{(1)}}{\Delta t}\right] = \mu\left[\frac{v_{i-1,j}^{(1)} - 2v_{i,j}^{(1)} + v_{i+1,j}^{(1)}}{(\Delta x)^2} + \frac{v_{i-1,j}^{(1)} - 2v_{i,j}^{(1)} + v_{i+1,j}^{(1)}}{(\Delta x)^2} + \frac{v_{i-1,j}^{(1)} - 2v_{i,j}^{(1)} + v_{i+1,j}^{(1)}}{(\Delta x)^2}\right] - \left[\frac{p_i^{(1)} - p_{i-1}^{(1)}}{\Delta x}\right] + f_{i,j}^{(1)}.$$

Along the X_2-axis, we have:

$$(2)\,\rho\left[\frac{v^{(2)}_{i,j+1}-v^{(2)}_{i,j}}{\Delta t}\right]=\mu\left[\frac{v^{(2)}_{i-1,j}-2v^{(2)}_{i,j}+v^{(2)}_{i+1,j}}{(\Delta x)^2}+\frac{v^{(2)}_{i-1,j}-2v^{(2)}_{i,j}+v^{(2)}_{i+1,j}}{(\Delta x)^2}+\frac{v^{(2)}_{i-1,j}-2v^{(2)}_{i,j}+v^{(2)}_{i+1,j}}{(\Delta x)^2}\right]-\left[\frac{p^{(2)}_{i}-p^{(2)}_{i-1}}{\Delta x}\right]+f^{(2)}_{i,j}$$

Along the X_3-axis, we have:

$$(3)\,\rho\left[\frac{v^{(3)}_{i,j+1}-v^{(3)}_{i,j}}{\Delta t}\right]=\mu\left[\frac{v^{(3)}_{i-1,j}-2v^{(3)}_{i,j}+v^{(3)}_{i+1,j}}{(\Delta x)^2}+\frac{v^{(3)}_{i-1,j}-2v^{(3)}_{i,j}+v^{(3)}_{i+1,j}}{(\Delta x)^2}+\frac{v^{(3)}_{i-1,j}-2v^{(3)}_{i,j}+v^{(3)}_{i+1,j}}{(\Delta x)^2}\right]-\left[\frac{p^{(3)}_{i}-p^{(3)}_{i-1}}{\Delta x}\right]+f^{(3)}_{i,j}$$

The scheme for the divergence equation is:

$$(4)\,\frac{v^{(1)}_{i,j+1}-v^{(1)}_{i,j}}{\Delta x}+\frac{v^{(2)}_{i,j+1}-v^{(2)}_{i,j}}{\Delta x}+\frac{v^{(3)}_{i,j+1}-v^{(3)}_{i,j}}{\Delta x}=0$$

Assigning $\Delta x = K$, then the scheme (1)-(4), is reduced to the following form:

$$(5)\begin{cases}\rho\left[\dfrac{v^{(1)}_{i,j+1}-v^{(1)}_{i,j}}{\Delta t}\right]=3\mu\left[\dfrac{v^{(1)}_{i-1,j}-2v^{(1)}_{i,j}+v^{(1)}_{i+1,j}}{K^2}\right]-\left[\dfrac{p^{(1)}_{i}-p^{(1)}_{i-1}}{K}\right]+f^{(1)}_{i,j};\\[2ex]\rho\left[\dfrac{v^{(2)}_{i,j+1}-v^{(2)}_{i,j}}{\Delta t}\right]=3\mu\left[\dfrac{v^{(2)}_{i-1,j}-2v^{(2)}_{i,j}+v^{(2)}_{i+1,j}}{K^2}\right]-\left[\dfrac{p^{(2)}_{i}-p^{(2)}_{i-1}}{K}\right]+f^{(2)}_{i,j};\\[2ex]\rho\left[\dfrac{v^{(3)}_{i,j+1}-v^{(3)}_{i,j}}{\Delta t}\right]=3\mu\left[\dfrac{v^{(3)}_{i-1,j}-2v^{(3)}_{i,j}+v^{(3)}_{i+1,j}}{K^2}\right]-\left[\dfrac{p^{(3)}_{i}-p^{(3)}_{i-1}}{K}\right]+f^{(3)}_{i,j};\ \text{and}\end{cases}$$

with the constitutive equation; for incompressibility, reducing to,

$$(6)\quad v^{(1)}_{i,j+1}-v^{(1)}_{i,j}+v^{(2)}_{i,j+1}-v^{(2)}_{i,j}+v^{(3)}_{i,j+1}-v^{(3)}_{i,j}=0$$

