Feedback Stabilization of Abstract Delay Systems on Banach Lattices

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***Abstract*—** In this paper, we examine the stabilization problem of systems described by partial differential equations and delay differential equations. The control of a partial differential equation with a time delay is a challenging problem with many applications that include physical, chemical, biological, economic, thermal, and fluid systems. The semigroup method is a unified approach to addressing systems that include ordinary differential equations, partial differential equations, and delay differential equations. Using semigroup theory, we introduce the concept of an abstract delay system that can be used to characterize the behavior of a wide class of dynamical systems. This paper examines the stabilization problem of an abstract delay system on a Banach lattice on the basis of semigroup theory. To tackle this problem, we take advantage of the properties of a non-negative C_0 semigroup on a Banach lattice. The objective of this paper is to propose a stabilization method for an abstract delay system on a Banach lattice. We derive a sufficient condition under which an abstract delay system is delay-independently stabilizable. Furthermore, we provide illustrative examples to verify the effectiveness of the proposed method.

***Keywords*—** Stabilization, Partial differential equation, Time delay, C_0 semigroup, Banach Lattice, Abstract Cauchy Problem, Infinite dimensional system

I. INTRODUCTION

PARTIAL differential equations arise from many physical, chemical, biological, thermal, and fluid systems which are characterized by both spatial and temporal variables [1]-[5]. Fig. 1 illustrates that the time evolution of a solution of a partial differential equation is depending on both spatial and temporal variables.

Time delays also arise in many dynamical systems because, in most instances, physical, chemical, biological, and economic phenomena naturally depend not only on the present state but also on some past occurrences [6]-[8]. Fig. 2 illustrates that the time evolution of the solution of a delay differential equation is depending on both the present and past solutions. The importance of the control of partial differential equations and delay differential equations is well recognized in a wide range of applications. Hence, this paper examines the stabilization problem of partial differential equations with time delays.

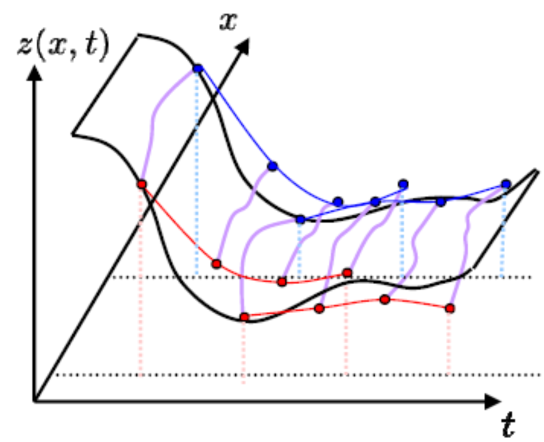

Fig. 1. A schematic view of the time response of a partial differential equation.

Partial differential equations and delay differential equations are known to be infinite-dimensional systems, while ordinary differential equations are finite-dimensional systems. The control of infinite-dimensional systems is a challenging problem attracting considerable attention in many research fields. It has been recognized that semigroups have become important tools in infinite-dimensional control theory over the past several decades. The semigroup method is a unified approach to addressing systems that include ordinary differential equations, partial differential equations, and delay differential equations. The behaviors of many dynamical systems including infinite-dimensional systems and finite-dimensional systems can be characterized by semigroup theory. The recent well-developed theory

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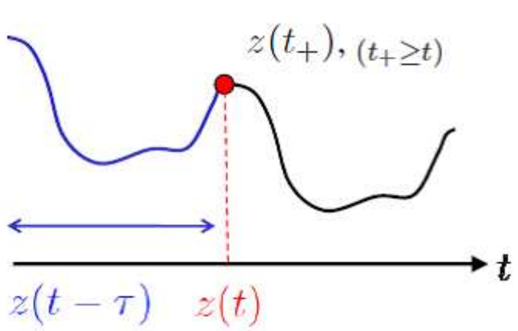

Fig. 2. A schematic view of the time response of a delay differential equation.

in such a framework has been accumulated in several books [9]-[13]. In this paper, using semigroup theory, we introduce the concept of an abstract delay system that can be used to describe the behavior of a wide class of dynamical systems. Fig. 3 shows that an abstract delay system is a generalized model that includes ordinary differential equations, partial differential equations, and delay differential equations. From this point of view, the control method proposed here for an abstract delay system is advantageous for its applicability to a wide class of dynamical systems.

The linear quadratic control problem for an abstract delay system has been studied in [15]-[17]. Furthermore, the H^∞ control problem for such a system has been examined in [18]. The problems addressed in [15]-[18] have been reduced to finding a solution of the corresponding operator Riccati equation in Hilbert spaces. The feedback stabilizability of an abstract delay system on a Banach space has been investigated in [19]. The analytic approach in [19] is based on the compactness of Banach spaces, while the problem in [15]-[18] is formulated in Hilbert spaces to make use of the properties of the inner product.

