Influence of memory time on the resonant behavior of an oscillatory system described by a generalized Langevin equation

K. Laas, R. Mankin, and E. Reiter

***Abstract*—** Motivated by subdiffusive motion of biomolecules observed in living cells, we study the output response of a system with memory described by a generalized Langevin equation under the impact of an external periodic force. The influence of fluctuations of environmental parameters on the dynamics of the system is modeled by a multiplicative dichotomous noise and by an internal Mittag-Leffler noise. The long-time behavior of the output response is obtained and the presence of stochastic resonance effects are analyzed. In the short memory time limit of Mittag-Leffler noise the dynamics of the system corresponds to a fractional oscillator driven by an internal noise with a power-law autocorrelation function. However, at intermediate and long memory times the dynamics has qualitative difference. Particularly, it is established that the critical memory exponent which marks a dynamical transition in the behavior of the system considered depends strongly on the ratio of the period of the external deterministic force to the memory time. The phenomena of the resonance versus the memory time as well as friction-induced transitions between different stochastic resonance regimes are also discussed.

***Keywords*—** Generalized Langevin Equation, memory-induced resonance, Mittag-Leffler noise, multiplicative noise, stochastic resonance, viscoelastic friction.

I. Introduction

STOCHASTIC, ordinary, or partial differential equations are very common tools in far from equilibrium systems in natural sciences and engineering where stochastic fluctuations are necessary for an appropriate description of the phenomenology involved [1]-[3]. Particularly, noise-induced phenomena in complex systems present a fascinating subject of investigation since, contrary to all intuition, environmental randomness may induce more order in the behavior of the system and thus generate unexpected effects. Among them we can mention the ratchet effect [1], [4], [5], [6], hypersensitive response [7], noise-enhanced stability [2], [8], [9], stochastic resonance [10], [11], and noise-induced multistability [12], [13]. It is well known that the conventional Brownian motion theory cannot account for anomalous diffusion processes, in which the mean-square displacement is not proportional to time. Examples of such systems are supercooled liquids, glasses, colloidal suspensions, polymer solutions [14], [15], viscoelastic media [16],[17], and amorphous semiconductors [18]. Particulary, diffusion of mRNA-s and ribosomes in the cytoplasm of living cells is anomalously slow [19], and large proteins behave similarly [20]. Even anomalous diffusive dynamics of atoms in biological macromolecules and intrinsic conformational dynamics of proteins can be subdiffusive [21]-[23]. The modeling of such anomalous stochastic diffusion processes has mainly been done using the generalized Langevin equation (GLE) approach [21], [24], [25], [26]. The dynamical equation for such systems is in most cases obtained by replacing the usual friction term by a generalized friction term with a power-law-type memory [21], [22], [26], [27]. Physically such a friction term has, due to the fluctuation-dissipation theorem, its origin in a non-Ohmic thermal bath whose influence on the dynamical system is described with a power-law correlated additive noise in the GLE [21], [25]. Although a GLE with a power-law-type friction kernel is very useful for modeling anomalous diffusion processes, the corresponding power-law correlated noises have some nonphysical properties, e.g. absence of a characteristic memory time and infinite variance. Thus, recently Viñales and Despósito have introduced a more general noise with a Mittag-Leffler correlation function (called Mittag-Leffler noise) in the GLE [28]. Notably, for certain values of the parameters that characterize this noise one can produce a power-law correlation function, a standard Ornstein-Uhlenbeck noise with an exponential correlation function, and a white noise. The behavior of the GLE with an additive noise has been investigated in some detail [24], [26], but it seems that analysis of the potential consequences of interplay between a multiplicative noise and memory effects is still rather rare in literature [29]-[32]. This is quite unjustified in view of the fact that the importance of multiplicative fluctuations and viscoelasticity for biological systems, e.g. living cells, has been well recognized [20], [33].

Thus motivated, we consider a GLE with a Mittag-Leffler memory kernel subjected to an external periodic force. The influence of the fluctuating environment is modeled by a multiplicative dichotomous noise and an additive Mittag-Leffler noise. The main contribution of the paper is as follows. We provide exact formulas for analytic treatment of the dependence of the mean particle displacement, in the long-

Katrin Laas, Assistant and PhD student, Insitute of Mathematics and Natural Sciences, Tallinn University, 25 Narva Road, 10120 Tallinn, Estonia (e-mail: katrin.laas@tlu.ee).

Romi Mankin, Professor, Insitute of Mathematics and Natural Sciences, Tallinn University, 25 Narva Road, 10120 Tallinn, Estonia (e-mail: romi.mankin@tlu.ee).

Eerik Reiter, Associate Professor, Institute of Physics, Tallinn University of Technology, 5 Ehitajate Road, 19086 Tallinn, Estonia (e-mail: eerik.reiter@ttu.ee).

time limit, $t \to \infty$, on system parameters. On the basis of those exact expressions we will show that stochastic resonance (SR) is manifested in the dependence of the response of the GLE upon both the dichotomous noise amplitude and the switching rate. Furthermore, we will show that the output signal of the GLE exhibits a resonance-like dependence on the characteristic memory time. Moreover, we have found a critical memory exponent below which friction-induced reentrant transitions appear between different SR regimes of the system. To avoid misunderstanding, let us mention that we use term the SR in a wide sense, meaning nonmonotonic behavior of the output signal or some function of it, e.g., moments, in response to noise parameters [11], [34].

The structure of the paper is as follows. In Section 2 we present the basic model investigated. Exact formulas for the mean particle displacement are derived in Section 3. In Section 4 we analyze the behavior of the output response, and expose the main results of this paper. Section 5 contains some brief concluding remarks.