Re-arranging (5), we have,

$$(7)\, v_{i,j+1}^{(k)} = \left[1-6\left(\frac{\mu}{\rho}\right)\left(\frac{\Delta t}{K^2}\right)\right]v_{i,j}^{(k)} + 3\left(\frac{\mu}{\rho}\right)\left(\frac{\Delta t}{K^2}\right)\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \left(\frac{1}{\rho}\right)\left(\frac{\Delta t}{K}\right)\left[p_i^{(k)} - p_{i-1}^{(k)}\right] + \frac{\Delta t}{\rho} f_{i,j}^{(k)};$$

for each $k = 1;2;3$

This implies that,

$$(8)\, v_{i,j+1}^{(k)} - v_{i,j}^{(k)} = -6\left(\frac{\mu}{\rho}\right)\left(\frac{\Delta t}{K^2}\right)v_{i,j}^{(k)} + 3\left(\frac{\mu}{\rho}\right)\left(\frac{\Delta t}{K^2}\right)\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \left(\frac{1}{\rho}\right)\left(\frac{\Delta t}{K}\right)\left[p_i^{(k)} - p_{i-1}^{(k)}\right] + \frac{\Delta t}{\rho} f_{i,j}^{(k)};\; k = 1;2;3$$

Likewise, we re-write (6) as follows:

$$(9)\sum_{k=1}^{3}\left[v_{i,j+1}^{(k)} - v_{i,j}^{(k)}\right] = 0$$

Rewriting (9), in terms of the right hand side of (8), we obtain,

$$-6\frac{\mu\Delta t}{\rho K^2} v_{i,j}^{(1)} + 3\frac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(1)} + v_{i+1,j}^{(1)}\right] - \frac{\Delta t}{\rho K}\left[p_i^{(1)} - p_{i-1}^{(1)}\right] + \frac{\Delta t}{\rho} f_{i,j}^{(1)}$$
$$-6\frac{\mu\Delta t}{\rho K^2} v_{i,j}^{(2)} + 3\frac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(2)} + v_{i+1,j}^{(2)}\right] - \frac{\Delta t}{\rho K}\left[\!\left[p_i^{(2)} - p_{i-1}^{(2)}\right]\!\right] + \frac{\Delta t}{\rho} f_{i,j}^{(2)}$$
$$-6\frac{\mu\Delta t}{\rho K^2} v_{i,j}^{(3)} + 3\frac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(3)} + v_{i+1,j}^{(3)}\right] - \frac{\Delta t}{\rho K}\left[\!\left[p_i^{(3)} - p_{i-1}^{(3)}\right]\!\right] + \frac{\Delta t}{\rho} f_{i,j}^{(3)} = 0$$

Therefore,

$$-6\frac{\mu\Delta t}{\rho K^2}\sum_{k=1}^{3} v_{i,j}^{(k)} + 3\frac{\mu\Delta t}{\rho K^2}\sum_{k=1}^{3}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \frac{\Delta t}{\rho K}\sum_{k=1}^{3}\left[\!\left[p_i^{(k)} - p_{i-1}^{(k)}\right]\!\right] = -\frac{\Delta t}{\rho}\sum_{k=1}^{3} f_{i,j}^{(k)}$$

This implies that,

$$\sum_{\mathrm{k}=1}^{3}\left[-6\frac{\mu\Delta t}{\rho K^2} v_{i,j}^{(k)} + 3\frac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \frac{\Delta t}{\rho K}\left[\!\left[p_i^{(k)} - p_{i-1}^{(k)}\right]\!\right]\right] = \sum_{k=1}^{3}\left[-\frac{\Delta t}{\rho} f_{i,j}^{(k)}\right].$$

From the preceding, we conclude that,

$$(10) -6\frac{\mu\Delta t}{\rho K^2} v_{i,j}^{(k)} + 3\frac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \frac{\Delta t}{\rho K}\left[\!\left[p_i^{(k)} - p_{i-1}^{(k)}\right]\!\right] = -\frac{\Delta t}{\rho} f_{i,j}^{(k)}$$

Therefore,solving for p_i in (10) ,the recursive algorithm for the region between Γ and Γ_0 is thus given by,

$$(11)\; p_i^{(k)} = -6\frac{\mu}{K} v_{i,j}^{(k)} + 3\frac{\mu}{K}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - K f_{i,j}^{(k)} + p_{i-1}^{(k)}.$$

Therefore, (7) and (11) jointly present a finite difference scheme for the numerical simulation of the Navier-Stokes flow between the boundaries Γ and Γ_0.

