Stabilization of a Delayed System by a Proportional Controller

Libor Pekař and Roman Prokop

***Abstract*—** Time-delay systems have been intensively studied for decades. Stability is one of the most important system dynamics properties and the task of stabilization is the main step of controller design. Closed loop characteristic equations of systems with input-output or internal delays contain quasipolynomials rather then polynomials. System poles determined by the solution of such equation have (in most cases) as the same meaning as for delay-free systems, thus they decide about system stability. The aim of this paper is to stabilize a selected system with internal delay by a proportional controller. The task can be equivalently formulated as a stabilization of a system with input-output delay. The analysis and derivations are based on the argument principle, i.e. on the Mikhaylov criterion, and on the required shape of the Mikhaylov plot. The analogy with the notions of the Nyquist criterion is also presented. Stability bounds for the controller parameter are found analytically through proven lemmas, propositions and theorems. Simulation examples clarify the obtained results.

***Keywords*—**Stabilization, time-delay system, characteristic quasipolynomial, argument principle, Mikhaylov plot, loop shaping.

I. Introduction

Systems with aftereffect or dead time, also called hereditary or time-delay systems (TDS), belonging to the class of infinite dimensional systems have been largely studied during last decades due to their interesting and important theoretical and practical features. A number of hypothetic or real-life processes, e.g. in a wide spectrum of natural sciences [1]-[5] or in pure informatics [6], is affected by delays which can have various forms. Linear time-invariant dynamic systems with distributed or lumped delays can be represented by the Laplace transfer function as a ratio of so-called *quasipolynomials* [7] in one complex variable [8]-[10], instead of polynomials which are usual in system and control theory. Quasipolynomials are formed as linear combinations of product of *s*-powers and exponential terms. Delay can significantly deteriorate the quality of feedback control performance, namely stability and periodicity.

The problem of conventional input-output delay in the closed loop has been an interesting topic in control theory since its nascence – indeed, the well known Smith predictor has been known for longer than five decades [11]. Since this pioneering work, many control approaches has been investigated and developed, e.g. [12]-[14]. Linear time delay systems in technological and other processes have been usually assumed to contain delay elements in input-output relations only, which results in shifted arguments on the right-hand side of differential equations. However, this conception is somewhat restrictive in effort to fit the real plant dynamics since in many cases; quantities and variables in inner feedbacks are of distributed or delayed nature, which yield delay elements on the left-hand side of a differential equation. Internal delays also appear in the feedback system when control plants with input-output transport delays, the dynamics of which is characterized using the Laplace transform by the *characteristic quasipolynomial*. This quasipolynomial decides (except some special cases) about the control system asymptotic stability because of the fact that its zeros are system poles with the same meaning as for polynomials; however, the number of poles is infinite.

A large number of conference and journal papers were dedicated to stability analysis of systems with delay elements on the left-hand side of a differential equation, e.g. in [7], [8], [11], [12], and to control design for those systems. see e.g. [13]-[15]. In this paper, we address the stabilization of a selected TDS by a proportional controller, which yields a problem of the stability analysis of the characteristic quasipolynomial. In contrast to some other papers, the presented contribution investigates the stability with respect to the single *non-delay* coefficient and not with respect to the delay. Presented derivations and calculations are based on the fact that the argument principle (i.e. the Mikhaylov criterion) holds for a class of quasipolynomials represented by the studied one as well [7]-[9]. The information about the admissible interval of the selectable real parameter can serve engineers to decide quickly about closed-loop system stability or to set a proportional controller parameter which appears in the characteristic quasipolynomial of a closed loop. Notice that the investigated quasipolynomial was analyzed already e.g. in [16]-[17]; however, these authors utilized different approaches.

The paper is organized as follows: In Chapter II, the solved problem is introduced. Argument principle, the cornerstone of the presented stability analysis, is described in Chapter III. Chapter IV represents the main part of the paper where quasipolynomial stability properties are derived and proven. Coherency with the Nyquist criterion together with simulation examples are demonstrated in Chapter V. Finally, conclusions follow.

The authors kindly appreciate the financial support which was provided by the Ministry of Education, Youth and Sports of the Czech Republic, in the grant No. MSM 708 835 2102.

II. Problem Statement

Linear time-invariant time delay systems (TDS) are generally described by the set of state space functional differential equations in the form

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} &= \sum_{i=1}^{N_H}\mathbf{H}_i\frac{\mathrm{d}\mathbf{x}(t-\eta_i)}{\mathrm{d}t} + \mathbf{A}_0\mathbf{x}(t) + \sum_{i=1}^{N_A}\mathbf{A}_i\mathbf{x}(t-\eta_i) + \mathbf{B}_0\mathbf{u}(t)\\ &+ \sum_{i=1}^{N_B}\mathbf{B}_i\mathbf{u}(t-\eta_i) + \int_0^L \mathbf{A}(\tau)\mathbf{x}(t-\tau)d\tau + \int_0^L \mathbf{B}(\tau)\mathbf{u}(t-\tau)d\tau \qquad (1)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

where $\mathbf{x} \in \nabla^n$ is a vector of state variables, $\mathbf{u} \in \nabla^m$ represents a vector of inputs, $\mathbf{y} \in \nabla^p$ stands for a vector of outputs, $\mathbf{A}_i$, $\mathbf{A}(\tau)$, $\mathbf{B}_i$, $\mathbf{B}(\tau)$, $\mathbf{C}$, $\mathbf{H}_i$ are matrices of compatible dimensions, $\eta_i \le L$ are lumped delays and integrals on the right-hand side express distributed delays. Model (1) can also sufficiently estimate the dynamics of high-order processes [17], [18]. Using the Laplace transform, a transfer function matrix is calculated from (1), see details e.g. in [12]. In a single-input single-output (SISO) case a transfer function as a ratio of so-called *quasipolynomials* is obtained. A quasipolynomial in s over ∇ reads

$$m(s) = s^n + \sum_{i=0}^{n}\sum_{j=1}^{h_i} m_{ij}s^i \exp\left(-s\vartheta_{ij}\right) \qquad (2)$$

