Discrete-time deterministic and stochastic triopoly game with heterogeneous players and delay

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Abstract– In this paper the discrete-time triopoly game with heterogeneous players has been studied. We take into consideration the deterministic and stochastic cases. A study for the local stability of the fixed points is carried out. The bifurcation flip and its normal form are analyzed. Also, the case when the system contains delay is discussed. Numerical simulations are performed for the above models. Finally, some conclusions and future prospects are provided.

Keywords– Triopoly game, Heterogeneous players, Flip bifurcation, Discrete-time dynamical system, Stochastic dynamical system, Strange attractor.

I. Introduction

Economic models and economic-mathematical modeling practice constituted an excellent instrument for studying the economic games, stimulating research in this area. Currently a number of modeling methods of economic and mathematical theory were used to study the evolution of the social-economic status parameters. From this perspective the study, in a dynamic environment, of the oligopoly market mechanism is an extremely important issue. Based on these considerations it is possible to approach the microeconomic problems working with a modern instrument, namely game theory.

The game theory has not changed the principle of rationality, but developed it by using strategically and informational complex, thus raising questions on the hypothesis of rational behavior for oligopoly type market structures [14].

The oligopoly market is an imperfect market structure that is found mostly in the actual economy, characterized by a limited number of company proceedings. The strategies of oligopoly companies are different and adapted to each actual situation on the market.

Augustin Cournot studied the oligopoly markets operation where each company acts knowing that the volume of production affects the market price [3]. He defined balance as a situation where each company chooses the output which can maximize its profit but taking into account the output forecast by the other companies, showing that such a balance leads to a price above the marginal productivity [4].

The oligopoly market structure showing the action of only three companies is called triopoly. This paper presents an oligopoly market analysis on the specific case of triopoly using game theory as a working instrument. The players choose simple expectations such as naive or complex as rational expectations. The players can use the same or different strategies.

Based on [1,2,5,9,11], in the present paper we consider a triopoly game with heterogenous players, where each player thinks with different strategy to maximize his output. We consider the first player to be boundedly rational, the second one is an adaptive player and the third one is a naive player. They all produce the same or homogeneous goods which are perfect substitutes and over them at discrete-time periods n=0,1,2,.. on a common market.

The paper is organized as follows. The discrete-time dynamical triopoly game with heterogenous players is described in Section 2. Section 3 provides the existence and the local stability of the fixed points, and the existence of the flip bifurcation and its normal form, as well. Section 4 presents the stochastic model. Section 5 present the deterministic and stochastic model with delay. Using Maple 12, some numerical simulations are carried out in Section 6. The strange attractor and Lyapunov exponent are measured numerically. Finally, some conclusions are offered.

II. The model

We consider a Cournot triopoly game, where q_i, $i = 1,2,3$ denotes the quantity supplied by i^{th} firm.

Also, let $P : \mathbb{R}_+ \to \mathbb{R}_+$ be a twice differentiable and non-increasing inverse demand function and $C_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,3$ the twice differentiable increasing cost functions. The profit functions $\Pi_i : \mathbb{R}^3_+ \to \mathbb{R}$ are defined by:

$$\Pi_i(q_1(n), q_2(n), q_3(n)) = P(q_1(n)+q_2(n)+q_3(n))q_i - C_i(q_i), \quad i = 1,2,3. \tag{1}$$

If $q_i(n)$, $i = 1,2,3$ are the outputs at the moment $n \in \mathbb{N}$, then in the moment $n+1$ the first player's output $q_1(n+1)$

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is given by:

$$q_1(n+1) = q_1(n) + \alpha q_1(n)\frac{\partial \Pi_1}{\partial q_1}(q_1(n), q_2(n), q_3(n)),$$
$$n = 0, 1, ... \tag{2}$$

the second player's output, $q_2(n+1)$, is:

$$q_2(n+1) = (1-\beta)q_2(n) + \beta r_2(q_1(n) + q_3(n)), \quad n = 0, 1, ... \tag{3}$$

and the third player's output, $q_3(n+1)$, is:

$$q_3(n+1) = r_3(q_1(n) + q_2(n)), \quad n = 1, 2, ... \tag{4}$$

In (2), α is a positive parameter which represents an adjustment coefficient of the first player's rationality.