The finite difference scheme referred to is,

$$(13)\begin{cases}(a)\ p_i^{(k)} = -6\dfrac{\mu}{K}v_{i,j}^{(k)} + 3\dfrac{\mu}{K}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - Kf_{i,j}^{(k)} + p_{i-1}^{(k)} \\ (b)\ v_{i,j+1}^{(k)} = \left[1 - 6\dfrac{\mu\Delta t}{\rho K^2}\right]v_{i,j}^{(k)} + 3\dfrac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \dfrac{\mu\Delta t}{\rho K}p_i^{(k)} + \dfrac{\Delta t}{\rho}f_{i,j}^{(k)}\end{cases}$$

where, $k = 1,2,3$ represent the three rectangular directions.

will later be tested for convergence and error stability.

For the fluid under consideration, we have,

$T := -p\mathbf{I} + 2\mu D(v)$, as the stress tensor; with $D(v) = \frac{1}{2}\left[\nabla v + \nabla^T v\right]$.

From the preceding equation, we conclude that,

$p_0^{(1)} = -p, \quad p_0^{(2)} = -p$ and $p_0^{(3)} = -p.$

We rewrite 13(b) to obtain,

$$(14)\ v_{i,j+1}^{(k)} = \left[1 - \frac{6\mu}{\rho}\alpha\right]v_{i,j}^{(k)} + \frac{3\mu}{\rho}\alpha\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] + \frac{K^2}{\rho}\alpha f_{i,k}^{(k)} - \frac{\mu K}{\rho}\alpha p_i^{(k)};$$

where, $\alpha = \dfrac{\Delta t}{K^2}$

Later, it will be shown that the scheme (14) converges for $0 < \alpha \le \dfrac{\rho}{6\mu}$; which is in terms of the physical constants for the fluid.

4 Stability for the scheme.

To study the stability of the scheme (13), we define the following appromation errors:

$e_{i,j+1}^{(k)} := v^{(k)}\left(\mathbf{w}_i, t_{j+1}\right) - v_{i,j+1}^{(k)}$

$e_{i,j}^{(k)} := v^{(k)}\left(\mathbf{w}_i, t_j\right) - v_{i,j}^{(k)}$

$e_{i-1,j}^{(k)} := v^{(k)}\left(\mathbf{w}_{i-1}, t_j\right) - v_{i-1,j}^{(k)}$

$e_{i+1,j}^{(k)} := v^{(k)}\left(\mathbf{w}_{i+1}, t_j\right) - v_{i+1,j}^{(k)}$

$e_i^p := p^{(k)}\left(\mathbf{w}_i\right) - p_i^{(k)}.$; where, $\mathbf{w}_i := \left(x_{1i}, x_{2i}, x_{3i}\right); v\left(\mathbf{w}_i, t_j\right)$ is the actual velocity; $p^{(k)}\left(\mathbf{w}_i\right)$ the actual pressure.

Therefore, the statement for the error of approximation for (14), is given by,

$$(15)\ e_{i,j+1}^{(k)} = \left[1 - \frac{6\mu}{\rho}\alpha\right]e_{i,j}^{(k)} + \frac{3\mu}{\rho}\alpha\left[e_{i-1,j}^{(k)} + e_{i+1,j}^{(k)}\right] + \frac{K^2}{\rho}\alpha f_{i,j} - \frac{\mu K}{\rho}\alpha e_i^p + \mathbf{T}_e,$$

where,

$\mathbf{T}_e := \mathbf{T}_e^v + \mathbf{T}_e^P$ is the truncation error;

However, by the Remarks 2.1(2),

$$\frac{\partial^4 v^{(1)}(\xi_1,t)}{\partial x_1^4} = \frac{\partial^4 v^{(2)}(\xi_2,t)}{\partial x_2^4} = \frac{\partial^4 v^{(3)}(\xi_3,t)}{\partial x_3^4} = 0 \text{ (also see [3])}$$