If $m_{nj} \neq 0$ for all j = 1, 2,..., h_n, a quasipolynomial is called *retarded*; otherwise it is of the *neutral* form.

The quasipolynomial denominator of a transfer function decides about system stability, except some cases when distributed delays cause that the transfer function has some common unstable poles and zeros. In this rare case, there however exists a stable system realization of form (1) avoiding mentioned common unstable roots [20]. Similarly as for delay-free systems, a TDS is stable if and only if all system poles are located in the open left-half complex plane. Notice that the spectrum of a TDS is infinite in general.

Control of TDS even in a simple feedback control loop as in Fig. 1 brings about serious stability inconveniences. Delay terms expressed in the process transfer function as exponential terms in either the numerator or denominator appear then in the characteristic quasipolynomial and thus the closed loop has an infinite spectrum.

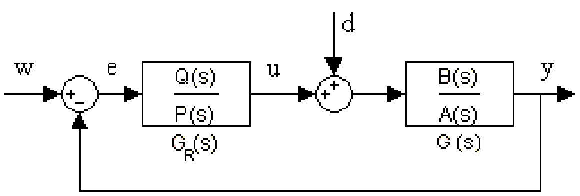

Fig. 1 Simple feedback control loop

The aim of this contribution is to find all possible proportional controllers $G_R(s) = q \neq 0$ which stabilize the system

$$G(s) = \frac{b(s)}{a(s)} = \frac{k}{s + a\exp(-\vartheta s)} \qquad (3)$$

where $a \neq 0 \in \nabla$; $k, \vartheta > 0 \in \nabla$ are fixed. Systems with internal delays (i.e. those with delay terms in the transfer function denominator) are also referred as *anisochronic*. In other words, we are to find lower and upper bounds (if possible) of r so that closed-loop system is stable.

The task is equivalent to the problem of searching $G_{R1}(s) = r$ which stabilizes the plant with input-output (transport) delay

$$G_1(s) = \frac{b_1(s)}{a_1(s)} = \frac{k_1\exp(-\tau s)}{s + a_1} \qquad (4)$$

since both characteristic equations are of the same retarded structure, i.e.

$$m(s) = s + a\exp(-\vartheta s) + kq = s + k_1 r\exp(-\tau s) + a_1 \qquad (5)$$

Hence, if one finds satisfying inequalities for q, it is possible to take substitutions $kq \to a_1$, $a \to k_1 r$, $\vartheta \to \tau$, which yields a solution of the second problem. However, parameters limitations introduced in (3) have to be taken into account.

III. Argument Principle

This chapter recalls a very important fact about TDS and quasipolynomial stability. It holds that for a general *retarded* quasipolynomial (2) the number N_U of unstable roots (i.e. those with non-negative real parts) is given by

$$N_U = \frac{n}{2} - \frac{1}{\pi}\underset{s=\omega \mathrm{j}, \omega\in[0,\infty)}{\Delta \arg}\, m(s) \qquad (6)$$

see [8]. It means that the well known argument principle, or the Mikhaylov stability criterion, holds for stable quasipolynomials (N_U = 0)

$$\underset{s=\omega \mathrm{j}, \omega\in[0,\infty)}{\Delta \arg}\, m(s) = \frac{n\pi}{2} \qquad (7)$$

see also [10].

This result is a powerful tool for retarded quasipolynomials stability analysis and thus for delayed feedback systems. Note that statement (7) does not say anything about the Mikhaylov curve for *unstable* quasipolynomials. Although formula (6) gives the answer about the overall argument change, calculation of N_U by analytic means can be very troublesome.

However, one can draw the Mikhaylov curve using software tools, e.g. Matlab-Simulink, and thus to observe its behavior graphically, or to calculate the curve analytically if it is possible. Information about the overall argument shift in unstable case is extremely important e.g. when using the Nyquist criterion for stabilization of internally delayed systems, as it is presented below in Chapter V. For example, a first order (n = 1) unstable retarded quasipolynomial can behave in the frequency domain e.g. like an unstable polynomial of the first ($\Delta \arg m(s) = -\pi/2$) or that of the third order with one ($\Delta \arg m(s) = \pi/2$) or three unstable roots ($\Delta \arg m(s) = -3\pi/2$).

IV. Stability Studies

Argument principle is now used to analyze stabilizing problem introduced above. The goal is to find the interval for q so that quasipolynomial (5) is asymptotically stable, whereas all the other parameters are fixed, using the criterion (7). Equivalently, the task is to set a proportional stabilizing feedback controller q when control a plant with internal delay ϑ. The loop-shape-like procedure is based on the requirement that the appropriate Mikhaylov curve for $\omega \in [0, \infty)$ must have the overall argument change equal to $\pi/2$, see Fig. 2. Stability properties of quasipolynomial (5) are presented in the form of proven lemmas, propositions and theorems.