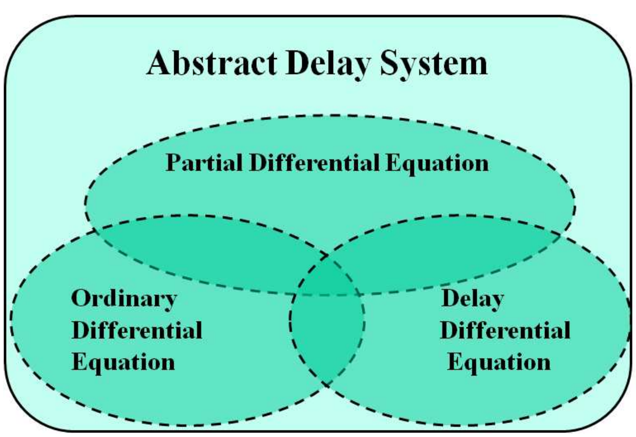

Fig. 3. An abstract delay system is a generalized model.

In this paper, we study the stabilization problem of an abstract delay system on a Banach lattice, which is a Banach space supplied with an order relation. To tackle this problem, we take advantage of the properties of a non-negative C_0 semigroup on a Banach lattice. The objective of this paper is to propose a stabilization method for an abstract delay system on a Banach lattice. We derive a sufficient condition under which an abstract delay system is delay-independently stabilizable. Furthermore, we provide illustrative examples to verify the effectiveness of the proposed method.

This paper is organized as follows. Some notation and terminology are given in Sec. II. The system considered here is defined in Sec. III. Moreover, Sec. III is devoted to the introduction of a stability criterion for an abstract delay system on a Banach lattice. The main results are provided in Sec. IV. In Sec. IV, we study the control design problem for an abstract delay system on the basis of the stability criterion provided in Sec. III. Then, we derive a sufficient condition for the stabilization of an abstract delay system under the assumption that the system has a non-negative delay operator. Furthermore, we provide illustrative examples to verify the effectiveness of the proposed method. Finally, some concluding remarks are given in Sec. V.

II. NOTATION AND TERMINOLOGY

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real numbers and non-negative real numbers, respectively. Let $\mathbb{N}_+$ denote the set of positive integers. Let X be a Banach space endowed with the operator norm $\|\cdot\|$. Let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from a Banach space X to another Banach space Y. Let $\mathcal{L}(X)$ be defined by $\mathcal{L}(X,X)$. Let $I_d \in \mathcal{L}(X)$ denote the identity operator on X.

Definition 1: A family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a C_0 semigroup if all the following properties hold:

(i) $T(0) = I_d$.
(ii) $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_+$.
(iii) The orbit maps $t \mapsto T(t)x$ are continuous maps from $\mathbb{R}_+$ into X for every $x \in X$.

Definition 2: Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on a Banach space X and let $D(\mathcal{A})$ be the subspace of X defined as

$$D(\mathcal{A}) := \left\{ x \in X : \lim_{h \searrow 0} \frac{1}{h}(T(h)x - x) \text{ exists} \right\}.$$

For every $x \in D(\mathcal{A})$, we define

$$\mathcal{A}x := \lim_{h \searrow 0} \frac{1}{h}(T(h)x - x).$$

The operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ is called the generator of the semigroup $(T(t))_{t\geq 0}$. In the following, let $(\mathcal{A}, D(\mathcal{A}))$ denote the operator $\mathcal{A}$ with domain $D(\mathcal{A})$.

Definition 3: Let $(\mathcal{A}, D(\mathcal{A}))$ be the generator of a C_0 semigroup $(T(t))_{t\geq 0}$.

$$\omega_0(\mathcal{A}) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\| \leq Me^{\omega t}, \forall t \in \mathbb{R}_+\}$$

is called the semigroup's growth bound.

Definition 4: Let $(\mathcal{A}, D(\mathcal{A}))$ be a closed operator on a Banach space X. The set

$$\rho(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda I_d - \mathcal{A} \text{ is bijective}\}$$

is called the resolvent set of $\mathcal{A}$, and the set

$$\sigma(\mathcal{A}) := \mathbb{C} \backslash \rho(\mathcal{A})$$

is called the spectrum of $\mathcal{A}$. For $\lambda \in \rho(\mathcal{A})$,

$$R(\lambda, \mathcal{A}) := (\lambda I_d - \mathcal{A})^{-1}$$

is called the resolvent of $\mathcal{A}$ at λ.

$$s(\mathcal{A}) := \sup \{\text{Real part of } \lambda : \lambda \in \sigma(\mathcal{A})\}$$

is called the spectral bound of $\mathcal{A}$.

Definition 5: A C_0 semigroup $(T(t))_{t\geq 0}$ with generator $(\mathcal{A}, D(\mathcal{A}))$ is said to be uniformly exponentially stable if $\omega_0(\mathcal{A}) < 0$.