Therefore,

$$\mathbf{T}_e^v := \begin{pmatrix} \frac{1}{2}\Delta t \frac{\partial^2 v^{(1)}\left(x^{(1)},\theta\right)}{\partial t^2} \\ \frac{1}{2}\Delta t \frac{\partial^2 v^{(1)}\left(x^{(2)},\theta\right)}{\partial t^2} \\ \frac{1}{2}\Delta t \frac{\partial^2 v^{(1)}\left(x^{(3)},\theta\right)}{\partial t^2} \end{pmatrix}.$$

Hence,

$$\mathbf{T}_e := \begin{pmatrix} \frac{1}{2}\Delta t \frac{\partial^2 v^{(1)}\left(x^{(1)},\theta\right)}{\partial t^2} \\ \frac{1}{2}\Delta t \frac{\partial^2 v^{(1)}\left(x^{(2)},\theta\right)}{\partial t^2} \\ \frac{1}{2}\Delta t \frac{\partial^2 v^{(1)}\left(x^{(3)},\theta\right)}{\partial t^2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}K \frac{\partial^2 p^{(1)}(\xi_1)}{\partial x_1^2} \\ \frac{1}{2}K \frac{\partial^2 p^{(2)}(\xi_2)}{\partial x_2^2} \\ \frac{1}{2}K \frac{\partial^2 p^{(3)}(\xi_3)}{\partial x_3^2} \end{pmatrix}; \text{ bounded in } [0,T]\times\Omega,$$

where,

$\theta \in \left(t_j, t_{j+1}\right); \xi_1 \in \left(x_{1i}, x_{1(i+1)}\right); \xi_2 \in \left(x_{2i}, x_{2(i+1)}\right)$ and $\xi_3 \in \left(x_{3i}, x_{3(i+1)}\right)$

There fore, by (13a),

$$(16)\ e_i^p := -\frac{6\mu}{K} e_{i,j}^{(k)} + \frac{3\mu}{K}\left[e_{i-1,j}^{(k)} + e_{i+1,j}^{(k)}\right] + Kf_{i,j} + \mathbf{T}_e$$

For a fluid particle; starting from the position of rest in Ω, accelerating towards Γ, we formulate and prove the following theorem:

Theorem 4.1:

For a fluid particle accelerating from the position of rest in Ω and accelerating towards Γ; with $v_{i,j+1}^{(k)} \in \left[0, v_N^{(k)}\right] \subset H^2(\Omega)$; $p_{i+1}^{(k)} \in \left[p_0^{(k)}, p_N^{(k)}\right] \subset L^2(\Omega)$;$t_{j+1} \in [0,T]$,$T < \infty$; $i, j = 1;2;3...,N; k = 1,2,3;$

the scheme (13) admits error stability for $0 < \alpha \le \frac{\rho}{6\mu}$.

Proof:

Firstly, all the norms in the proof are the $\|.\|_{H^2(\Omega)}$ norms.

On the other hand, by (16)

$$\|\mathbf{e}^{p}_{i+1,j}\| \le \frac{6\mu}{K}\|\mathbf{e}_{i,j}\| + \frac{3\mu}{K}\left[\|\mathbf{e}_{i-1,j}\| + \|\mathbf{e}_{i+1,j}\|\right] + K\|f_{i,j}\| + \|\mathbf{T}^{P}_{e}\|.$$

We rewrite the preceding inequality in the form,

$$(17)\ E^{p}_{i+1} \le \frac{12\mu}{K}E_j + KM_f + M^{p}_{e}, \text{ where,}$$

$$E^{p}_{i+1} := \max\|\mathbf{e}^{p}_{i+1,j}\|; M^{p}_{e} := \max\|\mathbf{T}^{P}_{e}\|; M_f := \max\|f_{i,j}\|$$

By (17), we have,

$$\begin{aligned}
E^{p}_{i+1} &\le \frac{12\mu}{K}E_j + KM_f + M^{p}_{e} \\
&\le \frac{12\mu}{K}\left[\frac{12\mu}{K}E_{j-1} + KM_f + M^{p}_{e}\right] + KM_f + M^{p}_{e} \\
&= \left(\frac{12\mu}{K}\right)^2 E_{j-1} + \frac{12\mu}{K}\left(KM_f + M^{p}_{e}\right) + KM_f + M^{p}_{e} \\
&\le \left(\frac{12\mu}{K}\right)^3 E_{j-2} + \left(\frac{12\mu}{K}\right)^2\left(KM_f + M^{p}_{e}\right) \\
&\qquad + \frac{12\mu}{K}\left(KM_f + M^{p}_{e}\right) + \left(KM_f + M^{p}_{e}\right) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\le \left(\frac{12\mu}{K}\right)^N E_0 + \left(\frac{12\mu}{K}\right)^{N-1}\left(KM_f + M^{p}_{e}\right) + \ldots. \\
&\ldots\ldots\ldots\ldots + \frac{12\mu}{K}\left(KM_f + M^{p}_{e}\right) + \left(KM_f + M^{p}_{e}\right)
\end{aligned}$$