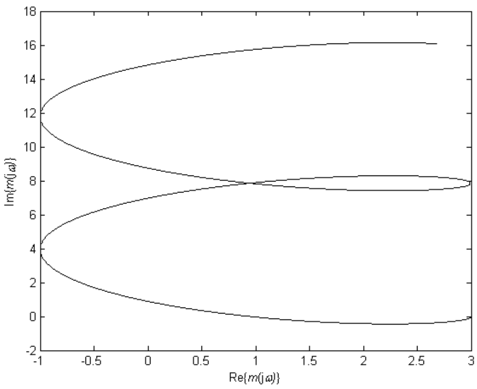

Fig. 2 The Mikhaylov curve of a stable quasipolynomial (5)

Lemma 1. For $\omega = 0$, the imaginary part of the Mikhaylov curve of quasipolynomial (5) equals zero and it approaches infinity for $\omega \to \infty$.

Proof. Decompose $m(\mathrm{j}\omega)$ into real and imaginary parts as

$$\operatorname{Re}\{m(\mathrm{j}\omega)\} = a\cos(\vartheta\omega) + kq \tag{8}$$

$$\operatorname{Im}\{m(\mathrm{j}\omega)\} = \omega - a\sin(\vartheta\omega) \tag{9}$$

Obviously

$$\operatorname{Im}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} = 0, \lim_{\omega\to\infty} \operatorname{Im}\{m(\mathrm{j}\omega)\} = \infty$$ □

Lemma 2. If (4) is stable, the following inequality holds

$$q > \frac{-a}{k} \tag{10}$$

and thus the Mikhaylov curve starts on the positive real axis.

Proof. If (5) is stable, the overall argument shift equals to $\pi/2$ according to (7). Moreover, Lemma 1 states that the imaginary part goes to infinity. These two requirements imply that for stable quasipolynomial is

$$\operatorname{Re}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} > 0 \tag{11}$$

By application of (11) onto (8) yields the condition (10). □

Lemma 2 represents the necessary stability condition and the lower bound for q. The curve can either pass through the first or the fourth quadrant for an infinitesimally small $\omega = \Delta > 0$, which is clarified in the following simple lemma.

Lemma 3. A point on the Mikhaylov curve of (5) lies in the first quadrant for an infinitesimally small $\omega = \Delta > 0$ if and only if

$$a\vartheta \le 1 \tag{12}$$

This point lies in the fourth quadrant if and only if

$$a\vartheta > 1 \tag{13}$$

Proof. (Necessity.) If the point on the curve goes to the first quadrant for an infinitesimally small $\omega = \Delta > 0$, then the change of function $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$ is positive or this function is increasing in $\omega = \Delta$. It is known fact that this is satisfied if either

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} > 0 \tag{14}$$

or there exists *even* $n \in$ | such that

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} = \ldots = \left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} = 0, \\ &\left.\frac{\mathrm{d}^{n}}{\mathrm{d}\omega^{n}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} > 0 \end{aligned} \tag{15}$$

(i.e. there is a local minimum of $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$) or there is *odd* $n \ge 3 \in$ | such that

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} = \ldots = \left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} = 0, \\ &\left.\frac{\mathrm{d}^{n}}{\mathrm{d}\omega^{n}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} \ne 0, \left.\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=\Delta} > 0 \end{aligned} \tag{16}$$

(i.e. there is a point of inflexion of $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$; however, the function is increasing in $\omega = \Delta$).

Analyze the previous three conditions. First, relation (14) w.r.t. (9) reads

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} = 1 - a\vartheta\cos(\vartheta\omega)\big|_{\omega=0} = 1 - a\vartheta > 0 \tag{17}$$

which is satisfied for $a\vartheta < 1$.

Second, condition (15) can be taken into account if

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0} = 0 \Leftrightarrow a\vartheta = 1 \tag{18}$$

hence

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\substack{\omega=0\\ a\vartheta=1}} = 0,\ \left.\frac{\mathrm{d}^3}{\mathrm{d}\omega^3}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\substack{\omega=0\\ a\vartheta=1}} > 0 \tag{19}$$

where the least non-zero *n*th derivation is odd, and thus (15) can not be satisfied for $a\vartheta = 1$; however, we can test (16). Indeed

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\substack{\omega=\Delta\\ a\vartheta=1}} > 0 \tag{20}$$

and thus function $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = \Delta$ is increasing.

Similarly, one can easily verify that if the Mikhaylov plot pass through the fourth quadrant first, then function $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ decreases in $\omega = 0$ when (13) holds .

(Sufficiency.) If conditions (12) or (13) are considered, particular derivations of $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ can be calculated, which guarantee, according to (14) – (16), whether there is a tendency of the Mikhaylov curve to go to the first or the fourth quadrant, respectively. □

The meaning of Lemma 3 is demonstrated in Fig. 3.

