A general and Dynamic production lot size Inventory model

Zaid T. Balkhi , and Ali S. Haj Bakry

***Abstract*—**A dynamic inventory model with deteriorating items in which each of the production ,the demand and the deterioration rates, as well as all cost parameters are assumed to be general functions of time is considered in this paper. Besides, shortages are allowed but are partially backordered. . Both inflation and time value of money are taken into account. The objective is to minimize the total net inventory cost . The relevant model is built , solved Necessary and sufficient conditions for a unique and global optimal solution are derived. An illustrative example is provided and numerically verified.

***Keywords*—**General production lot size, Inventory, Variable parameters, Optimality.

I. INTRODUCTION

INVENTORY is known as materials, commodities, products ,..etc, which are usually carried out in stocks in order to be consumed or benefited from when needed. According to Nahmias Book (Production and Operations Analysis (1997)), the investment in inventories in the United States held in the manufacturing, wholesale and retail sectors during the first quarter of 1995 was estimated to be $1.25 trillion **.** Therefore, there is a great need to perform special research on inventory control management for giant systems, in order to improve their efficiency and performance in such a way that the total of relevant inventory costs is minimized. Applying such research results are expected to save huge amounts of money that can be used for development, as it is the case in most first class countries.

In fact, many classical inventory models concern with single item and with the so called Economic Order Quantity(EOQ) models. Among these are Grubstormt and Erdem [21], Giri et al [19], And Sana and Chaudhuri [30]. An interesting problem, related to one of the assumptions of the classical Economic Production Quantity (EPQ) models, has received some attention in the literature. It concerns with the various unit costs involved, which are assumed to be *known and constant*. Among these are Cardenas and Barron [15], Resh et al [30], Hong et al [25], Chang [14], Lan et al [27], Chui [15] and Lai, et al [26]. A more dynamic inventory model was presented by Balkhi and Tadj [14], where they derived an EOQ model with deteriorating items and time varying demand rate, deterioration rate, and costs. Sugapriya and Jeyaraman [33] considered the variability of the holding cost in a non-instantaneous deterioration under a production inventory policy. Balkhi [5] conducted another study in which he treated the variability of parameters for an inventory model for deteriorating items under trade credit policy with partial backordering and an infinite time horizon. Changes in the values of the demand rate and production rate have been studied by Kumar et al [24] by assuming fuzzy values. Another form of variability in parameters of an EPQ inventory model is the learning phenomenon, which is a decrease in production costs of items as time progresses due to more familiarity with the production tools and procedures has been considered by Balkhi[9] . Alamri and Balkhi[1] considered forgetting phenomenon along with learning . Darwish [18] generalized the classical EPQ model by studying the relation between setup cost and length of the production cycle. An inventory model in which products deteriorate at a constant rate and in which demand, production rates are allowed to vary with time has been introduced by Balkhi and Benkhrouf [7]. Subsequently, Balkhi [6], [8], [10], [11], and [13] and Balkhi et al [12] have introduced several inventory models in each of which , the demand , production , and deterioration rates are arbitrary functions of times ,and in some of which , shortages are allowed but are completely backlogged. In each of the last mentioned seven papers, closed forms of the total inventory cost was established, a solution procedure was introduced and the conditions that guarantee the optimality of the solution for the considered inventory system were introduced. Recently, Balkhi[2] has treated a general (EPQ model with variable parameters. Also Balkhi and Foul [3] and [4] have applied a multi-item production inventory model to the Saudi Basic Industries Corporation (SABIC) .Though some of the above mentioned papers don not account for deterioration, the importance of items deteriorating in inventory modeling in now widely acknowledged, as shown by the recent survey of Goyal and Giri[21]