Definition 6: A Banach space X is called a Banach lattice if X is supplied with an order relation such that all the following conditions hold:

(i) $f \geq g \Rightarrow f + h \geq g + h$ for all $f, g, h \in X$.
(ii) $f \geq 0 \Rightarrow \lambda f \geq 0$ for all $f \in X$ and $\lambda \in \mathbb{R}_+$.
(iii) $|f| \geq |g| \Rightarrow \|f\| \geq \|g\|$ for all $f, g \in X$.

Definition 7: A C_0 semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X is said to be non-negative if

$$0 \leq x \in X \;\Rightarrow\; 0 \leq T(t)x, \text{ for all } t \geq 0.$$

An operator $T(x) \in \mathcal{L}(X)$ on a Banach lattice X is also said to be non-negative if $T(x) \geq 0$ whenever $0 \leq x \in X$.

III. PRELIMINARIES

In this section, we introduce the concept of an abstract delay system that can be used to describe the behavior of a wide class of dynamical systems. For a Banach space Y and a constant $\tau \in \mathbb{R}_+$, let $C([-\tau, 0], Y)$ denote the set of all continuous functions with domain $[-\tau, 0]$ and range Y. For a Banach space $X := C([-\tau, 0], Y)$, let $\Phi \in \mathcal{L}(X, Y)$ be a delay operator, and let $(\mathcal{B}, D(\mathcal{B}))$ be the generator of a C_0 semigroup on Y. With these notations, an abstract delay system is described by the following equation with an initial function $\varphi : [-\tau, 0] \to Y$:

$$\begin{cases} \dot{x}(t) = \mathcal{B}x(t) + \Phi(x(t-\tau)) & \text{for } t \geq 0, \\ x_0 = \varphi \in X. \end{cases} \tag{1}$$

A continuous function $x : [-\tau, \infty) \to Y$ is called a solution of (1) if all the following properties hold:

(i) $x(t)$ is right-sided differentiable at $t = 0$ and continuously differentiable for all $t > 0$.
(ii) $x(t) \in D(\mathcal{B})$ for all $t \geq 0$.
(iii) $x(t)$ satisfies (1).

Let C^r be the set of all r-times continuously differentiable functions. Let $(\mathcal{A}, D(\mathcal{A}))$ be the corresponding delay differential operator on X defined by

$$\begin{aligned} &\mathcal{A}f := \dot{f}, \\ &D(\mathcal{A}) := \{f \in C^1([-\tau, 0], Y) : f(0) \in D(\mathcal{B}) \\ &\qquad \text{and } \dot{f}(0) = \mathcal{B}f(0) + \Phi(f(-\tau))\}. \end{aligned} \tag{2}$$

Lemma 1 ([9]): The operator $(\mathcal{A}, D(\mathcal{A}))$ in (2) generates a C_0 semigroup $(T(t))_{t\geq 0}$ on X.

Lemma 2 ([9]): If $\varphi \in D(\mathcal{A})$, then the function $x : [-\tau, \infty) \to Y$ defined by

$$x(t) := \begin{cases} \varphi(t) & \text{if } -\tau \leq t \leq 0, \\ [T(t)\varphi](0) & \text{if } 0 < t, \end{cases}$$

is the unique solution of (1).

In the subsequent discussion, we assume that each Banach space X, Y in (1) is a Banach lattice.

Lemma 3 ([9]): If $\mathcal{B}$ generates a non-negative C_0 semigroup on Y and the delay operator $\Phi \in \mathcal{L}(X, Y)$ is non-negative, then the C_0 semigroup $(T(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ in (2) is also non-negative, and the following equivalence holds:

$$s(\mathcal{A}) < 0 \Leftrightarrow s(\mathcal{B} + \Phi) < 0.$$

Lemma 4 ([9]): Assume that $(T(t))_{t\geq 0}$ is a non-negative C_0 semigroup with generator $(\mathcal{A}, D(\mathcal{A}))$ on X. Then,

$$s(\mathcal{A}) = \omega_0(\mathcal{A}).$$

The following proposition directly follows from Lemmas 3 and 4.

Proposition 1: Under the assumption that $\mathcal{B}$ generates a non-negative C_0 semigroup on Y and the delay operator $\Phi \in \mathcal{L}(X, Y)$ is non-negative, the C_0 semigroup $(T(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ in (2) is uniformly exponentially stable if and only if the spectral bound $s(\mathcal{B} + \Phi) < 0$.

Note that the equality in Lemma 4 might not hold in general. This means that a C_0 semigroup $(T(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ is not necessarily uniformly exponentially stable even if the spectral bound is negative, i.e., $s(\mathcal{A}) < 0$. It can be seen from Proposition 1 that the non-negativity assumption enables us to determine the stability of an abstract delay system simply by examining the spectral bound.