Since, $E_0 = 0$, then,

$$(18)\ E_{j+1} \le \left(KM_f + M^{p}_{e}\right)\sum_{k=0}^{N-1}\left(\frac{12\mu}{K}\right)^k$$

By (15), we have,

$$\mathbf{e}_{i,j+1} = \left[1 - \frac{6\mu}{\rho}\alpha\right]\mathbf{e}_{i,j} + \frac{3\mu}{\rho}\alpha\left[\mathbf{e}_{i-1,j} + \mathbf{e}_{i+1,j}\right] + \frac{K^2}{\rho}\alpha f_{i,j} - \frac{\mu K}{\rho}\alpha \mathbf{e}^p_{i+1,j} + \mathbf{T}_e.$$

This implies that,

$$(20)\quad \left\|\mathbf{e}_{i,j+1}\right\| \le \left|1 - \frac{6\mu}{\rho}\alpha\right| \left\|\mathbf{e}_{i,j}\right\| + \frac{3\mu}{\rho}\alpha\left[\left\|\mathbf{e}_{i-1,j}\right\| + \left\|\mathbf{e}_{i+1,j}\right\|\right] + \frac{K^2}{\rho}\alpha\left\|f_{i,j}\right\| + \frac{\mu K}{\rho}\alpha\left\|\mathbf{e}^p_{i+1,j}\right\| + \left\|\mathbf{T}_e\right\|$$

We put: $E_{j+1} := \max\left\|\mathbf{e}_{i,j+1}\right\|$. Hence, $E_j := \max\left\|\mathbf{e}_{i,j}\right\| = \max\left\|\mathbf{e}_{i-1,j}\right\| = \max\left\|\mathbf{e}_{i+1,j}\right\|$

Rewriting (20), we obtain the inequality,

$$(21)\quad E_{j+1} \le \left|1 - \frac{6\mu}{\rho}\alpha\right| E_j + \frac{6\mu}{\rho}\alpha E_j + \frac{K^2}{\rho}\alpha\left\|f_{i,j}\right\| + \frac{\mu K}{\rho}\alpha\left\|\mathbf{e}^p_{i+1,j}\right\| + \left\|\mathbf{T}_e\right\|$$

For $1 - \frac{6\mu}{\rho}\alpha \ge 0$; that is, $0 < \alpha \le \frac{\rho}{6\mu}$, the inequality (21) assumes the form,

$$E_{j+1} \le E_j + \frac{K^2}{\rho}\alpha\left\|f_{i,j}\right\| + \frac{\mu K}{\rho}\alpha\left\|\mathbf{e}^p_{i+1,j}\right\| + \left\|\mathbf{T}_e\right\|;\ \text{that is,}$$

$$(22)\quad E_{j+1} \le E_j + \frac{K^2}{\rho}\alpha M_f + \frac{\mu K}{\rho}\alpha\left(KM_f + M_e^p\right)\sum_{k=0}^{N-1}\left(\frac{12\mu}{K}\right)^k + M_e;\ \text{by (18),}$$

We put $\varphi(\mathrm{K}) = \frac{K^2}{\rho}\alpha M_f + \frac{\mu K}{\rho}\alpha\left(KM_f + M_e^p\right)\sum_{k=0}^{N-1}\left(\frac{12\mu}{K}\right)^k + M_e$,

with (22) assuming the form,

$$(23)\quad E_{j+1} \le E_j + \varphi(\mathrm{K}).$$

Hence, by (23),
$$\begin{aligned} E_{j+1} &\le E_j + \varphi(\mathrm{K}) \\ &\le E_{j-1} + 2\varphi(\mathrm{K}) \\ &\le E_{j-2} + 3\varphi(\mathrm{K}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\le E_0 + N\varphi(\mathrm{K}) \end{aligned}$$

Therefore,

$$(24)\quad E_{j+1} \le N\varphi(\mathrm{K}),\ \text{since } E_0 = 0;\ \text{and error stability follows.}$$

Q.E.D.