In (3), $\beta \in [0, 1]$, is a parameter which represents an adjustment coefficient of the reaction function $r_2 : \mathbb{R}_+^2 \to \mathbb{R}_+$ which characterizes the adaptably of the second payer.

In (4), $r_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is the reaction function that characterizes the naive behavior of the third player.

In what follows we consider the linear case for the functions that define the system (2), (3), (4). Consider:

$$P(x) = a - bx, C_i(q_i) = c_i q_i, i = 1, 2, 3, \quad r_2(x) = r_3(x) = x. \tag{5}$$

From (1), (2), (3), (4) with (5) we obtain the discrete-time dynamical system [5]:

$$\begin{aligned}
&q_1(n+1) = q_1(n) + \alpha q_1(n)(a - c_1 - 2bq_1(n) - \\
&-bq_2(n) - bq_3(n)) \\
&q_2(n+1) = (1-\beta)q_2(n) + \frac{\beta}{2b}(a - c_2 - bq_1(n) - \\
&-bq_3(n)) \\
&q_3(n+1) = \frac{1}{2b}(a - c_3 - bq_1(n) - bq_2(n)), \\
&n = 0, 1, 2, ...
\end{aligned} \tag{6}$$

Application $F : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ associated to system (6) is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x + \alpha x(a - c_1 - 2bx - by - bz)) \\ (1-\beta)y + \frac{\beta}{2b}(a - c_2 - bx - bz) \\ \frac{1}{2b}(a - c_3 - bx - bz) \end{pmatrix}. \tag{7}$$

III. The qualitative analysis of (7)

The fixed points of (7) are given by the solutions of the system:

$$\begin{aligned}
x(a - 2bx - b(y+z) - c_1) = 0 \\
a - 2by - b(x+z) - c_2 = 0 \\
a - 2bz - b(x+y) - c_3 = 0.
\end{aligned} \tag{8}$$

If the parameters a, b, c_1, c_2, c_3 from (8) satisfy the relations:

$$\begin{aligned}
3c_1 - c_2 - c_3 < a \\
3c_2 - c_1 - c_3 < a \\
3c_3 - c_1 - c_2 < a
\end{aligned} \tag{9}$$

then the fixed points with the positive coordinates of (8) are:

$$E_1(0, q_{20}, q_{30}), \quad E_2 = (q_{11}, q_{21}, q_{31}), \tag{10}$$

where

$$\begin{aligned}
&q_{20} = \frac{a - 2c_2 + c_3}{3b}, q_{30} = \frac{a - 2c_3 + c_2}{3b}, \\
&q_{11} = \frac{a - 3c_1 + c_2 + c_3}{4b}, q_{21} = \frac{a - 3c_2 + c_1 + c_3}{4b}, \\
&q_{31} = \frac{a - 3c_3 + c_1 + c_3}{4b}.
\end{aligned}$$

The Jacobian matrix of application F is:

$$J(x, y, z) = \begin{pmatrix} 1 + \alpha(a - 4bx - b(y+z) - c_1) & -\alpha bx & -\alpha bx \\ -\frac{\beta}{2} & 1 - \beta & -\frac{\beta}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}. \tag{11}$$

From (11) and (10) we obtain:

Proposition 1. *(i) The characteristic equation of (11) in the point E_1 has the roots:*

$$\lambda_1 = 1 + \frac{\alpha(a + c_2 + c_3 - 3c_1)}{3}, \quad \lambda_{2,3} = \frac{1}{2} - \frac{\beta}{2} \pm \frac{1}{2}\sqrt{1 - \beta + \beta^2}.$$

(ii) If the model's parameters satisfy conditions (9) then $|\lambda_1| > 1$ and $|\lambda_{2,3}| < 1$. The point E_1 is a saddle point for the discrete-time dynamical system (6).