IV. STABILIZATION OF ABSTRACT DELAY SYSTEMS

Let $(\mathcal{B}, D(\mathcal{B}))$ be the generator of a C_0 semigroup on a Banach lattice Y. For a Banach lattice $X := C([-\tau, 0], Y)$, let $\Phi \in \mathcal{L}(X, Y)$ be a delay operator. In this section, we consider the stabilization problem of an abstract delay system described by

$$\begin{cases} \dot{x}(t) = \mathcal{B}x(t) + \Phi(x(t-\tau)) + \mathcal{C}u(t), \\ x_0 = \varphi \in X, \end{cases} \tag{3}$$

where $u(t) : t \in \mathbb{R}_+ \to Y$ is the control input, and $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a C_0 semigroup on Y.

Assumption 1: Φ is assumed to be non-negative.

Next, we consider the feedback stabilization problem of (3). Let $u(t)$ be given by

$$u(t) = \mathcal{K}x(t), \tag{4}$$

where $(\mathcal{K}, D(\mathcal{K}))$ is the generator of a C_0 semigroup on Y.

Definition 8: System (3) is said to be delay-independently stabilizable if there exists $u(t)$ in (4) such that the equilibrium point $x = 0$ of the resulting closed-loop system is uniformly exponentially stable.

Now, we state the following theorem.

Theorem 1: If there exists $\mathcal{K}$ such that $(\mathcal{B} + \mathcal{C}\mathcal{K})$ generates a non-negative C_0 semigroup and

$$s(\mathcal{B} + \mathcal{C}\mathcal{K} + \Phi) < 0 \tag{5}$$

is satisfied, then system (3) is delay-independently stabilizable.

Proof of Theorem 1: Under the assumption that Φ is non-negative and $(\mathcal{B} + \mathcal{C}\mathcal{K})$ generates a non-negative C_0 semigroup, we see from Proposition 1 that the resulting closed-loop system is uniformly exponentially stable if $s(\mathcal{B} + \mathcal{C}\mathcal{K} + \Phi) < 0$ holds.

An illustrative example is shown below. Let ℓ be a constant. We consider the following partial differential equation with a time delay, defined for $t \geq 0, x \in [0, \ell], s \in [-\tau, 0]$, as

$$\begin{aligned} \frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} &- d(x)z(x,t) \\ &+ b(x)z(x, t-\tau) + u(x,t), \end{aligned} \tag{6}$$

with the Dirichlet boundary condition

$$z(0,t) = z(\ell, t) = 0 \ \text{ for all } t \geq 0, \tag{7}$$

and with the initial condition

$$z(x,s) = h(x,s). \tag{8}$$

This equation can be interpreted as a model for the growth of a population in $[0, \ell]$. $z(x,t)$ is the population density at time t and space x. The term $\partial^2 z(x,t)/\partial x^2$ describes the internal migration. Moreover, the continuous functions $d(x)$ and $b(x)$ represent space-dependent death and birth rates, respectively. τ is the delay due to pregnancy. Let $d(x)$ and $b(x)$ be given as follows:

$$d(x) = 1 + \cos(8\pi x/\ell), \tag{9}$$

$$b(x) = 1 + 2\sin(\pi x/\ell). \tag{10}$$

Let $u(x,t)$ be given by

$$u(x,t) = -k(x)z(x,t). \tag{11}$$

To rewrite system (6) as an abstract delay system, we introduce the spaces $Y := C[0, \ell]$ and $X := C([-\tau, 0], Y)$. Moreover, we define the following operators

$$\Delta := \frac{d^2}{dx^2}, \tag{12}$$

$$D(\Delta) := \left\{ f \in C^2[0, \ell] : f(0) = f(\ell) = 0 \right\}, \tag{13}$$

$$\mathcal{B} := \Delta - M_d - M_k, \quad D(\mathcal{B}) := D(\Delta), \tag{14}$$

$$\Phi := M_b \phi_\tau \in \mathcal{L}(X, Y), \tag{15}$$

where M_d, M_k, and M_b are the multiplication operators induced by $d(x)$, $k(x)$, and $b(x)$, respectively. $\phi_\tau : X \to Y$ denotes the point evaluation in $t \in [-\tau, 0]$. We see from (14) and (15) that system (6) can be rewritten as an abstract delay system (1).

It is shown in [9] that Δ generates a non-negative C_0 semigroup. Since $e^{-t(M_d+M_k)}$ is non-negative, we see from the Trotter product formula [9] that $\mathcal{B}$ in (14) generates a non-negative C_0 semigroup. Moreover, we see from (10) and (15) that Φ is a non-negative operator. Consequently, it follows from Lemmas 3 and 4 that

$$\omega_0(\Delta + M_b - M_d - M_k) = s(\Delta + M_b - M_d - M_k).$$

In the following, we design $\mathcal{K}$ such that

$$s(\mathcal{B} + \Phi) < 0$$

is satisfied. Let δ be defined by

$$\delta := \inf_{x\in[0,\ell]} \left(d(x) + k(x) - b(x)\right).$$

If $\delta > 0$, then the operator $(\Delta + M_b - M_d - M_k + \delta)$ is dissipative. Hence, we obtain

$$\omega_0(\Delta + M_b - M_d - M_k) < -\delta.$$

This condition shows that if

$$b(x) - d(x) - k(x) < 0, \quad \text{for all } x \in [0, \ell],$$

then a solution of (6) is uniformly exponentially stable. For example, if we design

$$k(x) = 2 - d(x) + b(x), \tag{16}$$

then system (6) is delay-independently stable.