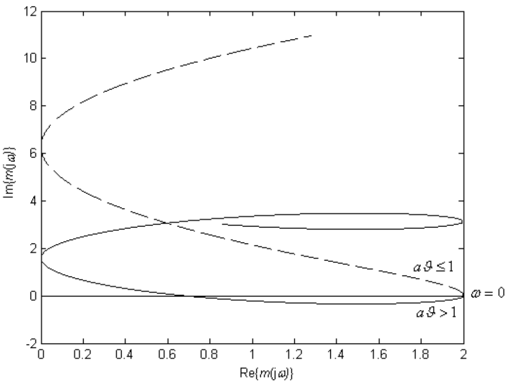

Fig. 3 Clarification of Lemma 3

Lemma 4. If the lower bound (10) holds and *a*, *k*, *q* are bounded, then $\mathrm{Re}\{m(\mathrm{j}\omega)\}$ is bounded for all $\omega > 0$.

Proof. Assume that $a > 0$. Then

$$-2a < -a + kq \le \mathrm{Re}\{m(\mathrm{j}\omega)\} = a\cos(\vartheta\omega) + kq \le a + kq \tag{21}$$

On the other hand, if $a < 0$

$$0 < a + kq \le \mathrm{Re}\{m(\mathrm{j}\omega)\} \le -a + kq < 2kq \tag{22}$$

where the left-hand sides of (21) and (22) and the right-hand one of (22) employ condition (10). The case when $a = 0$ can be discarded due to definition (5) of the quasipolynomial. □

The requirement of bounded parameters is natural with regard to their physical meaning as process quantities or controller gains.

Lemma 5. If (10) holds, there it exists an intersection of the Mikhaylov plot with the imaginary axis for some $\omega > 0$ if and only if

$$a > 0 \ \text{and}\ |kq| \le a \tag{23}$$

Proof. (Necessity.) Show a contradiction, hence if $a < 0$ and (13) holds, then $0 < a + kq \le \mathrm{Re}\{m(\mathrm{j}\omega)\}$ according to Lemma 4 and thus there is no intersection with the imaginary axis.

(Sufficiency.) Consider $a > 0$. If $|kq| \le a$, there must exists $\omega > 0$ such that $a\cos(\vartheta\omega) = kq$, hence, $\mathrm{Re}\{m(\mathrm{j}\omega)\} = 0$. □

Searching of the stability upper bound will be made in two branches, so that conditions (9) and (10) are solved separately. The following theorem presents the necessary and sufficient stability condition for the former case.

Theorem 1. If (12) holds, then quasipolynomial (5) is asymptotically stable if and only if condition (10) is satisfied.

Proof. (Necessity.) See Lemma 2.

(Sufficiency.) Lemma 2 indicates that if (10) is satisfied, the Mikhailov curve starts on the positive real axis for $\omega = 0$. According to Lemma 1 the imaginary part of the curve goes to infinity and Lemma 4 states that for bounded parameters, the curve is bounded in the real axis. Now for the stability it is sufficient to certify that for $a\vartheta \le 1$ the Mikhailov plot does not leave either the first and the fourth quadrant, or the first and the second quadrant, since then the overall phase shift is $\pi/2$.

Indeed, Lemma 4 and Lemma 5 state that if $a < 0$, there is no intersection with the imaginary axis and thus the plot lies in the first and the fourth quadrant. Otherwise, if $0 < a \le 1/\vartheta$, an intersection with the imaginary axis can exist because of Lemma 5. Thus, it ought to be verified that there is no intersection with the real axis. Consider two cases:

1) If $\sin(\vartheta\omega) \ge 0,\ \omega > 0$, then

$$\begin{aligned}\mathrm{Im}\{m(\mathrm{j}\omega)\} &= \omega - a\sin(\vartheta\omega)\\ &\ge \omega - \frac{\sin(\vartheta\omega)}{\vartheta} = \omega\left(1 - \frac{\sin(\vartheta\omega)}{\vartheta\omega}\right) > 0\end{aligned} \tag{24}$$

2) If $\sin(\vartheta\omega) < 0,\ \omega > 0$, we induce a contradiction. Hence, assume that there exists $\omega > 0$ such that $\sin(\vartheta\omega) < 0$ and $\mathrm{Im}\{m(\mathrm{j}\omega)\} = 0$. Then

$$a = \frac{\omega}{\sin(\vartheta\omega)} \tag{25}$$

which yields $\sin(\vartheta\omega) > 0$ and thus we have a contradiction. □

The both stable cases in the second part of the proof of Theorem 1 are pictured in Fig. 4.

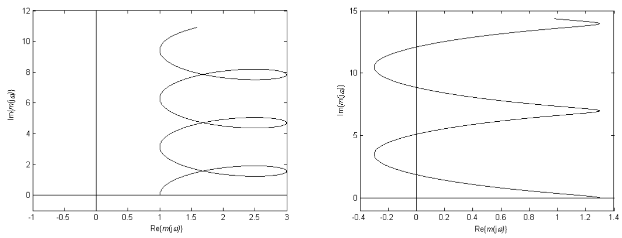

Fig. 4 When (12) holds, a stable Mikhaylov plot lies in the first and the fourth quadrant (a) or in the first and second quadrant (b).

Now consider the second case, i.e. $a\vartheta > 1$. The following result reinforces condition (10).

*Definition 1***.** Let (10) holds. The *crossover frequency* ω_C is defined as

$$\omega_C := \min\{\omega : \omega > 0, \mathrm{Im}\{m(\mathrm{j}\omega)\} = 0\} \tag{26}$$

for some $a \neq 0,\ \vartheta > 0$. In other words, it represents the least solution of (25).

The frequency is graphically displayed in Fig. 5.