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The goal of this paper is to generalize the models of the above mentioned papers in various ways. First, the cost parameters in our model are general functions of time instead of being linear functions or constant. Second, each of the production, demand rates are assumed to be general functions of time. Third, shortages are allowed, so that part of these shortages is lost and the rest are backlogged. The part of the shortage that is lost is proportional to the waiting time. Fourth ,the on-hand inventories deteriorate while are effectively in stock, and the deterioration rate is a general function of time instead of being a linear function or constant. Fifth, we consider the effect inflation and the time value of money to on the total relevant cost. Sixth, we present practical examples to show how to implement the solution methodology to real problems. In general ,this paper deals with a very general (EPQ) inventory model for which many of the models available in the literature are special cases. Our analysis deepens, broadens, and enriches the available theoretical studies, in particular the mathematical results related to (EPQ) inventory models.The rest of the paper is organized as follows. Following this introduction, we introduce the model assumptions and notations. The problem is formulated in section III. We solve the problem in section IV, and in sections V and VI, we prove the optimality and the uniqueness of the solution. An illustrative example is presented and verified in section VII. Section VII concludes the paper.

II. Assumptions and Notations

The inventory model assumptions and notations are as follows:

1. A single item is produced at the beginning of the cycle and held in stock.
2. Shortages are allowed but only a fraction of the stock out is backordered and the rest are lost.
3. All costs are affected by inflation rate and time value of money. We shall denote by r1 the inflation rate and by r2 the discount rate representing the time value of money so that

r = r2 – r1 is the discount rate net of inflation

The parameters of the model are general functions of time and are denoted as follows:

D(t): Demand rate at time t.
P(t): Production rate at time t.
θ(t): Deterioration rate at time t.
I(t): Inventory level at time t.
c(t): Item production cost at time t.
h(t): Holding cost per unit per unit of time at time t.
b(t): Shortage cost per unit per unit of time at time t for backordered items.
l(t): Shortage cost per unit per unit of time at time t for lost items.
k(t): Setup cost at time t.

β(τ) = $e^{-\tau}$, is the rate of backordered items and τ = T_3 – t is the waiting time up to the new production where shortages start to be backlogged. Note that, **β**(τ) is a decreasing function of τ, which reflects the fact that less waiting time implies more backordered items.

The proposed inventory system operates as follows. The cycle starts at time t = 0 and the inventory accumulates at a rate P(t) - D(t) - θ(t) I(t) up to time t= T_1 where the production stops. After that, the inventory level starts to decrease due to demand and deterioration at a rate D(t) - θ(t)I(t) up to time t = T_2, where shortages start to accumulate at a rate **β**(τ) D(t) up to time t=T_3 .Production restarts again at time t = T_2 and ends at time t = T with a rate P(t)-D(t) to recover both the previous shortages in the period [T2,T3] and to satisfy demand in the period [T3,T4] . The process is repeated. In this respect and in order to recover the backordered items within the period $[T_2,T_3]$ and to satisfies the demand in the period $[T_3,T_4]$ we require that:

$P(t) > [1+\beta(T_3-t)]D(t) = [1+\beta(\tau)]D(t)$ .The behavior of the inventory levels is shown in Fig 1.

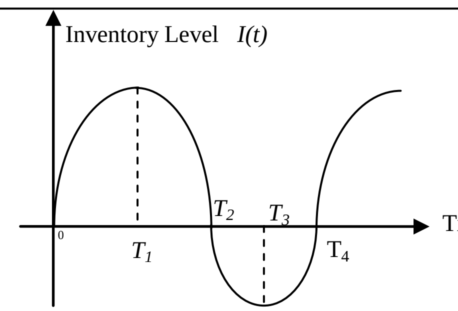

Fig 1.The Behavior of a General and Dynamic Production Lot size Inventory Model

III. Problem Formulation

The changes of the inventory level I(t) is governed by the following differential equation

$$\frac{dI(t)}{dt} = P(t) - D(t) - \theta(t)I(t)\,;\ 0 \le t \le T_1 \quad (1)$$

$$\frac{dI(t)}{dt} = -D(t) - \theta(t)I(t)\,;\ T_1 \le t \le T_2 \quad (2)$$

$$\frac{dI(t)}{dt} = -\beta(\tau)D(t)\,;\ T_2 \le t \le T_3 \quad (3)$$

$$\frac{dI(t)}{dt} = P(t) - D(t)\,;\ T_3 \le t \le T_4 \quad (4)$$

With the boundary conditions I(0)=0, I(T_2) = 0,

I(T_2) = 0, I(T_4) = 0 respectively.
The solutions of the above differential equations are