The effectiveness of controller (16) is verified by numerical simulations. To solve equation (6) using a numerical algorithm, we must discretize equation (6) into the finite difference equation. The Crank-Nicolson method [21] is a finite difference method used for numerically solving a partial differential equation. It is a second-order method in time and space, and is numerically stable. For the sake of completeness, a brief description of the Crank-Nicolson method applied to this problem is provided in the subsequent discussion.

We divide the space and time into $M \in \mathbb{N}_+$ steps and $N \in \mathbb{N}_+$ steps, respectively. This means that each step size is given by $\Delta x := \ell/(M-1)$ and $\Delta t := t_f/(N-1)$, where t_f denotes the terminal time of the simulation. By means of the discretization, $z(x,t), (0 \le t \le t_f)$ can be described as $z_{i,j}$ $(i = 1, \cdots, M, j = 1, \cdots, N)$, where the subscripts i and j denote the space and time, respectively. For other variables, we adopt such notation without explanation. Note that equation (6) can be discretized as follows:

$$\begin{aligned}\frac{z_{i,j+1} - z_{i,j}}{\Delta t} &= \frac{1}{2}\left\{\frac{z_{i+1,j+1} - 2z_{i,j+1} + z_{i-1,j+1}}{\Delta x^2}\right. \\ &\quad \left. + \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta x^2}\right\} - \{d(x_i) + k(x_i)\}\, z_{i,j} \\ &\quad + b(x_i) z_{i,j-\tau/\Delta t}\end{aligned} \tag{17}$$

Taking Dirichlet boundary condition (7) into account, we see that (17) yields the following equation:

$$\mathbf{A}\mathbf{z}_{j+1} = (\mathbf{C} - \mathbf{D} - \mathbf{K})\mathbf{z}_j + \mathbf{B}\mathbf{z}_{j-\tau/\Delta t}, \tag{18}$$

where let r be defined by

$$r := \frac{\Delta t}{2\Delta x^2},$$

and let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D} \in \mathbb{R}^{M\times M}$, and $\mathbf{z}_j \in \mathbb{R}^M$ be defined as

$$\mathbf{A} := \begin{bmatrix} 1+2r & -r & 0 & 0 & \cdots & 0 \\ -r & 1+2r & -r & 0 & \ddots & \vdots \\ 0 & -r & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & -r & 0 \\ \vdots & \ddots & 0 & -r & 1+2r & -r \\ 0 & \cdots & 0 & 0 & -r & 1+2r \end{bmatrix},$$

$$\mathbf{C} := \begin{bmatrix} 1-2r & r & 0 & 0 & \cdots & 0 \\ r & 1-2r & r & 0 & \ddots & \vdots \\ 0 & r & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & r & 0 \\ \vdots & \ddots & 0 & r & 1-2r & r \\ 0 & \cdots & 0 & 0 & r & 1-2r \end{bmatrix},$$

$$\mathbf{D} := \Delta t \begin{bmatrix} d(x_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & d(x_2) & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & d(x_M) \end{bmatrix},$$

$$\mathbf{K} := \Delta t \begin{bmatrix} k(x_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & k(x_2) & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & k(x_M) \end{bmatrix},$$

$$\mathbf{B} := \Delta t \begin{bmatrix} b(x_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & b(x_2) & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & b(x_M) \end{bmatrix},$$

$$\mathbf{z}_j := \begin{bmatrix} z_{1,j} \\ z_{2,j} \\ \vdots \\ z_{M-1,j} \\ z_{M,j} \end{bmatrix}.$$

Consequently, it follows from (18) that

$$\mathbf{z}_{j+1} = \mathbf{A}^{-1}\{(\mathbf{C} - \mathbf{D} - \mathbf{K})\mathbf{z}_j + \mathbf{B}\mathbf{z}_{j-\tau/\Delta t}\}. \quad (19)$$

Therefore, we see that $\mathbf{z}_j(t)_{(j=1,\cdots,N)}$ is calculated recursively by (19), for a given initial state $h(x,s)$. The parameters employed in the numerical simulations are listed in Table I.

M	101 [steps]
N	101 [steps]
ℓ	10 [m]
t_f	10 [sec]
Δx	0.1 [m]
Δt	0.1 [sec]
τ	1 [sec]
$h(x,s)$	$\lvert\sin(2\pi x/\ell)\rvert$ for all $s \in [-\tau, 0]$

TABLE I
THE PARAMETERS EMPLOYED IN THE NUMERICAL SIMULATIONS.