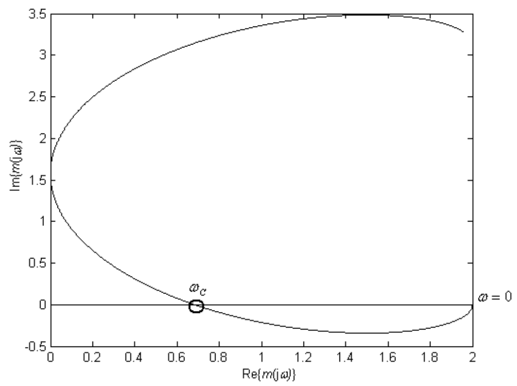

Fig. 5 Crossover frequency

Theorem 2. If (13) holds, then quasipolynomial (5) is asymptotically stable if and only if

$$q > \frac{-a\cos(\vartheta\omega_C)}{k} \tag{27}$$

Proof. (Necessity.) Lemma 1 and Lemma 2 state that the Mikhaylov curve for stable quasipolynomial (5) starts on the positive real axis. Condition (13) guaranties that the initial movement of the curve in the imaginary axis is negative, see Lemma 3. Thus, the curve has to pass through the fourth followed by the first quadrant. In other words, the first crossing with the real axis on the frequency $\omega_C > 0$ has to satisfy

$$\begin{aligned} \mathrm{Im}\{m(\mathrm{j}\omega_C)\} &= \omega_C - a\sin(\vartheta\omega_C) = 0 \\ \mathrm{Re}\{m(\mathrm{j}\omega_C)\} &= a\cos(\vartheta\omega_C) + kq > 0 \end{aligned} \tag{28}$$

which gives (27) directly.

(Sufficiency.) If (13) holds, then $a > 0$ and

$$q > \frac{-a\cos(\vartheta\omega_C)}{k} \geq \frac{-a}{k} \tag{29}$$

and thus the Mikhaylov curve for quasipolynomial (4) starts on the positive real axis according to Lemma 2 and the initial change of the curve in the imaginary axis is negative, see Lemma 3. Condition (27) then agrees with the fact that the curve crosses positive real axis first, as it is obvious from (9). Since the curve is bounded in the real part and the imaginary part goes to infinity (see Lemma 1 and Lemma 4), the overall phase shift is $\pi/2$ and thus the quasipolynomial is stable. □

An example of a stable quasipolynomial according to Theorem 2 is presented in Fig. 6.

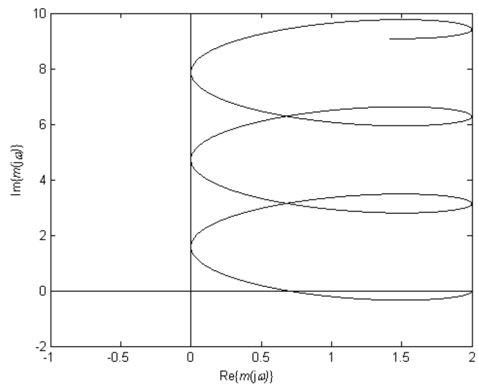

Fig. 6 Stable Mikhaylov criterion according to Theorem 2

V. Alternative: Nyquist criterion

This chapter proposes a link between the Mikhaylov criterion used above with the Nyquist criterion. Both criteria are based on the argument principle which holds for retarded quasipolynomials as in (7). The Nyquist criterion gives information about the closed-loop stability according to the knowledge of the overall phase shift of the open-loop transfer function denominator. Consider closed loop TDS transfer function

$$G_{WY}(s)=\frac{Y(s)}{W(s)}=\frac{G_R(s)G(s)}{1+G(s)G_R(s)}=\frac{G_0(s)}{1+G_0(s)} \tag{30}$$

If transfer functions are expressed as ratios of quasipolynomials, $G(s)=b(s)/a(s)$, $G_R(s)=q(s)/p(s)$, then

$$G_{WY}(s)=\frac{\dfrac{q(s)b(s)}{p(s)a(s)}}{\dfrac{p(s)a(s)+q(s)b(s)}{p(s)a(s)}} \tag{31}$$

where $p(s)a(s)+q(s)b(s)$ is the characteristic quasipolynomial of the closed-loop system. Since the plant transfer function is strictly proper, the highest s-power n of $p(s)a(s)+q(s)b(s)$ equals that of $p(s)a(s)$. Sign $\underset{s=\omega j,\omega\in[0,\infty)}{\Delta \arg} p(s)a(s)=l\pi/2$, then, if the closed loop is stable, then must hold that

$$\underset{s=\omega j,\omega\in[0,\infty)}{\Delta \arg}\left(1+G_0(s)\right)=(n-l)\frac{\pi}{2} \tag{32}$$

according to (7), (30) and (31). The Nyquist criterion thus states that: If the Nyquist plot of the open-loop transfer function $G_0(s)$ encircles the critical point [-1;0j] $[(n-l)/2]$-times (and it does not cross the point), the closed-loop system is asymptotically stable. One circuit means the phase shift equals π.

Applying the criterion onto the problem (3), one can easily deduce that the closed-loop system is stable if

$$\underset{s=\omega j,\omega\in[0,\infty)}{\Delta \arg}\left(1+G_0(s)\right)=(1-l)\frac{\pi}{2} \tag{33}$$

The question is how to utilize the results of Theorem 1 and Theorem 2. Theorem 1 states that if the plant has $a\vartheta\le 1$, the closed-loop system is asymptotically stable if and only if $q>-a/k$. Assume the limit unstable case $q=-a/k$. It is possible to calculate that this is equivalent to

$$\left[1+G_0(\mathrm{j}\omega)\right]_{\omega=0}=0 \tag{34}$$

In other words, the Nyquist plot $G_0(\mathrm{j}\omega)$ starts in the critical point [-1;0j]. Condition (10) then moves the plot so that (33) is satisfied.