I. Model

We start from the traditional GLE model in one selected direction for a particle of the mass ($m = 1$) in the fluctuating harmonic potential

$$V(X,t) = \left(\omega^2 + Z(t)\right)\frac{X^2}{2} \tag{1}$$

subjected to a linear friction with a memory kernel $\eta(t)$, an additive periodic force, and an internal random force $\xi(t)$ of zero mean:

$$\ddot{X} + \int_0^t \eta\left(t-t'\right)\dot{X}(t')\,dt' + \frac{\partial}{\partial X}V(X,t) = A_0 \sin(\Omega t) + \xi(t), \tag{2}$$

where $\dot{X}(t) \equiv dX/dt$, $X(t)$ is the particle displacement, and A_0 and Ω are the amplitude and the frequency of the harmonic driving force, respectively. The random force $\xi(t)$ is Gaussian and fully characterized by its autocorrelation function satisfying the fluctuation-dissipation relation

$$\langle \xi(t)\xi(t') \rangle = k_B\, T\, \eta\left(|t-t'|\right) \tag{3}$$

which in turn is a consequence of the fluctuation-dissipation theorem [35]. In Eq. (3), T is the absolute temperature of the heat bath and k_B is the Boltzmann constant. It is well known that if the correlation function (3) is a Dirac delta function, the stochastic process *X(t)* described by Eq. (2) with $Z(t) = A_0 = 0$ is Markovian and its dynamics can be straightforwardly obtained [36]. However, in order to describe the non-Markovian dynamics of an anomalously diffusing particle one must take into account the memory effects manifested by a long tail noise. Usually a power-law correlation function is employed to model such processes [21], [22], [26]. As in Ref. [28], in this paper we assume a more general correlation function modeled as

$$\eta(t) = \frac{\gamma}{\tau^\alpha} E_\alpha\left[-\left(\frac{|t|}{\tau}\right)^\alpha\right], \tag{4}$$

where τ acts as the characteristic memory time, γ is a constant (called friction constant), and the exponent α can be taken as $0 < \alpha < 1$, which is determined by the dynamical mechanism of the physical process considered. The $E_\alpha(y)$ function denotes the Mittag-Leffler function [37], which behaves as a stretched exponential for short times and as inverse power law in the long time regime. Note that if $\alpha = 1$, the correlation function (3) with Eq. (4) reduces to an exponential form which describes a standard Orstein-Uhlenbeck process [36]. In the limit $\tau \to 0$ the proposed correlation function reproduces a power-law correlation function

$$\langle \xi(t)\xi(t') \rangle = \frac{\gamma}{\Gamma(1-\alpha)|t-t'|^\alpha}, \tag{5}$$

where $\Gamma(y)$ is the gamma function, which has been previously used to model viscoelastic properties of a medium [16], [26]. Moreover, taking the limit $\alpha \to 1$ in Eq. (5) we obtain that the noise $\xi(t)$ corresponds to white noise and consequently to non-retarded friction. Fluctuations of the frequency ω of the binding harmonic field are expressed as a dichotomous noise $Z(t)$. The dichotomous process $Z(t)$ [38] is a random stationary Markovian process that consists of jumps between two values a and $-a$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities $p_S(a) = p_S(-a) = 1/2$. The mean value of $Z(t)$ and the correlation function are

$$\langle Z(t)\rangle = 0, \qquad \langle Z(t)Z(t')\rangle = a^2 e^{-\nu|t-t'|}, \tag{6}$$

where $\tau_c = 1/\nu$ is the correlation time. The probabilities $W_n(t)$ that $Z(t)$ is in the state $n \in \{1,2\}$, $z_1 = a, z_2 = -a$, at the time *t* evolve according to the master equation

$$\frac{d}{dt}W_n(t) = \nu \sum_{m=1}^{2} S_{nm} W_m(t), \tag{7}$$

where

$$S_{nm} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{8}$$

The transition probabilities $T_{ij} = p(z_i, t+t' \mid z_j, t)$ between the states z_n can be represented by means of the transition matrix T_{ij} of the dichotomous process as follows

$$T_{ij} = \delta_{ij} + \left(1 - e^{-\nu t'}\right) S_{ij}, \tag{9}$$

where δ_{ij} is the Kronecker symbol. The dichotomous process is a particular case of the Kangaroo process [39]. Note that the noise *Z(t)* is assumed as statistically independent from the noise $\xi(t)$.

II. Exact solution

To find the first moment of *X*, we use the Shapiro-Loginov procedure [29], [40], which for an exponentially correlated noise *Z(t)* yields

$$\frac{d}{dt}\langle Zg \rangle = \left\langle Z \frac{dg}{dt} \right\rangle - \nu \langle Zg \rangle, \tag{10}$$

where *g* is an arbitrary function of the noise, $g = g(Z)$. From Eqs. (1)-(6) and (10) we thus obtain an exact linear system of first-order integrodifferential equations for four variables, $x_1 \equiv \langle X \rangle$, $x_2 \equiv \langle \dot{X} \rangle$, $x_3 \equiv \langle ZX \rangle$, and $x_4 \equiv \langle Z\dot{X} \rangle$:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\omega^2 x_1 - x_3 - \int_0^t \eta\,(t-t')\, x_2\,(t')\, dt' + A_0 \sin(\Omega\, t), \\
\dot{x}_3 &= x_4 - \nu x_3, \\
\dot{x}_4 &= -a^2 x_1 - \omega^2 x_3 - \nu x_4 \\
&\quad - e^{-\nu t} \int_0^t \eta\,(t-t')\, x_4\,(t') e^{\nu t'} dt'.
\end{aligned} \tag{11}$$