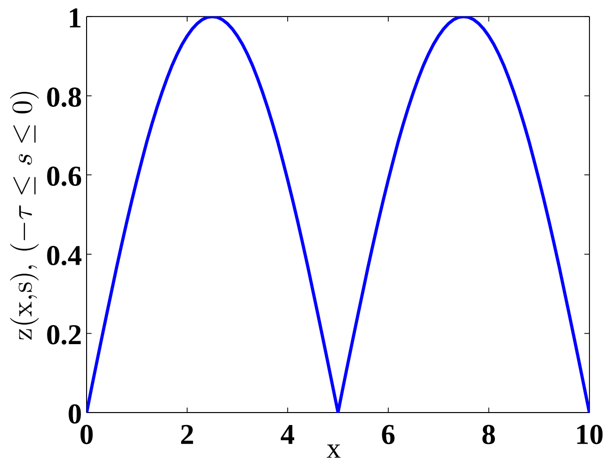

Fig. 4. The initial condition $h(x,s)$.

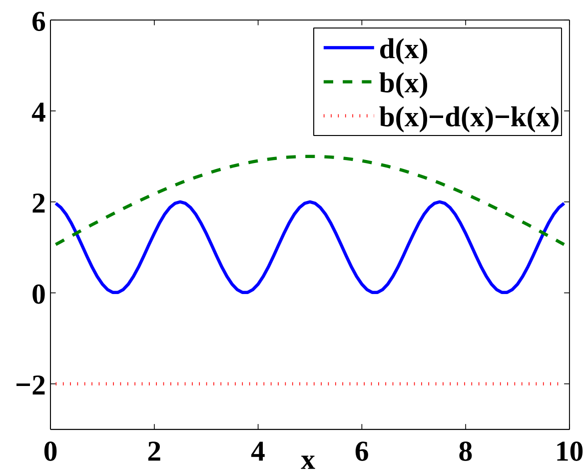

Fig. 5. The space-dependent death and birth rates.

Fig. 4 shows that the initial state $h(x,s)$ is given by $h(x,s) = |\sin(2\pi x/\ell)|$ for all $s \in [-\tau, 0]$. Fig. 5 shows the space-dependent death and birth rates, $d(x)$ and $b(x)$, respectively. Moreover, we see from Fig. 5 that the condition $b(x) - d(x) - k(x) < 0$ is satisfied for all $x \in [0, \ell]$.

The simulation results with the proposed method are shown in Figs. 6-7. Fig. 6 shows the free response of $z(x,t)$ without control. We see that the population density increases over time. Fig. 7 shows the time response of $z(x,t)$ in which the controller (16) is employed. We see that the population density is uniformly stabilized at $z = 0$. The figures reveal the effectiveness of controller (16).

In the following, we consider system (6) with the Neumann boundary condition

$$\frac{\partial z(0,t)}{\partial x} = \frac{\partial z(\ell,t)}{\partial x} = 0 \quad \text{for all } t \geq 0,$$

and with the same initial condition as (8). In this case, we can also rewrite the system as an abstract delay system by introducing the following operators:

$$\Delta := \frac{d^2}{dx^2},$$
$$D(\Delta) := \left\{ f \in C^2[0,\ell] : \frac{df}{dx}(0) = \frac{df}{dx}(\ell) = 0 \right\},$$
$$\mathcal{B} := \Delta - M_d - M_k, \quad D(\mathcal{B}) := D(\Delta),$$
$$\Phi := M_b\phi_\tau \in \mathcal{L}(X,Y),$$

Similarly, we see that if

$$b(x) - d(x) - k(x) < 0, \quad \text{for all } x \in [0,\ell]$$

is satisfied, then the system is uniformly exponentially stable. To perform numerical simulations for

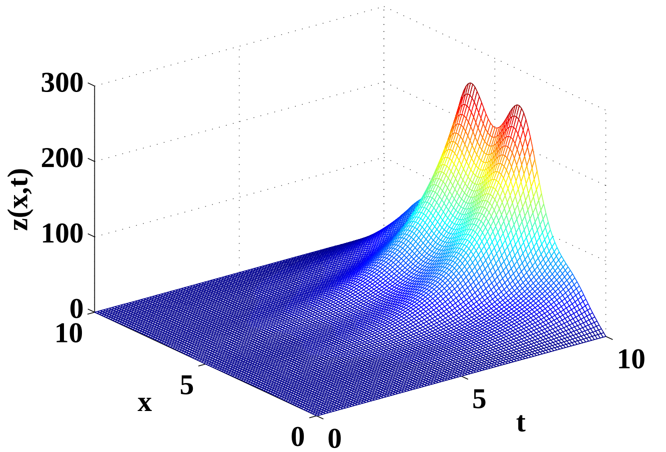

Fig. 6. Free response of $z(x,t)$ without control with the Dirichlet boundary condition.