According to Theorem 2, if $a\vartheta>1$, the crossover frequency ω_C has to be found first, consequently, the closed-loop system is asymptotically stable if and only if $q>-a\cos(\vartheta\omega_C)/k$. Again, the limit unstable case $q=-a\cos(\vartheta\omega_C)/k$ means that the Nyquist plot crosses the critical point [-1;0j] at frequency ω_C. Stability condition (27) moves the curve so that the required number of encircles is reached.

A. Gain Margin and Phase Margin

Using the Nyquist plot, a factor of stability can be measured by the gain margin and the phase margin which characterize the "distance" of the plot from the critical point.

The *gain margin* means an amplification of $G_0(s)$ so that the closed loop becomes unstable (or equivalently, remains stable). Generally, we can have upper $A_{m,\max}$ and lower $A_{m,\min}$ gain margin, defined as

$$\begin{aligned}
&A_{m,\min}:=\max_{\omega_{CP,\min}}\left(-1/\operatorname{Re}\{G_0(\mathrm{j}\omega_{CP,\min})\}\right)\\
&\omega_{CP,\min}:=\{\omega:\{\operatorname{Im}G_0\}=0,\ \operatorname{Re}\{G_0(\mathrm{j}\omega)\}\in(-\infty,-1)\cup(0,\infty)\}\\
&A_{m,\max}:=\min_{\omega_{CP,\max}}\left(-1/\{\operatorname{Re}G_0(\mathrm{j}\omega_{CP,\max})\}\right)\\
&\omega_{CP,\max}:=\{\omega:\operatorname{Im}\{G_0\}=0,\operatorname{Re}\{G_0(\mathrm{j}\omega)\}\in(-1,0)\}
\end{aligned} \tag{35}$$

where $\omega_{CP,\min}$ and $\omega_{CP,\max}$ are the appropriate *phase crossing frequencies* (through the imaginary axis), see Fig. 7. Then the acceptable open-loop gain interval satisfying closed loop stability is

$$q\in\left(A_{m,\min},A_{m,\max}\right) \tag{36}$$

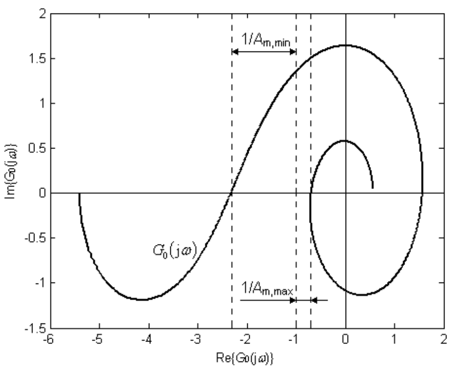

Fig. 7 Gain margin

By comparison (10), (27) and (36), there is a lower bound $A_{m,\min}$ only corresponding to either $-a/k$ or $-a\cos(\vartheta\omega_C)/k$ in our case given by (3).

The *phase margin* is defined as

$$\varphi_m:=\pi+\varphi=\pi+\arg G_0(\mathrm{j}\omega_{CM}),\omega_{CA}:=\{\omega:|G_0(\mathrm{j}\omega)|=1\} \tag{37}$$

where ω_{CA} is the *gain (amplitude) crossing frequency*, see Fig. 8. Examples combining the Mikhaylov criterion with the Nyquist criterion for the problem (3) follow.

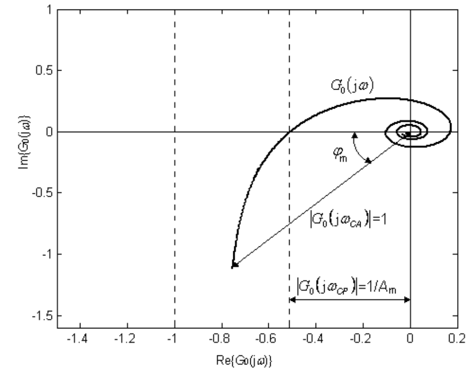

Fig. 8 Phase margin

B. Examples

Example 1. Let $k = 1, a = -1, \vartheta = 1$ are parameters of the TDS plant which is unstable as reveals from the Mikhaylov plot of $a(s)$ in Fig. 9. Notice that the phase shift is $\Delta \arg a(s) = -\pi / 2$, i.e. l = -1.

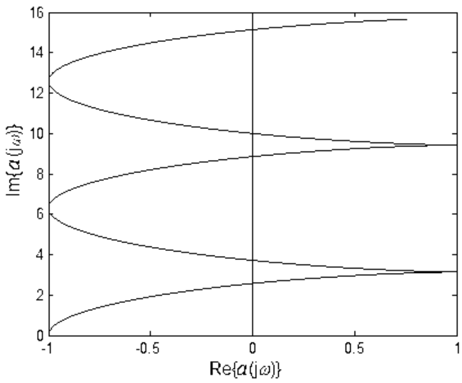

Fig. 9 Mikhaylov plot of $a(s)$ in Example 1

Since $a\vartheta < 1$, Theorem 1 states that the closed loop system with a proportional controller q is stable if and only if $q > 1$. Choose q = 2, then the closed-loop characteristic quasipolynomial has the Mikhaylov plot as in Fig. 10.