The solution of equations (11) can be formally represented in the form

$$x_i = \sum_{k=1}^{4} H_{ik}(t) x_k(0) + A_0 \int_0^t H_{i2}(t-t') \sin(\Omega\, t') dt', \tag{12}$$

where the constants of integration $x_k(0)$ are determined by initial conditions. The relaxation functions $H_{ik}(t)$ with the initial conditions $H_{ik}(0) = \delta_{ik}$ can be obtained by means of the Laplace transformation technique. Particularly, we find that

$$\begin{aligned}
\hat{H}_{11}(s) &= [\omega^2 + (s+\nu)^2 + (s+\nu)\hat{\eta}(s+\nu)] \\
&\quad \times \frac{\left(s + \hat{\eta}(s)\right)}{D(s)}, \\
\hat{H}_{21}(s) &= \frac{a^2 - \omega^2\left[\omega^2 + (s+\nu)^2 + (s+\nu)\hat{\eta}(s+\nu)\right]}{D(s)}, \\
\hat{H}_{31}(s) &= -\frac{a^2 [s + \hat{\eta}(s)]}{D(s)}, \\
\hat{H}_{41}(s) &= -\frac{a^2 (s+\nu)(s+\hat{\eta}(s))}{D(s)}, \\
\hat{H}_{12}(s) &= \frac{\omega^2 + (s+\nu)^2 + (s+\nu)\,\hat{\eta}(s+\nu)}{D(s)}, \\
\hat{H}_{22}(s) &= \frac{s\,[\omega^2 + (s+\nu)^2 + (s+\nu)\,\hat{\eta}(s+\nu)]}{D(s)}, \\
\hat{H}_{32}(s) &= -\frac{a^2}{D(s)}, \\
\hat{H}_{42}(s) &= -\frac{(s+\nu)a^2}{D(s)}, \\
\hat{H}_{13}(s) &= -\frac{s + \nu + \hat{\eta}(s+\nu)}{D(s)}, \\
\hat{H}_{23}(s) &= -\frac{s\left[s + \nu + \hat{\eta}(s+\nu)\right]}{D(s)}, \\
\hat{H}_{33}(s) &= \frac{\left[s^2 + \omega^2 + s\,\hat{\eta}(s)\right]\left[s + \nu + \hat{\eta}(s+\nu)\right]}{D(s)}, \\
\hat{H}_{43}(s) &= \frac{a^2 - \omega^2\left[s^2 + \omega^2 + s\,\hat{\eta}(s)\right]}{D(s)}, \\
\hat{H}_{14}(s) &= -\frac{1}{D(s)}, \\
\hat{H}_{24}(s) &= -\frac{s}{D(s)}, \\
\hat{H}_{34}(s) &= \frac{s^2 + \omega^2 + s\,\hat{\eta}(s)}{D(s)}, \\
\hat{H}_{44}(s) &= \frac{(s+\nu)\left[s^2 + \omega^2 + s\,\hat{\eta}(s)\right]}{D(s)}.
\end{aligned} \tag{13}$$

where

$$D(s) = \left(s^2 + s\,\hat{\eta}(s) + \omega^2\right) \times \left[\omega^2 + (s+\nu)^2 + (s+\nu)\,\hat{\eta}(s+\nu)\right] - a^2 \tag{14}$$

with

$$\hat{\eta}(s) = \frac{\gamma\, s^{\alpha-1}}{1+(\tau\, s)^{\alpha}}, \tag{15}$$

and $\hat{H}_{ik}(s)$ is the Laplace transform of $H_{ik}(t)$, i.e.,

$$\hat{H}_{ik}(s) = \int_0^{\infty} e^{-s\,t} H_{ik}(t)\, dt. \tag{16}$$

One can check the stability of solution (12), which, according to the results of Ref. [41], means that the solution s_j of the equation $D(s) = 0$ cannot have roots with a positive real part. This requirement is met if the inequality

$$a^2 < a_{cr}^2 = \omega^2\left[\omega^2 + \nu^2 + \frac{\gamma\,\nu^{\alpha}}{1+(\tau\,\nu)^{\alpha}}\right] \tag{17}$$

holds. Henceforth in this work we shall assume that condition (17) is fulfilled. Thus in the long time limit, $t \to \infty$, the memory about the initial conditions will vanish as

$$\sum_{k=1}^{4} H_{1k}(t) x_k(0) = \frac{\gamma\,\hat{H}_{12}(0)\, x_1(0)}{\Gamma(1-\alpha)\, t^{\alpha}} + O\left(t^{-(1+\alpha)}\right) \tag{18}$$

and the first moment $\langle X(t)\rangle$ of the particle displacement is given by

$$\langle X(t)\rangle_{as} \equiv \langle X(t)\rangle_{|t\to\infty} = A_0 \int_0^t H_{12}(t-t')\sin(\Omega\, t'). \tag{19}$$