$$I(t) = e^{-g(t)} \int_0^t \{P(u) - D(u)\} e^{g(u)} du, \; 0 \le t \le T_1 \quad (5)$$

$$I(t) = e^{-g(t)} \int_t^{T_2} D(u) e^{g(u)} du; \; T_1 \le t \le T_2 \quad (6)$$

$$I(t) = -\int_{T_2}^{t} \beta(\tau) D(t) dt; \; T_2 \le t \le T_3 \quad (7)$$

$$I(t) = -\int_t^{T_4} \{P(u) - D(u)\} du, \; T_3 \le t \le T_4 \quad (8)$$

Respectively, where g(t) = θ(t) ⇔ g(t) = $\int \theta(t) dt$ with *g(0)=0*.

Next, we derive the present worth for each type of cost:
Present worth of holding cost (PWHC). Items are held in stock in the two periods [0, T_1] and [T_1, T_2].so we have

$$PWHC = \int_0^{T_1} [H(T_1) - H(t)]\{P(t) - D(t)\} e^{g(t)} d\,t + \int_{T_1}^{T_2} [H(t) - H(T_1)] D(t) e^{g(t)} dt \quad (9)$$

Where

$$H(t) = \int h(t) e^{-r(t) - g(t)} dt \quad (10)$$

With *H(0)=0*

Present worth of shortage cost for backordered items (PWSCB). Shortages occur over two periods, [T_2,T_3] and [T_3,T_4]. Which we denote by $PWSCB_1$ and $PWSCB_2$ respectively. Now ,

$$PWSCB_1 = \int_{T_2}^{T_3} b(t) e^{-rt} I(t) dt$$

$$= \int_{T_2}^{T_3} b(t) e^{-rt} \left(\int_{T_2}^{t} \beta(T_3 - u) D(u) du\right) dt \cdot$$

Integrating by parts, we get:

$$PWSCB_1 = \int_{T_2}^{T_3} [B(T_3) - B(t)] \beta(T_3 - t) D(t) dt \quad (11)$$

Similarly $PWSCB_2 = \int_{T_3}^{T_4} b(t) e^{-rt} I(t) dt$

$$= \int_{T_3}^{T_4} b(t) e^{-rt} \left(\int_t^{T_4} \{P(u) - D(u)\} du\right) dt$$

Integrating by parts, we get:

$$PWSCB_2 = \int_{T_3}^{T_4} [B(t) - B(T_3)]\{P(t) - D(t)\} dt \quad (12)$$

Present worth of storage cost for lost items (PWSCL).

In a small time period (*dt*) we lose a fraction $[1 - \beta(\tau)] D(t) dt$, hence:

$$PWSCL = \int_{T_2}^{T_3} l(t) e^{-rt} [1 - \beta(T_3 - t)] D(t) dt \quad (13)$$

Present worth of item production cost (PWPC). Since production occurs during the two periods [0, T_1] and [T_3,T_4],we have:

$$PWPC = \{\int_0^{T_1} c(t) P(t) e^{-rt} dt + \int_{T_3}^{T_4} c(t) P(t) e^{-rt} dt\} \quad (14)$$

Note that the last cost includes both consumed and deteriorated items.

Present worth of the set-up cost (PWSUC) .The set-up of new production occurs twice during any cycle, the first is at *t* = 0, and the second is at *t* = T_4. Therefore, the present worth of the set-up cost

$$PWSUC = k(0) e^{-r(0)} + k(T_3) e^{-rT_3}$$

$$= k(0) + k(T_3) e^{-rT_3} \quad (15)$$

Hence, the total relevant cost per unit time as a function of T_1, T_2, T_3, T_4 which we shall denote by **TCU(**T_1, T_2, T_3, T_4**)**is given by