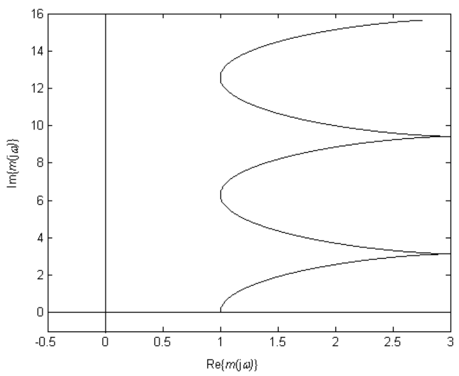

Fig. 10 Mikhaylov plot of $m(s)$ in Example 1

Obviously, $A_m = A_{m,\min} = 0.5$, because $0.5q = a\vartheta$ (i.e. the stability border). The open-loop Nyquist plot is displayed in Fig. 11. Condition (33) yields the stability condition $\Delta \arg(1 + G_0(s)) = \pi$ which is fulfilled and one can verify that $\varphi_m = 1.326$ [rad] = 76 °.

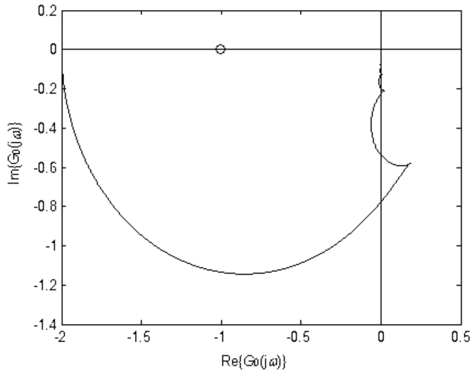

Fig. 11 Nyquist plot for Example 1

Finally, the corresponding stable closed-loop response in is in Fig. 12 (input equals 1).

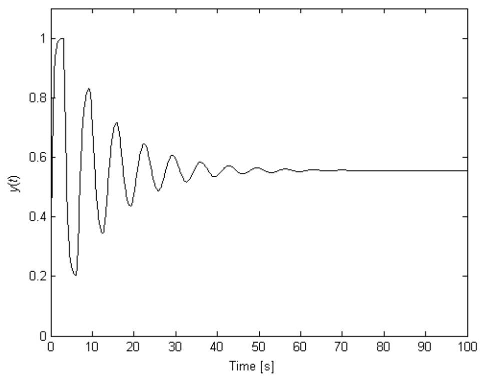

Fig. 12 Closed-loop response for Example 1

Example 2. Now consider the second case, e.g. $k = 1$, $a = 2$, $\vartheta = 3$. Again, the plant is unstable, see Fig. 13, because $\Delta \arg a(s) = -3\pi / 2$.

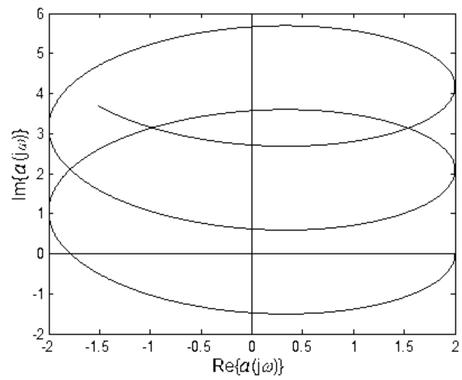

Fig. 13 Mikhaylov plot of $a(s)$ in Example 2

The crossover frequency can be found as $\omega_C = 0.934$ which gives the stability condition according to (27) as $q > 1.886$. Take $q = 2.5$, i.e. $A_m = A_{m,\min} = 0.754$. The Mikhaylov plot of $m(s)$ is pictured in Fig. 14 and, obviously, the closed-loop system is stable.

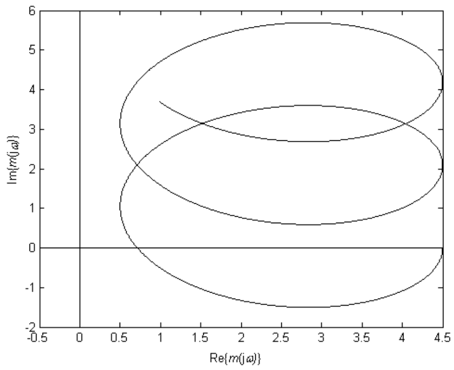

Fig. 14 Mikhaylov plot of $m(s)$ in Example 2

The corresponding Nyquist plot must have $\Delta \arg(1 + G_0(s)) = 2\pi$ as derived in (33). Indeed, Fig. 15 verifies again that the system is stabilized.

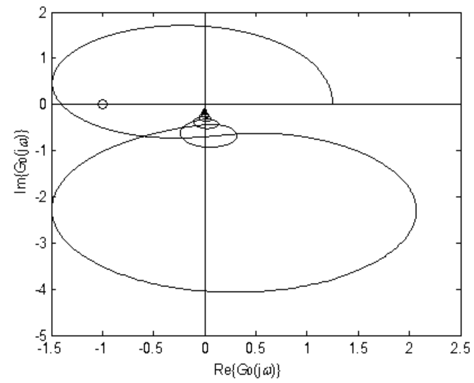

Fig. 15 Nyquist plot for Example 2

In time domain, the stable closed-loop transfer function is shown in Fig. 16.