From Eq. (19) it follows that the complex susceptibility $\chi(\Omega)$ of the dynamical system (2) is given by

$$\chi(\Omega) = \hat{H}_{12}(-i\,\Omega). \tag{20}$$

Now, Eq. (19) can be written by means of the complex susceptibility as

$$\langle X\rangle_{as} = A\sin\left(\Omega\, t + \Theta\right) \tag{21}$$

with the output amplitude

$$A = A_0|\chi|. \tag{22}$$

Using Eqs. (13)-(15) we obtain for the amplitude A (the response function) of the output signal that

$$A^2 = A_0^2 \frac{f_1^2 + f_3^2}{\left(f_1 f_2 - f_3 f_4 - a^2\right)^2 + \left(f_1 f_4 + f_2 f_3\right)^2}, \tag{23}$$

where

$$f_1 := \omega^2 + \nu^2 - \Omega^2 + \frac{\gamma\left(\nu^2+\Omega^2\right)^{\frac{\alpha}{2}}\left[\cos(\alpha\,\varphi) + \tau^{\alpha}\left(\nu^2+\Omega^2\right)^{\frac{\alpha}{2}}\right]}{1+\tau^{2\alpha}(\nu^2+\Omega^2)^{\alpha} + 2\,(\nu^2+\Omega^2)^{\frac{\alpha}{2}}\,\tau^{\alpha}\cos(\alpha\,\varphi)},$$

$$f_2 := \omega^2 - \Omega^2 + \frac{\gamma\,\Omega^{\alpha}\left[\cos\left(\frac{\alpha\,\pi}{2}\right) + \tau^{\alpha}\Omega^{\alpha}\right]}{1+(\tau\,\Omega)^{2\alpha} + 2\,(\tau\,\Omega)^{\alpha}\cos\left(\frac{\alpha\,\pi}{2}\right)},$$

$$f_3 := 2\,\Omega\,\nu + \frac{\gamma\left(\nu^2+\Omega^2\right)^{\frac{\alpha}{2}}\sin(\alpha\,\varphi)}{1+\tau^{2\alpha}\left(\nu^2+\Omega^2\right)^{\alpha} + 2\left(\nu^2+\Omega^2\right)^{\frac{\alpha}{2}}\tau^{\alpha}\cos(\alpha\,\varphi)},$$

$$f_4 := \frac{\gamma\,\Omega^{\alpha}\sin\left(\frac{\alpha\,\pi}{2}\right)}{1+(\tau\,\Omega)^{2\alpha} + 2\,(\tau\,\Omega)^{\alpha}\cos\left(\frac{\alpha\,\pi}{2}\right)},$$

$$\varphi := \arctan\left(\frac{\Omega}{\nu}\right). \tag{24}$$

Note that the phase shift Θ can be represented as

$$\Theta = \arctan\left[\frac{a^2 f_3 + f_4\left(f_1^2 + f_3^2\right)}{a^2 f_1 - f_2\left(f_1^2 + f_3^2\right)}\right]. \tag{25}$$

The analytical expressions (23)-(25) belong to the main results of this work.

III. RESULTS

By the use of Eqs. (23) and (24) we can now explicitly obtain the behavior of $A(\tau)$ for any combination of the system parameters α, γ, a, Ω, and ω. Figure 1 depicts the behavior of the response A versus the characteristic memory time τ for different values of the noise switching rate ν and the memory exponent α. In this figure (panels (a) and (b)), one observes resonance versus τ, which apparently gets more and more pronounced as the memory exponent α decreases or as the switching rate ν increases. Thus, as a rule, there exists an optimal memory time at which the response of the output signal to the external periodic force has a maximal value. However, there are certain ranges of system parameters for which the behavior of $A(\tau)$ can be qualitatively different. A plot (Fig. 1(c)) of the output response A vs τ for different values of the driving frequency Ω shows a strong suppression of A at intermediate values of the memory time τ. It is remarkable that in the cases of the system parameters applied in Fig. 1(c) the resonance peaks are relatively small.

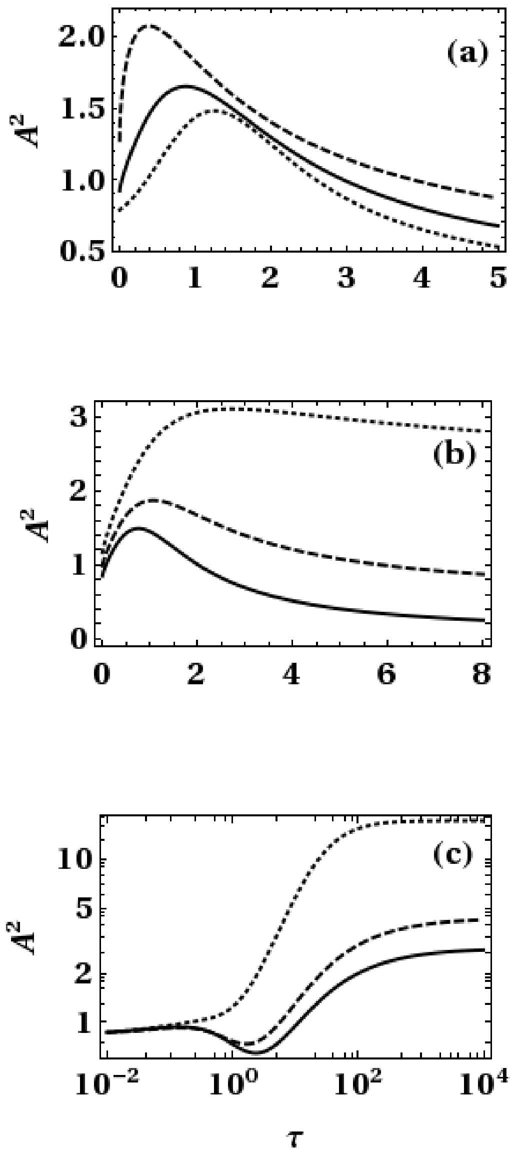

Fig. 1. Dependence of the response function A^2 on the characteristic memory time τ at $A_0 = \omega = 1$, $\gamma = 0.7$, $a^2 = 0.4$. Panel (a): $\Omega = 1$, $\nu = 0.1$; solid line: $\alpha = 0.7$; dashed line: $\alpha = 0.5$; dotted line: $\alpha = 0.9$. Panel (b): $\Omega = 1$, $\alpha = 0.7$; solid line: $\nu = 0.05$; dashed line: $\nu = 0.15$; dotted line: $\nu = 0.3$. Panel (c): $\alpha = 0.6$, $\nu = 0.1$; solid line: $\Omega = 1.18$; dashed line: $\Omega = 1.2$; dotted line $\Omega = 1.28$.