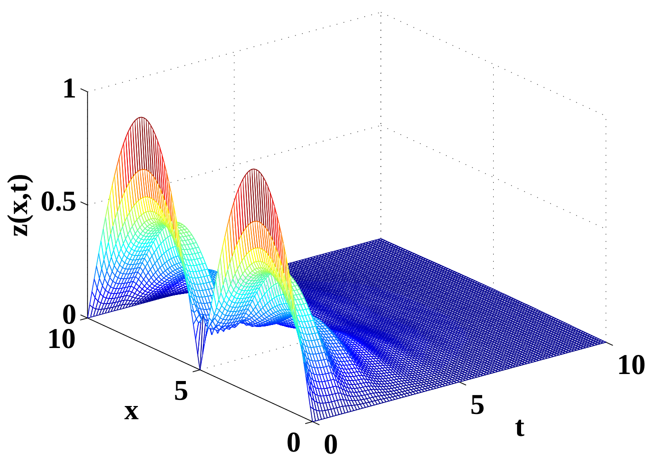

Fig. 7. Time history of $z(x,t)$ controlled by (16) with the Dirichlet boundary condition.

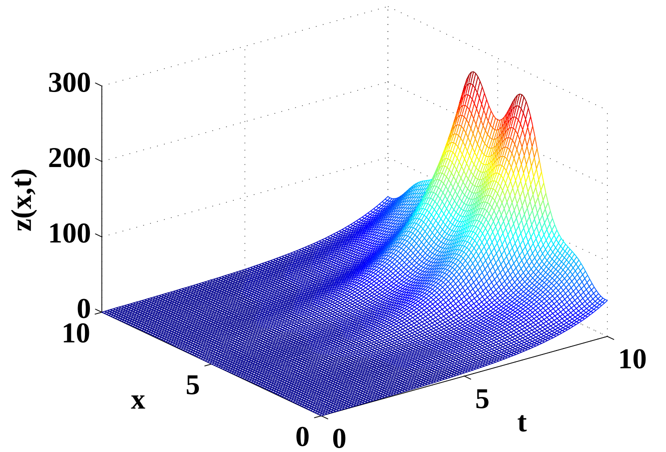

Fig. 8. Free response of $z(x,t)$ without control with the Neumann boundary condition.

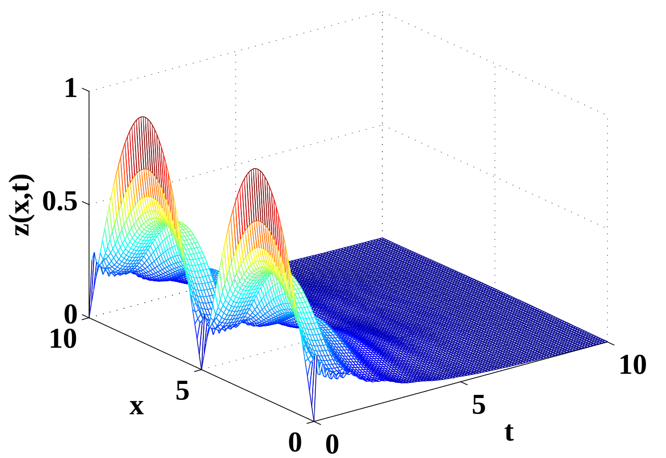

Fig. 9. Time history of $z(x,t)$ controlled by (16) with the Neumann boundary condition.

system (6) with the Neumann boundary condition, we also obtain the discretized equation as in (19), where $\mathbf{A}$ and $\mathbf{C}$ are changed as follows:

$$\mathbf{A} = \begin{bmatrix} 1+2r & -2r & 0 & 0 & \cdots & 0 \\ -r & 1+2r & -r & 0 & \ddots & \vdots \\ 0 & -r & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & -r & 0 \\ \vdots & \ddots & 0 & -r & 1+2r & -r \\ 0 & \cdots & 0 & 0 & -2r & 1+2r \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1-2r & 2r & 0 & 0 & \cdots & 0 \\ r & 1-2r & r & 0 & \ddots & \vdots \\ 0 & r & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & r & 0 \\ \vdots & \ddots & 0 & r & 1-2r & r \\ 0 & \cdots & 0 & 0 & 2r & 1-2r \end{bmatrix}.$$

Likewise, $\mathbf{z}_j(t)_{(j=1,\cdots,N)}$ is calculated recursively by (19), for a given initial state $h(x,s)$.

The simulation results with the Neumann boundary condition are shown in Figs. 8-9. Fig. 8 shows the free response of $z(x,t)$ without control. We see that the population density increases over time.

Fig. 9 shows the time response of $z(x,t)$ in which the controller (16) is employed. We see that the population density is uniformly stabilized at $z = 0$. The figures also reveal the effectiveness of controller (16).