$$TCU = \frac{1}{T_4}\{PWHC_1 + PWHC_2 + PWSCB_1 + PWSCB_2 + PWSCL + PWPC + PWSUC\}$$

$$TCU = \frac{1}{T_4}\{\int_0^{T_1} [H(T_1) - H(t)]\{P(t) - D(t)\} e^{g(t)} dt + \int_{T_1}^{T_2} [H(t) - H(T_1)] D(t) e^{g(t)} dt + \int_{T_2}^{T_3} [B(T_3) - B(t)] \beta(T_3 - t) D(t) dt + \int_{T_3}^{T_4} [B(t) - B(T_3)]\{P(t) - D(t)\} dt + \int_{T_2}^{T_3} l(t) e^{-rt} [1 - \beta(T_3 - t)] D(t) dt + \int_0^{T_1} c(t) P(t) e^{-rt} dt + \int_{T_3}^{T_4} c(t) P(t) e^{-rt} dt + k(0) + k(T_3) e^{-rT_3} \quad (16)$$

Our problem is to find the optimal values of T_1, T_2, T_3, T_4 that minimize TCU(T_1, T_2, T_3, T_4) given by (16) subject to the following constraint:

$$0 < T_1 < T_2 < T_3 < T_4 \quad (17)$$

$$\int_{T_2}^{T_3} \beta(T_3 - t) D(t) dt = \int_{T_3}^{T_4} [P(t) - D(t)] dt$$

$$e^{-g(T_1)}\int_0^{T_1}\{P(t)-D(t)\}e^{g(t)}dt = e^{-g(T_1)}\int_{T_1}^{T_2} D(t)e^{g(t)}dt$$

Which are, respectively, equivalent to

$$C_1 : \int_{T_2}^{T_3} \beta(T_3-t)D(t)dt - \int_{T_3}^{T_4}[P(t)-D(t)]dt = 0 \quad (18)$$

$$C_2 : \int_0^{T_1} P(t)e^{g(t)}dt - \int_0^{T_2} D(t)e^{g(t)}dt = 0 \quad (19)$$

Note that constraint (17) is a natural constraint since otherwise our problem would have no meaning. Constraint (18) comes from the fact that, the inventory levels given by (6) & (8) must be equal at $t = T_3$ whereas constraint (19) comes from the fact that the inventory levels given by (2) & (4) must be equal at $t = T_1$

Thus, our problem (call it (P)) is given by

Minimize TCU(T_1,T_2,T_3,T_4) subject to

(17), (19) & (19) (P)

IV. Problem Solution

To solve problem (P), we first ignore (17). This can be justified by the reasons that; if (17) does not hold, then the whole problem would have no meaning. However, we shall not consider any solutions that do not satisfy (17). Thus, our new problem is:

Minimize TCU(T_1,T_2,T_3,T_4) subject to (18) & (19) (P_1)

Note that (P_1) is an optimization problem with two equality constraints, so it can be solved by the Lagrange Techniques . Now ,let L($T_1,T_2,T_3,T_4,\lambda_1,\lambda_2$) be our Lagrangian then,

L($T_1,T_2,T_3,T_4,\lambda_1,\lambda_2$)=

TCU(T_1,T_2,T_3,T_4) + $\lambda_1 C_1 + \lambda_2 C_2$ (20)

The necessary conditions for having optima are:

$$\frac{dL}{dT_1}=0, \frac{dL}{dT_2}=0, \frac{dL}{dT_3}=0, \frac{dL}{dT_4}=0,$$

$$\frac{dL}{d\lambda_1}=0, \frac{dL}{d\lambda_2}=0 \quad (21)$$

From (16), (18) & (19) we have:

$$\frac{dL}{dT_1} = \frac{1}{T_4}\int_0^{T_1}[H'(T_1)[P(t)-D(t)]e^{g(t)}dt -$$

$$\frac{1}{T_4}\int_{T_1}^{T_2} H'(T_1)D(t)e^{g(t)}dt + \frac{1}{T_4}e^{-rT_1}c(T_1)P(T_1)$$

$$+\lambda_1.0 + \lambda_2 P(T_1)e^{g(T_1)} = 0 \quad (22)$$

From (19)we have:

$$\frac{1}{T_4}e^{-rT_1}c(T_1)P(T_1) + \lambda_2 P(T_1)e^{g(T_1)} = 0 \Leftrightarrow$$