Proposition 2. *(i) The matrix (11) evaluated at the point E_2 is:*

$$J(E_2) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}
&a_{11} = 1 - \frac{\alpha}{2}k, \quad a_{12} = -\frac{\alpha}{4}k, \quad a_{13} = -\frac{\alpha}{4}k, \\
&a_{21} = -\frac{\beta}{2}, a_{22} = 1 - \beta, a_{23} = -\frac{\beta}{2}, a_{31} = -\frac{1}{2}, \\
&a_{32} = -\frac{1}{2}, a_{33} = 0, k = a - 3c_1 + c_2 + c_3.
\end{aligned} \tag{13}$$

(ii) The characteristic equation of $J(E_2)$ is:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{14}$$

where

$$\begin{aligned}
&A_1 = -2 + \beta + \frac{k}{2}\alpha, A_2 = \frac{4 - 5\beta}{4} - \frac{k(5 - 3\beta)}{8}\alpha, \\
&A_3 = \frac{\beta}{4} + \frac{k(1-\beta)}{8}\alpha.
\end{aligned} \tag{15}$$

(iii) The fixed point E_2 is locally asymptotically stable if and only if the following relations:

$$\begin{aligned}
&D_1 = k(2\beta - 5)\alpha + 2(8 - 5\beta) > 0 \\
&D_2 = 12\beta(\beta + 4) + (24 + 4\beta - 4\beta^2)k\alpha + \\
&+(3 + 2\beta - \beta^2)k^2\alpha^2 > 0
\end{aligned} \tag{16}$$

hold.

The inequalities (16) are obtained applying Schur Theorem [8] for equation (14).

The necessary and sufficient conditions as the roots of the equation have their absolute values less than one are:

$$\begin{array}{l} 3+A_1-A_2-3A_3>0 \\ 1-A_1+A_3(A_1-A_3)>0 \\ 1-A_1+A_2-A_3>0. \end{array} \tag{17}$$

From (16) with (37) we get (17).

We analyze system (6), if α is a parameter and $a, c_1, c_2, c_3, b, \beta$ are fixed parameters.

Consider $\alpha(s)=\alpha_0+s$, where

$$\alpha_0=\frac{2(8-5\beta)}{5-2\beta}, \tag{18}$$

$s\in \mathbb{R}$, and $A=J(E_2)|_{\alpha=\alpha(s)}$.

Proposition 3. *(i) The value $s=0$ defines a flip bifurcation for system (6) in E_2.*

(ii) If $s=0$, the eigenvectors that satisfy the system $Ap=-p$, $A^Tp^*=-p^*$, $<p,p^*>=1$ have the components:

$$\begin{array}{l} p_1=-\dfrac{3}{8}\alpha_0 k, p_2=\dfrac{5}{8}\alpha_0 k, p_3=\dfrac{1}{8}\alpha_0 k(2\beta-5), \\ p_1^*=G_1p_3^*, p_2^*=G_2p_3^*, p_3^*=\dfrac{1}{p_1G_1+p_2G_2+p_3}, \\ G_1=-\dfrac{\alpha_0 k}{12\beta}, G_2=\dfrac{48\beta-\alpha_0^2k^2}{48\beta}. \end{array} \tag{19}$$

(iii) The projection of (6) on the center manifold in the point E_2 has the normal form:

$$\xi(n+1)=-\xi(n)+\frac{1}{6}d\xi(n)^3 \tag{20}$$

where

$$\begin{array}{l} d=3p_3^*(-4\alpha_0 bp_1r_1-\alpha_0 b(p_1r_2+p_2r_1)- \\ -\alpha_0 b(p_1r_3+p_3r_1)) \\ r_1=\dfrac{3}{\alpha_0 k}B(p,p), r_2=-\dfrac{1}{\alpha_0 k}B(p,p), \\ r_3=-\dfrac{1}{\alpha_0 k}B(p,p) \\ B(p,p)=-4b\alpha_0 p_1^2-2\alpha_0 bp_1p_2-\alpha_0 bp_1p_3. \end{array} \tag{21}$$