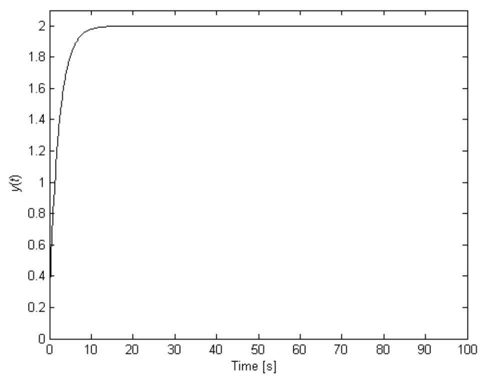

Fig. 16 Closed-loop transfer function for Example 2

VI. Conclusion

In this paper, we have addressed the stabilization of a system with internal delay (so-called anisochronic) by a proportional controller. The problem yields a stability analysis of a selected first order quasipolynomial. The aim has been to find acceptable upper and lower limits for a non-delay parameter; however, only the lower one could be derived. The analysis has been based on the argument principle, i.e. the Mikhaylov stability criterion, in order to keep the desired shape of the Mikhaylov curve. The analogy with the Nyquist stability criterion and its measures – the gain and phase margins – has been shown. Simulation examples figure clarify the proposed methodology and they are supported by many figures. The analytic tools utilized in this contribution can be employed when studying other retarded quasipolynomials as well.

References

[1] V. B. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Dordrecht: Cluwer Academy, 1999.

[2] F. M. Koumboulis, N. D. Kouvakas, and P. N. Paraskevopoulos, "Analytic modeling and metaheuristic PID control of a neutral time delay test case central heating system", *WSEAS Trans. on Systems and Control*, vol. 3, no. 11, 2008, pp. 967-981. ISSN 1991-8763.

[3] L. Garbai and L. Barna, "Modelling of non-steady-state conditions in a gas boiler heated room," in *Proc. 3rd IASME/WSEAS Int. Conf. Heat Transfer, Thermal Engineering and Environment*, Corfu, Greece, August 20-22, 2006, pp. 6-11.

[4] J. Morávka and K. Michálek, "Anisochronous model of the metallurgical RH Process," *Trans. of the VŠB – Technical University of Ostrava, Mechanical Series*, vol. 14, no. 2, 2008, pp. 91–96.

[5] L. Pekař, R. Prokop, and P. Dostálek, "Circuit Heating Plant Model with Internal Delays," *WSEAS Transaction on Systems*, vol. 8., issue 9, September 2009, pp. 1093-1104. ISSN 1109-2777.

[6] L. Bushnell, "Editorial: Networks and Control," *IEEE Control System Magazine*, Vol.21, No.1, pp. 22-99, 2001.

[7] L. E. El'sgol'ts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviated Arguments*. New York: Academic Press, 1973.

[8] H. Górecki, S. Fuksa, P. Grabowski, and A Korytowski, *Analysis and Synthesis of Time Delay Systems*. New York: John Wiley & Sons, 1989. ISBN 978-0471276227.

[9] J. E. Marshall, H. Górecki, A. Korytowski, and K. Walton, *Time Delay Systems, Stability and Performance Criteria with Applications*. Ellis Horwood, 1992.

[10] P. Zítek and A. Víteček, *Control Design of Systems with Delays and Nonlinearities* (in Czech). Prague: ČVUT publishing, 1999.

[11] O. J. M. Smith, "Closer control of loops with death time," *Chem. Eng. Prog.*, vol. 53, no. 5, 1957, pp. 217-219.

[12] J-P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, Issue 10, 2003, pp. 1667-1694.

[13] R. Matušů and R. Prokop, "Control of systems with time-varying delay: a comparison study," in *Proc. 12th WSEAS Int. Conf. Automatic Control, Modelling and Simulations*, Catania, Italy, 2010, pp. 125-130.

[14] E. M. Jafarov, "Robust sliding mode control of multivariable time-delay systems," in *Proc. 11th WSEAS Int. Conf. Automatic Control, Modelling and Simulations*, Istanbul, Turkey, 2009, pp. 430-437.

[15] L. Pekař, R. Prokop, "Control of delayed integrating processes using two feedback controllers – RMS approach," in *Proc. 7th WSEAS Int. Conf. System Science and Simulation in Engineering,* Venice, Italy, 2008, pp. 35-40.

[16] E. Beretta and Y. Kuang, "Geometric stability switch criteria in delay differential systems with delay depended parameters," *SIAM Journal on Mathematical Analysis*, vol. 3, no. 5, 2002, pp. 1144-1165.

[17] K. L. Cooke and Z. Grossman, “Discrete delay, distributed delay and stability switches,” *Journal of Mathematical Analysis and Applications*, vol. 86, 1982, pp. 592-627.
[18] T. Vyhlídal and P. Zítek, “Control system design based on a universal first order model with time delays,” *Acta Polytechnica*, vol. 44, no. 4-5, 2001, pp. 49-53.
[19] L. Pekař and R. Prokop, „An approach for relay based identification of anisochronic models,“ in *Proc. 27th IASTED Int. Conf. Modeling, Identification and Control*, Innsbruck, Austria, 2008 [CD-ROM].
[20] J. J. Loiseau, “Algebraic tools for the control and stabilization of time-delay systems,” *Annual Reviews in Control*, vol. 24, 2000, pp.135-149.