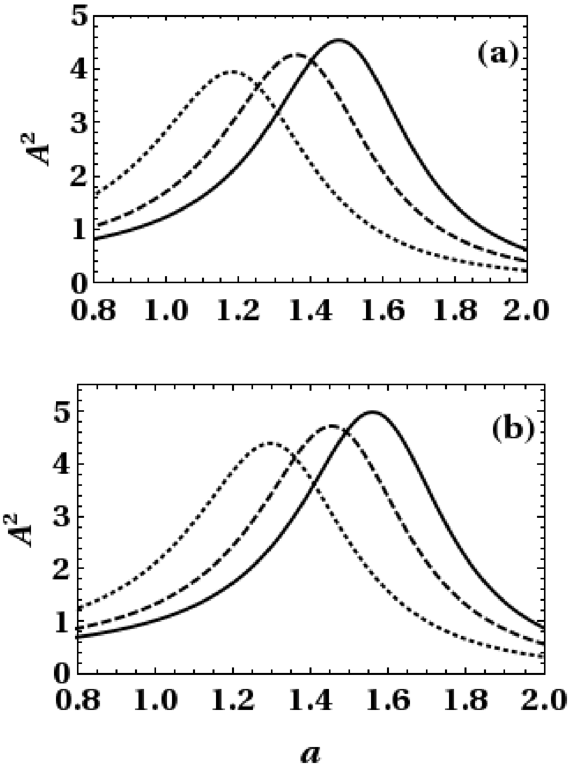

Fig. 2. SR for the response function A vs the multiplicative noise amplitude a at $A_0 = \omega = 1$, $\Omega = 1.8$, $\nu = \alpha = 0.1$. Panel (a): solid line: $\gamma = 1.3$, $\tau = 0.1$; dashed line: $\gamma = 1.5$, $\tau = 0.1$; dotted line: $\gamma = 1.8$, $\tau = 0.1$. Panel (b): solid line: $\gamma = 1.3$, $\tau = 1.0$; dashed line: $\gamma = 1.5$, $\tau = 1.0$; dotted line: $\gamma = 1.8$, $\tau = 1.0$.

Our next task is to examine the dependence of the response A on the noise amplitude a. In Fig. 2 we depict the behavior of $A(a)$ for various values of the system parameters γ and τ. As is shown in Fig. 2, all curves exhibit a resonance-like maximum at some values of a, i.e., a typical SR phenomenon appears with increase of a. The existence of such a SR effect depends strongly on other system parameters. From Eqs. (23) and (24) one can easily find the necessary and sufficient conditions for the emergence of SR due to noise amplitude variations. Namely, nonmonotonic behavior of $A(a)$ appears