V. CONCLUSION

In this study, the stabilization problem of a partial differential equation with a time delay was examined using semigroup theory. We first introduced the concept of an abstract delay system that can be used to characterize the behavior of a wide class of dynamical systems. Next, we examined the stabilization problem of an abstract delay system on a Banach lattice on the basis of the properties of a non-negative C_0 semigroup. In Sec. IV, we derived a sufficient condition for the stabilization of an abstract delay system under the assumption that the system has a non-negative delay operator. An abstract delay system is a generalized model that includes ordinary differential equations, partial differential equations, and delay differential equations. From this point of view, the control method proposed here for an abstract delay system is advantageous for its applicability to a wide class of dynamical systems. Illustrative examples revealed that the stabilization method proposed here is useful for designing a controller to stabilize a partial differential equation with a time delay.

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REFERENCES

[1] B. Wiwatanapataphee, K. Chayantrakom and Y.-H. Wu, Mathematical Modelling and Numerical Simulation of Fluid-Magnetic Particle Flow in a Small Vessel, *International Journal of Mathematical Models and Methods in Applied Sciences*, 2007, pp. 209-215.

[2] S. R. Sabbagh-Yazdi, N. E. Mastorakis and B. Bayat, Assessment and Application of 3D Galerkin Finite Volume Explicit Solver for Seepage and Uplift in Dam Foundation, *International Journal of Mathematical Models and Methods in Applied Sciences*, 2007, pp. 285-293.

[3] A. Boucherif, One-dimensional parabolic equation with a discontinuous nonlinearity and integral boundary conditions, *International Journal of Mathematical Models and Methods in Applied Sciences*, 2008, pp. 8-17.

[4] H. Sano, Boundary Stabilization of a String with Two Rigid Loads: Calculation of Optimal Feedback Gain Based on a Finite Difference Approximation, *International Journal of Mathematical Models and Methods in Applied Sciences*, 2008, pp. 513-522.

[5] T. N. Le, Y. K. Suh and S. Kang, Efficient Mixing in Microchannel by using Magnetic Nanoparticles, *International Journal of Mathematical Models and Methods in Applied Sciences*, 2009, pp. 58-67.

[6] F. N. Koumboulis, N. D. Kouvakas and P. N. Paraskevopoulos, Analytic Modeling and Metaheuristic PID Control of a Neutral Time Delay Test Case Central Heating System, *WSEAS Transactions on Systems and Control*, Vol. 3, pp. 967-981, 2008.

[7] K. Zakova, One Type of Controller Design for Delayed Double Integrator System, *WSEAS Transactions on Systems and Control*, Vol. 3, pp. 62-69, 2008.

[8] T. Hashimoto and T. Amemiya, Stabilization of Linear Time-varying Uncertain Delay Systems with Double Triangular Configuration, *WSEAS Transactions on Systems and Control*, Vol. 4, pp. 465-475, 2009.

[9] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, Vol. 194, Springer-Verlag, New York, 2000.

[10] R. Nagel, ed., One-Parameter Semigroups of Positive Operators, Lecture Notes in Mathematics, Vol. 1184, Springer-Verlag, Berlin, 1986.

[11] A. Bátkai and S. Piazzera, Semigroups for Delay Equations, Research Notes in Mathematics, Vol. 10, Wellesley, MA: A K Peters Ltd., 2005.

[12] R. F. Curtain and H. J. Zwart, Stability and Stabilization of Infinite Dimensional Systems with Applications, Texts in Applied Mathematics, Vol. 21, Springer-Verlag, New York, 1995.

[13] Z.-H. Luo, B.-Z. Guo and O. Morgul, Stability and Stabilization of Infinite Dimensional Systems with Applications, Springer-Verlag, London, 1999.

[14] M. B. Branco and N. Franco, Study of algorithms for decomposition of a numerical semigroup, *International Journal of Mathematical Models and Methods in Applied Sciences*, 2007, pp. 106-110.

[15] A. J. Pritchard and D. Salamon, The linear-quadratic control problem for retarded systems with delays in control and observation, *IMA Journal of Mathematical Control and Information*, Vol. 2, pp. 335-362, 1985.

[16] A. J. Pritchard and D. Salamon, The linear quadratic control problem for infinite dimensional systems with unbounded input and output operators, *SIAM Journal on Control and Optimization*, Vol. 25, pp. 121-144, 1987.

[17] A. J. Pritchard and S. Townley, Robustness optimization for uncertain infinite-dimensional systems with unbounded input, *IMA Journal of Mathematical Control and Information*, Vol. 8, pp. 121-133, 1991.

[18] A. Kojima and S. Ishijima, Formulas on preview and delayed H^∞ control, *IEEE Transactions on Automatic Control*, Vol. 51, pp. 1920-1937, 2006.

[19] S. Hadd and Q.-C. Zhong, On Feedback Stabilizability of Linear Systems With State and Input Delays in Banach Spaces, *IEEE Transactions on Automatic Control*, Vol. 54, pp. 438-451, 2009.

[20] H. H. Schaefer, Banach Lattices and Positive Operators, Springer-Verlag, New York, 1974.

[21] J. Crank and P. Nicolson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, *Advances in Computational Mathematics*, Vol. 6, 1996, pp. 207-226.

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