$$\lambda_2 = -\frac{e^{-rT_1}c(T_1)}{e^{g(T_1)}T_4}, \text{ or}$$

$$\lambda_2 = \frac{-c(T_1)e^{-rT_1-g(T_1)}}{T_4} \quad (23)$$

$$\frac{dL}{dT_2} = \frac{1}{T_4}[H(T_2)-H(T_1)]D(T_2)e^{g(T_2)} -$$

$$\frac{1}{T_4}[B(T_3)-B(T_2)]\beta(T_3-T_2)D(T_2)$$

$$-\frac{1}{T_4}e^{-rT_2}l(T_2)[1-\beta(T_3-T_2)]D(T_2) -$$

$$\lambda_1\beta(T_3-T_2)D(T_2) - \lambda_2 D(T_2)e^{g(T_2)} = 0 \quad (24)$$

The above equation can be simplified as follows:

$$\frac{1}{T_4}[H(T_2)-H(T_1)]e^{g(T_2)} - \frac{1}{T_4}[B(T_3)-B(T_2)]\beta(T_3-T_2)$$

$$\frac{1}{T_4}e^{-rT_2}l(T_2)[1-\beta(T_3)] - \lambda_1\beta(T_3-T_2) - \lambda_2 e^{g(T_2)} = 0 \quad (25)$$

$$\frac{dL}{dT_3} = \frac{1}{T_4}\frac{dW}{dT_3} + \lambda_1\frac{dC_1}{dT_3} + \lambda_2\frac{dC_2}{dT_3} = 0 \quad .$$

Recalling the constraint C_1 and noting that

$$\beta(T_3-t) = e^{-(T_3-t)}; \beta'(T_3-t) = -e^{-(T_3-t)}; \beta''(T_3-t)$$

$$= e^{-(T_3-t)} = \beta(T_3-t) \quad ,$$

the last equation is equivalent to

$$-\frac{1}{T_4}\int_{T_2}^{T_3}\beta(T_3-t)[B(T_3)-B(t)]D(t)dt +$$

$$\frac{1}{T_4}\int_{T_2}^{T_3} e^{-rt}l(t)\beta(T_3-t)D(t)dt -$$

$$\frac{1}{T_4}e^{-rT_3}c(T_3)P(T_3) + \frac{1}{T_4}k'(T_3)e^{-rT_3} -$$

$$\frac{1}{T_4}rk(T_3)e^{-rT_3} + \lambda_1[\int_{T_2}^{T_3} -\beta(T_3-t)D(t)dt -$$

$$2D(T_3)+P(T_3)] = 0 \quad (26)$$

To facilitate computations of $\frac{dL}{dT_4}$,let

$TCU = \frac{W}{T_4}$,then from(16),(18)&(19)we have

$$\frac{dL}{dT_4} = \frac{\frac{dw}{dT_4}T_4 - w}{T_4^{\,2}} + \lambda_1\frac{dC_1}{dT_4} + \lambda_2\frac{dC_2}{dT_4} = 0$$

This gives:

$$\frac{1}{T_4^{\,2}}(T_4\{[B(T_4)-B(T_3)][P(T_4)-D(T_4)]+$$
$$e^{-rT_4}c(T_4)P(T_4)\}-w)-\lambda_1[P(T_4)-D(T_4)]=0$$

Which in turn gives

$$\lambda_1=\frac{[B(T_4)-B(T_3)][P(T_4)-D(T_4)]}{T_4[P(T_4)-D(T_4)]}+\frac{e^{-rT_4}c(T_4)P(T_4)-TCU}{T_4[P(T_4)-D(T_4)]} \qquad (27))$$

Which leads to

$$TCU=\frac{w}{T_4}=[B(T_4)-B(T_3)][P(T_4)-D(T_4)]+$$
$$e^{-rT_4}c(T_4)P(T_4)-\lambda_1T_4[P(T_4)-D(T_4)] \qquad (28)$$

Where *w* is taken from (16). Note that , (28) gives the minimum total cost in terms of λ_1, T_3 & T_4

Equations (18),(19),(23),(25),(26),(28), are 6 equations with 6 variables .Namely $T_1,T_2,T_3,T_4,\lambda_1,\lambda_2$ so that the solution of these equations (if it exists) gives the critical points of L($T_1,T_2,T_3,T_4,\lambda_1,\lambda_2$) from which ($T_1,T_2,T_3,T_4$) is the corresponding critical point of TCU(T_1,T_2,T_3,T_4).