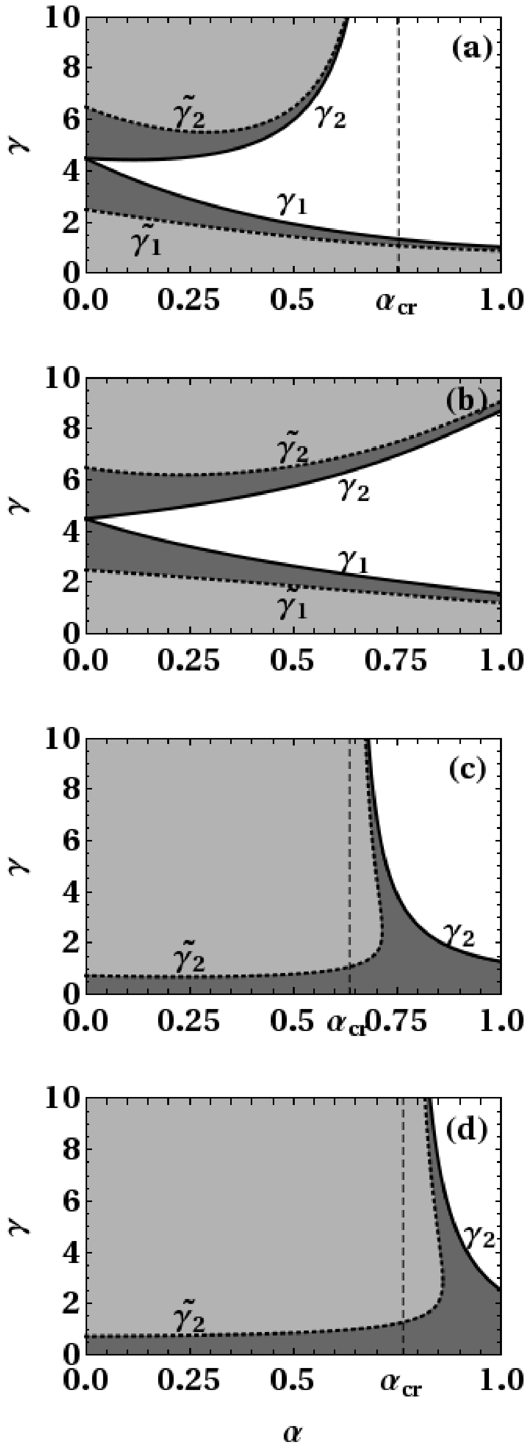

Fig. 3. A plot of the phase diagrams for SR in the $\gamma - \alpha$ plane at $A_0 = \omega = 1$, $\nu = 0$. In the unshaded region resonance of A s the multiplicative noise amplitude a is impossible. In the light grey region the function *A(a)* exhibits a maximum at $a_m > \omega^2$, i.e., at a_m the first moment of the particle displacement $\langle X(t) \rangle$ is unstable, see Eq. (17). In the dark grey domain (the stability region) a stochastic resonance of A vs a occurs. The thin dashed line depicts the position of the critical memory exponent α_{cr}. Panel (a): $\Omega = 1.8$, $\tau = 0.25$, $\alpha_{cr} = 0.753$; panel (b): $\Omega = 1.8$, $\tau = 0.8$; panel (c): $\Omega = 0.6$, $\tau = 0.25$, $\alpha_{cr} = 0.636$; panel (d): $\Omega = 0.6$, $\tau = 0.8$, $\alpha_{cr} = 0.764$.

in the stability region, $0 < a < a_{cr}$ (see Eq. (17)), for the parameter regime where the following inequalities hold:

$$f_3 f_4 < f_2 f_1 < f_3 f_4 + a_{cr}^2. \tag{26}$$

In this case the response *A(a)* reaches the maximum at

$$a_m^2 = f_1 f_2 - f_3 f_4. \tag{27}$$

In Fig. 3 conditions (26) are illustrated for the case of adiabatic multiplicative noise (i.e, $\nu \to 0$) in the parameter space (*γ*, *α*) with four panels. The dark grey shaded domains in the figure correspond to those regions of the parameters *γ* and *α*, where SR versus *a* is possible. Note that in the light grey regions the response *A(a)* formally also exhibits a resonance-like maximum, but in those regions the first moment $\langle X(t) \rangle$ is unstable at the resonance regime and that renders formula (23) physically meaningless. The boundaries $\gamma_{1,2}(\alpha)$ of the regions where SR vs a is possible are determined by the inequality $a_m^2 > 0$ with Eq. (27). From Eqs. (24) and (27) it follows that

$$\gamma_{1,2} = \left(\Omega^2 - \omega^2\right) \times \frac{\left\{\left[\cos\left(\frac{\alpha\pi}{2}\right) + (\tau\Omega)^\alpha\right]^2 + \sin^2\left(\frac{\alpha\pi}{2}\right)\right\}}{\Omega^\alpha\left[\cos\left(\frac{\alpha\pi}{2}\right) \pm \sin\left(\frac{\alpha\pi}{2}\right) + (\tau\Omega)^\alpha\right]}. \tag{28}$$

Two findings can be pointed out. First, if the memory time *τ* is sufficiently small, $\tau < 1/\Omega$, there exists a critical memory exponent α_{cr}, which marks a sharp transition in the behavior of system dynamics. At α_{cr}, one of the boundaries $\gamma_{1,2}(\alpha)$ between the resonance and no-resonance regions tends to infinity. From Eq. (28) it follows that the critical memory exponent α_{cr} is determined as a solution of the following equation:

$$(\tau\Omega)^{\alpha_{cr}} - \sqrt{2}\sin\left[\frac{\pi}{4}(2\alpha_{cr} - 1)\right] = 0, \quad \tau < \frac{1}{\Omega}. \tag{29}$$

It is obvious that $\alpha_{cr} \geq 1/2$ and its value depends only on the product of the driving frequency Ω and the memory time *τ*.