Remarks 4.2:

(1) Choosing $\left|\frac{2\mu}{\mathrm{K}}\right| < 1;$ with $0<\alpha \le \frac{\rho}{6\mu},$ we have,

$$\varphi(\mathrm{K}) \le \frac{K^2\alpha M_f}{\rho} + \frac{\mu K\alpha}{\rho}\left(KM_f + M_e^p\right)\sum_{k=0}^{\infty}\left(\frac{2\mu}{\mathrm{K}}\right)^k$$

$$+\mathrm{M_e}$$

$$\le \frac{\mathrm{K}^2 M_f}{6\mu} + \frac{K}{6}\left(KM_f + M_e^p\right)\frac{K}{K-12\mu} + M_e$$

Therefore,

$$(25)\ E_{j+1} \le N\left[\frac{\mathrm{K}^2 M_f}{6\mu} + \frac{K^2\left(KM_f + M_e^p\right)}{6\left(K-12\mu\right)} + M_e\right];\text{and,choosing}$$

the smaller K would decrease the scheme error upper bound.

(2) For $f(x,t) \equiv 0$(conservative force) which is the case in [2], then $M_f = 0,$ and,

$$(26)\ E_{j+1} \le \frac{N}{6}\left(\frac{K^2 M_e^p}{K-12\mu}\right) + M_e;\ \text{for } K > 12\mu$$

5 The scheme for the flow through the boundary Γ.

Any point on the permeable surface Γ may be represented by the general coordinates, $(\tau_1, \tau_2, \mathbf{n}),$ with τ_1 and τ_2 tangential to Γ, and, $\mathbf{n}$ normal to the surface(see Section 2.5 of [5])

Hence, on the boundary $\Gamma,$ the equation ,

$\rho\, \partial_t \mathbf{v}(x,t) \quad = \mu\Delta\mathbf{v}(x,t) - \nabla p(x,t) + \mathbf{f}(x,t),$ assumes the form,

$\sigma\partial_t \mathbf{v}(x,t)\mathbf{n} = \gamma_0\nabla.\left[\gamma_0 T\mathbf{n}\right] + \gamma_0\mathbf{fn};$ with $\mathbf{n}$ as the normal to Γ.

$$= \gamma_0\nabla.\left[-\gamma_0 p\mathbf{In} + 2\mu\gamma_0 D(\mathbf{v})\mathbf{n}\right] + \gamma_0\mathbf{fn}$$

$$= -\gamma_0 p\mathbf{I}\left[\nabla_s.\mathbf{n} + \mathbf{n}.\frac{\partial\mathbf{n}}{\partial n}\right] + 2\mu\gamma_0 D(\mathbf{v})\left[\nabla_s.\mathbf{n} + \mathbf{n}.\frac{\partial\mathbf{n}}{\partial n}\right] + \gamma_0\mathbf{fn}$$

$$= -\gamma_0 p\mathbf{In} + 2\mu\gamma_0 D(\mathbf{v})\mathbf{n} + \gamma_0\mathbf{fn};$$

where σ is the surface fluid density and,$\gamma_0 T = -\gamma_0 p\mathbf{I} + 2\mu\gamma_0 D,$ the usual stress tensor, on the inner container surface,Γ. Also, note that while ∇_s is tangential, $\mathbf{n}$ is normal and, hence, $\nabla_s.\mathbf{n} = 0.$

Finally,

$$\mathbf{n}.\left(\sigma\ \partial_{\mathrm{t}}\mathbf{v}\left(x,t\right)\mathbf{n}\right)=\sigma\partial_t\eta_v$$
$$=\mathbf{n}.\left(-\gamma_0 p\mathbf{In}+2\mu\gamma_0 D\left(\mathbf{v}\right)\mathbf{n}\right)+\mathbf{n}.\gamma_0\mathbf{fn}$$
$$=-\gamma_0 p+\mathbf{n}.\left(2\mu\gamma_0 D\left(\mathbf{v}\right)\mathbf{n}\right)+\mathbf{n}.\gamma_0\mathbf{fn}$$
$$=-\gamma_0 p-2\mu\kappa\eta_v+\mathbf{n}.\gamma_0\mathbf{fn},\ \text{by (17) of [6].}$$

Therefore,

$$(23)\quad \sigma\partial_t\eta_v=-\gamma_0 p-2\mu\kappa\eta_v+\mathbf{n}.\gamma_0\mathbf{fn}$$

Equation (23) in the normal direction to Γ; represents flows across Γ, and it is to be discretized to approximate the value of η_{v}.