V. Optimality of Solution

In this section, we derive conditions that guarantee the existence, uniqueness, and global optimality of solution to problem *(P)* For that purpose, we first establish sufficient conditions under which the Hessian matrix of the Lagrangean function L$(T_1^*, T_2^*, T_3^*, T_4^*)$, calculated at any critical point $(T_1^*, T_2^*, T_3^*, T_4^*)$of L , is positive definite. To compute the Hessian matrix of L we consider the following notations

$$\frac{\partial^2 L}{\partial T_i^2}=L_{T_i^2}\ , \quad \frac{\partial^2 L}{\partial T_i\partial T_j}=L_{T_iT_j}\ , \quad i,j=1,2,3,4$$

Then the related computations showed that L$(T_1^*, T_2^*, T_3^*, T_4^*)$ has he following form

$$L(T_1^*, T_2^*, T_3^*, T_4^*)=\begin{bmatrix} L_{T_1^2} & L_{T_1T_2} & 0 & L_{T_1T_4} \\ L_{T_1T_2} & L_{T_2^2} & L_{T_2T_3} & 0 \\ 0 & L_{T_2T_3} & L_{T_3^2} & L_{T_3T_4} \\ L_{T_1T_4} & 0 & L_{T_3T_4} & L_{T_4^2} \end{bmatrix}_{(T_1^*, T_2^*, T_3^*, T_4^*)}$$

By Balkhi and Bebkherouf [7]Stewart [32] and Emet(19) ,this symmetric matrix is positive semi-definite if

$$\left|L_{T_1^2}\right| \ge \left|L_{T_1T_2}\right|+\left|L_{T_1T_4}\right| \qquad (29)$$

$$\left|L_{T_2^2}\right| \ge \left|L_{T_1T_2}\right|+\left|L_{T_2T_3}\right| \qquad (30)$$

$$\left|L_{T_3^2}\right| \ge \left|L_{T_2T_3}\right|+\left|L_{T_3T_4}\right| \qquad (31)$$

$$\left|L_{T_4^2}\right| \ge \left|L_{T_1T_4}\right|+\left|L_{T_3T_4}\right| \qquad (32)$$

Thus, the above arguments lead to the following theorem .

Theorem 1. *Any existing solution of (P_1) is a minimizing solution to (P_1) if this solution satisfies (29) through (32).*

Next we shall show that any minimizing solution of *(P_1)* is unique. To see this, we note, from (21), that *each of* T_1,T_2,T_3,T_4 can implicitly be determined as a function of T_1, say $T_1=f_1(T_1)=T_1\ , T_2=f_2(T_1), T_3=f_3(T_11), T_4=f_4(T_1)$
Our argument in showing the uniqueness of the solution is based on the idea that the general value of TCU given by (16) must coincide with the minimum value of TCU given by (28).That is we must have

W(T_1, $f_2(T_1)$, $f_3(T_1)$, $f_4(T_1)$)/ $f_4(T_1)$ -
TCU(T_1, $f_2(T_1)$, $f_3(T_11)$, $f_4(T_1)$)=0
$\Leftrightarrow$ W(T_1, $f_2(T_1)$, $f_3(T_1)$, $f_4(T_1)$) -
$f_4(T_1)$.TCU(T_1, $f_2(T_1)$, $f_3(T_1)$, $f_4(T_1)$)) = 0 (33)

where *W*(T_1, $f_2(T_1)$, $f_3(T_1)$, $f_4(T_1)$)/ $f_4(T_1)$ is taken from (28) and *TCU*(T_1, $f_2(T_1)$, $f_3(T_11)$, $f_4(T_1)$) is taken from (16).