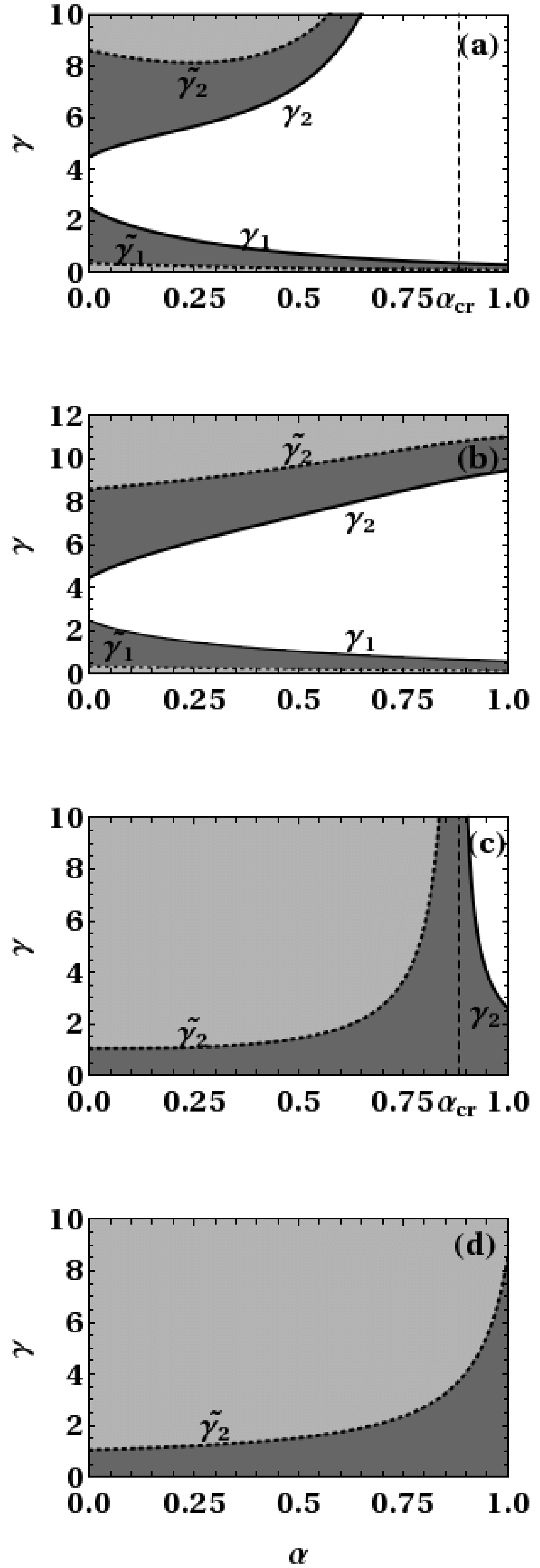

Fig. 4. The phase diagrams for SR vs a in $\gamma-\alpha$ plane at $A_0 = \omega = 1$ and $\nu = 1.0$. Panel (a): $\Omega = 1.8$, $\tau = 0.25$, $\alpha_{cr} = 0.879$; panel (b): $\Omega = 1.8$, $\tau = 0.8$; panel (c): $\Omega = 0.6$, $\tau = 0.25$, $\alpha_{cr} = 0.884$; panel (d): $\Omega = 0.6$, $\tau = 0.8$. In the unshaded region SR versus the noise amplitude a is impossible. In the light grey region the function *A(a)* exhibits a maximum at $a_m > a_{cr}$, see Eq. (17). In the dark grey domain SR vs a occurs (in the stability region, $a_m < a_{cr}$). The thin dashed line depicts the position of the critical memory exponent α_{cr}.

Particulary, in the limit $\tau \to 1/\Omega$ the critical exponent α_{cr} tends to 1. Here we emphasize that in the case of $\tau > 1/\Omega$ such a transition of system dynamics is absent. The second finding is that depending on the driving frequency Ω, two different cases can be discerned. (i) For $\Omega < \omega$, resonance vs a appears in the stability region for all values of γ when $\tau > 1/\Omega$, but if $\tau < 1/\Omega$, there is by $\alpha > \alpha_{cr}$ an upper border $\gamma(\alpha)$, which tends to infinity at α_{cr}, above which the resonance is absent (Fig. 3(c)). (ii) In the case of $\Omega > \omega$, the interesting peculiarity of the diagrams is that there are two disconnected regions (the shaded areas in Figs. 3(a) and 3(b)) where the resonance can appear. Thus, in this case a variation in the values of the friction parameter γ induces reentrant transitions between different dynamical regimes. Namely, an increase of γ can induce transitions from a regime where SR vs a is possible to a regime where SR is absent, but SR appears again through a reentrant transition at higher values of γ.

Next we consider the general case, $\nu \neq 0$ (see Eqs. (24), (26), and (27)). In this case the regions in the parameter space ($\gamma-\alpha$) where SR versus the noise amplitude a is possible are also determined by the inequality $a_m^2 > 0$. Thus the boundaries $\gamma_{1,2}(\alpha)$ of the resonance regions are determined by the positive solutions of the equation

$$\begin{aligned}&\gamma^2\Omega^\alpha\left(\Omega^2+\nu^2\right)^{\frac{\alpha}{2}}\left[g_1g_2-\sin\left(\frac{\alpha\pi}{2}\right)\sin(\alpha\varphi)\right]\\&+\gamma\left\{\Omega^\alpha g_4\left[g_1\left(\omega^2-\Omega^2+\nu^2\right)-2\,\Omega\nu\sin\left(\frac{\alpha\pi}{2}\right)\right]\right.\\&\left.+\left(\Omega^2+\nu^2\right)^{\frac{\alpha}{2}}g_2g_3\left(\omega^2-\Omega^2\right)\right\}\\&+\left(\omega^2-\Omega^2\right)\left(\omega^2-\Omega^2+\nu^2\right)g_3g_4=0,\end{aligned} \tag{30}$$

where

$$g_1=\cos\left(\frac{\alpha\pi}{2}\right)+(\tau\,\Omega)^\alpha,$$

$$g_2=\cos(\alpha\varphi)+\tau^\alpha\left(\nu^2+\Omega^2\right)^{\frac{\alpha}{2}},$$

$$g_3 = g_1^2 + \sin^2\left(\frac{\alpha\pi}{2}\right),$$
$$g_4 = g_2^2 + \sin^2(\alpha\varphi). \quad (31)$$

It can be seen from Eqs. (24), (30) and (31) that the minimal value of a critical memory exponent, $\alpha_{cr} \geq \alpha_{cr_{min}}$, at which one of the boundaries $\gamma_{1,2}(\alpha)$ tends to infinity corresponds to the case of vanishing memory time, $\tau \to 0$, i.e. (see also [31])

$$\alpha_{cr_{min}} = \frac{\pi}{\pi + 2\arctan\left(\frac{\Omega}{\nu}\right)}. \quad (32)$$

Particularly, in the adiabatic case ($\nu \to 0$) $\alpha_{cr_{min}}$ = 0.5 and in the fast-noise limit ($\nu \to \infty$) $\alpha_{cr_{min}}$ tends to 1. One readily sees from Eqs. (30) and (31) that a critical exponent α_{cr} exists if and only if the following inequality holds:

$$\tau < \tau_c \equiv \frac{2}{\nu + \sqrt{5\nu^2 + 4\Omega^2}}, \quad (33)$$

i.e. if the memory time is sufficiently small.