In terms of our rectangular coordinates,

$$(24)\quad \eta_{\mathrm{v}}\left(0\right)=v_{i,j+1}^{(1)};\ \text{when } \mathbf{n} \text{ is parallel to the } X_1-\text{axis.}$$

Note:

(i) If $\eta_v\left(0\right)=v_{i,j+1}^{(1)}$, then, $\mathbf{n}.\gamma_0\mathbf{fn}=0$.

(ii) If $\eta_v\left(0\right)=v_{i,j+1}^{(2)}$, then $\mathbf{n}.\gamma_0\mathbf{fn}=-\rho g_{x_2}$,

where, g_{x_2} is the acceleration component due to gravity.

(iii) However, normal stresses at the boundary, Γ, are non-zero and described by the expression,

$$(25)\quad \mathbf{n}.\mathbf{Tn}=-\gamma_0 p-2\mu\eta_v\kappa$$

We then discretize (14) as follows:

$$(26)\quad o\left[\frac{\eta_{\mathrm{i,j+1}}-\eta_{i,j}}{\Delta t}\right]=-2\mu\kappa\eta_{i,j}-\gamma_0 p_i.+\rho g_s+o\left(h^2\right)$$

Rewriting (25), we obtain,

$$(27)\quad \eta_{\mathrm{i,j+1}}=\left(1-\frac{2\mu\kappa\Delta t}{\sigma}\right)\eta_{i,j}-\frac{\Delta t}{\sigma}\gamma_0 p_i+\rho g_s;\ \text{ where, } \gamma_0 p\left(s_i,t_{j+1}\right) \text{ is approximated by } \gamma_0 p_{i+1}.$$

Assuming that the nornal direction on Γ is along the X_1-axis, then we rewrite (27) in the form,

$$(28)\begin{cases}\eta_{i,j+1}=\left(1-\dfrac{2\mu\kappa\Delta t}{\sigma}\right)\eta_{i,j}-\dfrac{\Delta t}{\sigma}\gamma_0 p_i;with\ g_{x_1}=0,\\ where,\\ \eta_{i,j}:=v_{i,j+1}^{(1)};\gamma_0 p_i:=p_i^{(1)}.\end{cases}$$

Assuming a normal direction(parallel to the X_1-axis) to the boundary surface Γ, then from (28), we obtain the boundary computational scheme:

$\eta_{\mathrm{i,j+1}} = \left(1 - \frac{2\mu\kappa\Delta t}{\sigma}\right) v^{(1)}_{i,j+1} - \frac{\Delta t}{\sigma} p^{(1)}_i$; which is readily usable for the approximation of the boundary velocity for the permeable boundary fluid.

Finally, the finite difference scheme of the permeable boundary Navier-Stokes flow is given by,

$$(29)\begin{cases} (a)\ p_i^{(k)} = -6\dfrac{\mu}{K} v_{i,j}^{(k)} + 3\dfrac{\mu}{K}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - Kf_{i,j}^{(k)} + p_{i-1}^{(k)} \\ (b)\ v_{i,j+1}^{(k)} = \left[1 - 6\dfrac{\mu\Delta t}{\rho K^2}\right] v_{i,j}^{(k)} + 3\dfrac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \dfrac{\mu\Delta t}{\rho K} p_i^{(k)} + \dfrac{\Delta t}{\rho} f_{i,j}^{(k)} \\ (c)\ \eta_{i,j+1} = \left(1 - \dfrac{2\mu\kappa\Delta t}{\sigma}\right) v_{i,j+1}^{(1)} - \dfrac{\Delta t}{\sigma} p^{(1)}{}_i;\ \ for\ \ 0 < \Delta t \le \dfrac{\sigma}{2\mu\kappa} \end{cases}$$

Remarks 5.1

(1) It is now logical that we should "marry" the convergence criteria for the components of the scheme (29).While the components 29(a) and 29(b) converge for $0 < \frac{\Delta t}{K^2} \le \frac{\rho}{6\mu}$; the boundary permeation scheme converges for $0<\Delta t \le \frac{\sigma}{2\mu\kappa}$.

(2) Ultimately, we assert that the scheme (28) converges for, $0<\Delta t \le \min\left(\frac{\sigma}{2\mu\kappa}, \frac{\rho K^2}{6\mu}\right)$.