Note that any minimizing solution of *(P_1)* (if it exists) is unique (hence global minimum) if equation (33), as an equation of T_1, has a unique solution. This fact has been shown by Balkhi([5] ,[8] ,[9] and is illustrated by Fig 2 .Hence , the above arguments lead to the following theorem

Theorem 2. *Any existing solution of (P_1) for which (29) through (32.hold is the uniqueand global optimal solution to(P_1) .*

Next we shall verify our model by the following illustrative example

VII. Illustrative Example and its Verification

We have verified our model by the following illustrative example

$$D(t)=at+a_0\,,\theta(t)=\theta\,,P(t)=p_0e^{pt}\,,k(0)=k_1,$$
$$k(T_3)=k_2\,,C(t)=c_0e^{ct}\,,h(t)=h_0e^{ht},$$
$$b(t)=b_0e^{bt},\ l(t)=l_0e^{lt}\,,\beta(T_3-t)=e^{-(T_3-t)}$$

In order to verify the theoretical results of introduced model, twenty different numerical sets of the parameter values were chosen to be verified for this example

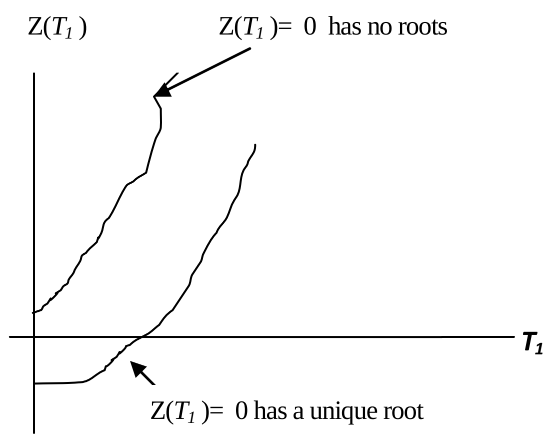

Fig 2. The roots of the equation $Z(T_1) = 0$

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The numerical results are shown by the two tables below , namely Table I & Table II. The first table displays the values of the parameters, whereas the second table shows the optimal solution that correspond to these values. These twenty sets are arranged in an increasing order of the total net relevant cost in order to facilitate the sensitivity analysis. From the above numerical results, one can easily deduce the following. Production rate and the cost parameters have the major influence on the value of TCU. Also, the difference between production and demand rates also has major influence on the value of TCU. The influence of inflation and deterioration rate on the TCU is minor compared to the influence of other parameters, but they are cannot be ignored.

VIII. Conclusion

In this paper, we have considered a general production lot size inventory model in which each of the demand, production, and deterioration as well as all cost parameters are known and general functions of time. Shortages are allowed but are partially backordered. Both inflation and time value of money are incorporated in all cost components. The objective is to minimize the overall total relevant inventory cost. We have built an exact mathematical model and introduced a solution procedure by which we could determine the optimal stopping and restarting production times in any cycle. Then, quite simple and feasible sufficient conditions that guarantee the uniqueness and global optimality of the obtained solution are established. An illustrative example which explains the applicability of the theoretical results are also introduced and numerically verified .Most of previously related models that have been introduced by previous authors are special cases of our model. This seems to be the first time where such a general (EPQ) is mathematically treated and numerically verified.

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Table-I . Twenty Different Sets of the Model Parameters' Values