When the inequality (33) holds, it follows from the analysis of Eqs. (30) and (31) that depending on the driving frequency Ω, three different regimes of the dynamical system (2) can be discerned: (i) for $\Omega^2 < \omega^2$, SR vs a appears in the stability region for all values of γ when $\alpha < \alpha_{cr}$, but if $\alpha > \alpha_{cr}$, there is an upper border $\gamma(\alpha)$ above which SR is absent (Fig. 4(c)). (ii) In the case of $\omega^2 < \Omega^2 < \omega^2 + \nu^2$ for $\alpha < \alpha_{cr}$ the resonance exists only if $\gamma > 2(\Omega^2 - \omega^2)$; in the region $\alpha > \alpha_{cr}$ the resonance is absent. (iii) At the driving frequency regime $\Omega^2 > \omega^2 + \nu^2$, if $\alpha < \alpha_{cr}$ there are two disconnected regions (Fig. 4(a)), where SR vs a is possible. An important observation here is that the region where the resonance is not possible grows as the noise switching rate ν increases (cf. Figs. 3(a) and 4(a)).

Note that in the case of a long memory time, $\tau > \tau_c$, the critical memory exponent α_{cr} is absent and the dynamical system (2) behaves qualitatively similarly to the case of $\alpha < \alpha_{cr}$ (see Figs. 4(b) and 4(d)).

Finally, the phenomenon of SR is not restricted to nonmonotonic dependence of A on the noise amplitude a. Figure 5 depicts the behavior of the response A versus the noise switching rate ν for different values of the memory exponent α and the memory time τ. In this figure, one observes resonance versus ν, which apparently gets more and more pronounced as the memory exponent α decreases or as the memory time τ increases. It is remarkable that in contrast to the case A vs a, SR vs ν depends on the exponent α very weakly: as α increases from 0 to 1 only a slight deformation of the curves $A(\nu)$ can be observed.

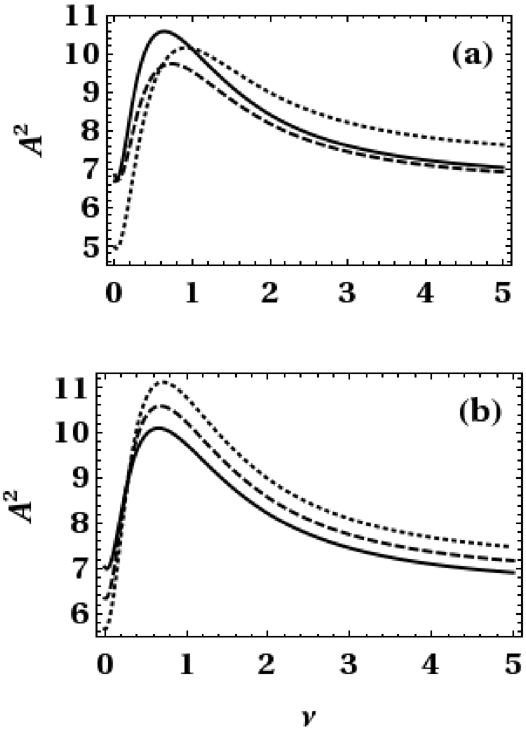

Fig. 5. SR for A^2 versus the noise switching rate ν, computed from Eqs. (23) and (24) at various values of the memory time τ and the memory exponent α. System parameter values: $A_0 = \omega = 1$, $a^2 = 0.3$, $\Omega = 0.8$, and $\gamma = 0.05$. Panel (a): $\tau = 0.1$; solid line: $\alpha = 0.1$; dashed line: $\alpha = 0.5$; dotted line: $\alpha = 0.9$. Panel (b): $\alpha = 0.3$; solid line: $\tau = 0.1$; dashed line: $\tau = 1.0$; dotted line: $\tau = 10$.

IV. CONCLUSION

In the present work we have studied, in the long-time regime, the response function of a GLE with a Mittag-Leffler memory kernel for the friction term. The influence of the fluctuating medium is modeled by a multiplicative dichotomous noise and by an additive Mittag-Leffler noise. The Shapiro-Loginov formula [40] with the Laplace transformation technique allows us to find an exact expression for the long-time behavior of the mean particle displacement.

As the main result we have established that the output signal of the GLE depends crucially on the characteristic memory time of the friction-kernel. It is remarkable that the output response of the GLE shows a resonance-like dependence on the characteristic memory time. Moreover, we have found a nonmonotonic dependence of the response function on the amplitude a and the switching rate ν of the multiplicative noise (i.e. SR). Particularly, we have shown the existence of a band gap for the values of the friction coefficient γ between two regions of $(\gamma - \alpha)$ phase diagrams where SR vs the multiplicative noise amplitude a is possible, the corresponding friction-induced reentrant transitions between these different dynamical regimes, and a related critical memory exponent which marks a dynamical transition in the behavior of the system. Finally, to avoid misunderstandings, let us point out a qualitative difference between the results on SR vs a in this work and in Refs. [29]. Namely, in the case of a power law memory kernel considered in [29], at the adiabatic limit of the multiplicative noise the critical memory exponent α_{cr} is a constant, $\alpha_{cr} = 1/2$, but in the case considered in this work α_{cr} depends crucially on the product of the driving frequency Ω and the memory time τ, and is absent for $\tau > 1/\Omega$. We believe that the results obtained are of interest also in cell biology, where issues of memory and multiplicative colored noise can be crucial [20], [21], [33]. A further detailed study is, however, necessary – especially an investigation of the behavior of second moments [20], [22].

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