(3) From the preceding remark, we observe that the choice of Δt is subject to the physical contants of the fluid. This makes sense as the boundary permeation time interval is expexted to be very short.

6 The scheme for the flow inside the region $\Omega_{\cdot 0}$.

After the flow through the boundary surface Γ, the fluid "spills" into the region Ω_0. This may require some modification to the scheme (29). We, thus, have the following scheme:

$$(30)\begin{cases}(a)\ p_i^{(k)} = -6\dfrac{\mu}{K}v_{i,j}^{(k)} + 3\dfrac{\mu}{K}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - Kf_{i,j}^{(k)} + p_{i-1}^{(k)} \\ (b)\ v_{i,j+1}^{(k)} = \left[1 - 6\dfrac{\mu\Delta t}{\rho K^2}\right]v_{i,j}^{(k)} + 3\dfrac{\mu\Delta t}{\rho K^2}\left[v_{i-1,j}^{(k)} + v_{i+1,j}^{(k)}\right] - \dfrac{\mu\Delta t}{\rho K}p_i^{(k)} + \dfrac{\Delta t}{\rho}f_{i,j}^{(k)}\end{cases}$$

where, $\mathbf{p}_i := \begin{pmatrix}\gamma_0 p_i \\ 0 \\ 0\end{pmatrix}$; $\mathbf{v}_{i,j} := \begin{pmatrix}\eta_{i,j+1} \\ 0 \\ 0\end{pmatrix}$.; $\mathbf{f}_{i,j} := \begin{pmatrix}-\gamma_0 p_i - 2\mu\eta_{i,j+1}\kappa \\ -g_{x_3} \\ 0\end{pmatrix}$, for a short time duration before non "incident" components were to re-develop.

Remarks 6.1:

(1) In the scheme (30), $p_i^{(1)} = \gamma_0 p_i; v_{i,j}^{(1)} = \eta_{i,j+1}$ and $f_{i,j}^{(1)} = -\gamma_0 p_i - 2\mu\eta_{i,j+1}\kappa.$

(2) By **Remarks** 5.1(2), the scheme will converge for $0<\Delta t \le \min\left(\dfrac{\sigma}{2\mu\kappa}, \dfrac{\rho K^2}{6\mu}\right)$.

7. Conclusion.

Although the discretization of the boundary permeation fluid flow may still be regarded as “splitting hair”, our preceding analysis does indicate that it is still feasible. The role played by the physical constants of the fluid $(\sigma, \rho,$ and $\mu)$, may definitely not be ignored. Obviously, the larger the inner container curvature κ ,the more “viable” the boundary discretization scheme becomes.

8 References:

[1] Burden, R.L. and Faires, J.D.: *Numerical Analysis (7th Edition).* Wadsworth Group (Brookes Cole)(2001): Pacific Grove, CA, USA

[2] Hlomuka, Joe: The linearized non-stationary problem for the permeable boundary Navier-Stokes flows. *Applied Mathematics and Computation,* Vol.158/3;(2004), pp.717-727.

[3] Hlomuka, Joe: On the finite difference scheme for a non-linear evolution problem, with a non-linear dynamic boundary condition. *International Journal of Nonlinear Sciences and Numerical*

Simulation ; Vol. 7/2;(2006),pp.149-154

[4] Hlomuka, Joe.:Solvability conditions for the nonlinear, non-stationary problem of the permeable boundary Navier-Stokes flows. *International Journal of Nonlinear Operators Theory and Applications*, Vol. 1/1;(2006),pp.1-15.

[5] Hlomuka, Joe V. and Sauer,N: Stability of Navier-Stokes flows through permeable boundaries. *Navier-Stokes equations: Theory and numerical methods,*
Ed. R. Salvi; Marcel Dekker, Inc.(2001): New York, Basel. pp.33-43.

[6] Hlomuka, Vuka J. :Stability of a boundary permeation model for Navier-Stokes fluids.
M.Sc. thesis: University of Pretoria (South Africa)
Publikationsansicht(2002);pp.41-42

[7] Ladyzhenskaya,O.L. :*Mathematical theory of viscous incompressible flows*: Gordon & Breach (1963): New York.

[8] Mathews, J.H. : *Numerical methods for mathematics, science and Engineering(2nd Edition)*
Prentice- Hall International, Inc.(1992): Englewood Cliffs,N,J.,USA