Set	a	a_0	p	p_0	c	c_0	h	h_0	k_1	k_2	l	l_0	b	b_0	r	θ
Set 1	1.5	10	0.22	20	0.5	1	0.25	0.15	50	150	0.05	0.2	0.2	0.3	0.01	0.08
Set 2	2	10	0.3	18	0.5	5	0.35	0.3	150	70	0.4	1	0.2	0.8	0.15	0.15
Set 3	2	10	0.3	18	0.5	5	0.35	0.3	200	200	0.4	1	0.2	0.8	0.15	0.15
Set 4	1	15	0.2	30	1	1	0.2	0.1	100	100	0.1	0.1	0.1	0.1	0.01	0.01
Set 5	1	15	0.2	30	1.5	1	0.2	0.1	100	100	0.1	0.1	0.03	0.1	0.01	0.01
Set 6	1.2	15	0.2	30	1	1	0.25	0.15	100	200	0.1	0.2	0.1	0.1	0.01	0.01
Set 7	1	10	0.22	20	0.5	3	0.25	0.15	200	150	0.2	0.2	0.2	0.3	0.05	0.08
Set 8	2	10	0.3	20	0.5	5	0.25	0.15	200	150	0.2	0.2	0.2	0.3	0.05	0.08
Set 9	3	10	0.4	18	0.5	8	0.35	0.3	150	70	0.4	5	0.2	0.8	0.15	0.15
Set 10	2	10	0.3	20	0.5	5	0.25	0.15	200	200	0.2	1	0.2	0.8	0.05	0.15
Set 11	28	23	0.9	40	0.3	40	0.35	15	800	600	0.4	8	0.2	10	0.03	0.15
Set 12	25	25	0.9	40	0.4	30	0.35	15	900	870	0.4	10	0.2	8	0.03	0.1
Set 13	28	23	1	35	0.3	45	0.3	10	800	600	0.6	8	0.2	10	0.03	0.15
Set 14	30	23	1	35	0.3	45	0.3	10	800	600	0.6	8	0.1	15	0.03	0.2
Set 15	28	23	0.9	40	0.4	30	0.35	15	800	600	0.4	10	0.2	8	0.03	0.1
Set 16	10	30	0.4	50	0.5	20	0.35	15	300	270	0.4	10	0.2	8	0.05	0.1
Set 17	30	23	1	35	0.6	45	0.3	10	1000	500	0.4	8	0.1	5	0.08	0.15
Set 18	10	30	0.4	50	0.4	30	0.3	25	1000	870	0.4	10	0.2	8	0.05	0.1
Set 19	25	25	0.4	50	0.4	30	0.3	15	1000	870	0.4	10	0.2	8	0.05	0.1
Set 20	30	23	1	35	0.6	45	0.3	10	1000	500	0.3	8	0.1	15	0.03	0.2

Table-II . Optimal Results for the Above Twenty Sets of Table I.

Set	λ_1	λ_2	T_1	T_2	T_3	T_4	Net TCU
Set 1	0.5493	-0.459	0.6045	1.5236	1.5825	2.7912	**122.73**
Set 2	4.0028	-2.8773	0.72063	1.71409	1.76621	2.00714	**184.288**
Set 3	3.90886	-2.6899	0.86425	2.01963	2.05162	2.20951	**204.186**
Set 4	1.0054	-0.8441	0.4161	0.8832	1.0598	1.78133	**205.36**
Set 5	1.6018	-1.3971	0.4213	0.8943	1.0098	1.3353	**238.42**
Set 6	0.974	-0.8295	0.4815	1.0172	1.1564	1.9324	**250.991**
Set 7	1.6604	-1.3776	0.5062	1.3251	1.4397	2.6263	**259.37**
Set 8	4.09641	-2.838	0.55635	1.3144	1.62465	2.16453	**299.762**
Set 9	6.24315	-4.8429	0.57998	1.28339	1.51247	1.85507	**327.971**
Set 10	3.6712	-2.6755	0.50375	1.34669	1.64704	2.17368	**332.141**
Set 11	44.5251	-29.031	0.68049	1.10141	1.35123	1.49505	**3454.83**
Set 12	33.4422	-22.932	0.58976	0.92854	1.32596	1.53403	**3710.8**
Set 13	50.1382	-37.04	0.78139	1.17311	1.26675	1.33434	**3782.58**
Set 14	47.1033	-32.363	0.81915	1.23016	1.37762	1.47252	**4428.88**
Set 15	29.7952	-19.777	0.50374	0.81141	1.48881	1.73796	**5034.47**
Set 16	2.74969	-9.84316	0.060587	0.124855	0.917797	2.07541	**6213.5**
Set 17	58.7418	-37.338	0.80441	1.18413	1.47368	1.62303	**8568.35**
Set 18	2.64418	-11.1222	0.111335	0.227364	1.58336	2.77343	**12560**
Set 19	2.02763	-11.0667	0.229738	0.479007	1.90116	2.87108	**13241.9**
Set 20	50.457	-28.935	0.55234	0.8437	1.66743	1.90786	**17143.1**