Proof. (i) From (8) and (16) we obtain $D_1=k(2\beta-5)s$, $D_2=(3+2\beta-\beta^2)k^2(s^2+2\alpha_0 s)$. If $s>0$, then $D_1>0$, $D_2>0$. Thus, the fixed point E_2 is locally asymptotically stable. If $s\in(-2\alpha_0,s)$ then $D_1<0$, $D_2>0$. The fixed point E_2 is unstable. If $s=0$, $D_1=0$, $D_2>0$, the characteristic equation (14) has the root $\lambda_1=-1$ and the others have their absolute values less than one. Therefore, $s=0$ is a flip bifurcation.

(ii) The relations (19) are obtained by straight calculation.

(iii) Using [12], coefficient d is obtained.

We analyze the discrete-time dynamical system (6), if β is parameter and a, c_1, c_2, c_3, b, α are fixed parameters.

Consider $\beta(s)=\beta_0+s$, where

$$\beta_0=\frac{16-5k\alpha}{10-k\alpha}, \tag{22}$$

$s\in\mathbb{R}$ and $s=J(E_2)|_{\beta=\beta(s)}$.

Proposition 4. *(i) The value $s=0$ defines a flip bifurcation for system (6) in E_2.*

(ii) If $s=0$, the vectors that satisfy the relations $Bp=-p$, $B^Tp^*=-p^*$, $<p,p^*>=1$ have the components:

$$\begin{array}{l} p_1=-\dfrac{3}{8}\alpha k, \quad p_2=\dfrac{5}{8}\alpha k, \quad p_3=\dfrac{1}{8}\alpha k(2\beta_0-5), \\ p_1^*=H_1p_3^*, \quad p_2^*=H_2p_3^*, \quad p_3^*=\dfrac{1}{p_1H_1+p_2H_2+p_3}, \\ H_1=-\dfrac{\alpha k}{12\beta_0}, \quad H_2=\dfrac{48\beta_0-\alpha^2k^2}{48\beta_0}. \end{array} \tag{23}$$

(iii) The projection of system (6) on the center manifold in E_2 has the normal form:

$$\xi(n+1)=-\xi(n)+\frac{1}{6}d_1\xi(n)^3$$

where

$$\begin{array}{l} d_1=3p_3^*(-4\alpha bp_1r_1-\alpha b(p_1r_2+p_2r_1)- \\ -\alpha b(p_1r_3+p_3r_1)) \\ r_1=\dfrac{3}{\alpha k}B_1(p,p), r_2=-\dfrac{1}{\alpha k}B_1(p,p), \\ r_3=-\dfrac{1}{\alpha k}B_1(p,p) \\ B_1(p,p)=-4b\alpha p_1^2-2\alpha bp_1p_2-\alpha bp_1p_3. \end{array}$$

The proof can be done in the similar way as the proof of Proposition 3.

IV. The stochastic model

Using the methods from [7,13] we analyze the stochastic perturbed of (6).

Let $(\Omega,\mathcal{F})$ be a measurable space, where Ω is a set whose elements will be noted by ω and $\mathcal{F}$ is a $\sigma-$algebra of subsets of Ω. We denote by $\mathcal{B}(\mathbb{R})$ $\sigma-$algebra of Borelian subsets of $\mathbb{R}$. *A random variable* is a measurable function $X:\Omega\to\mathbb{R}$ with respect to the measurable spaces $(\Omega,\mathcal{F})$ and $(\mathbb{R},\mathcal{B}(\mathbb{R}))$.

A probability measure P on the measurable space $(\Omega,\mathcal{F})$ is a $\sigma-$additive function defined on $\mathcal{F}$ with values in $[0,1]$ such that $P(\Omega)=1$. The triplet $(\Omega,\mathcal{F},P)$ is called probability space.

An arbitrary family $\xi(n,\omega)=\xi_n$ of random variables, defined on Ω with values in $\mathbb{R}$, is called stochastic process. We denote by $\xi(n,\omega)=\xi(n)(\omega)$ for any $n\in\mathbb{N}$ and $\omega\in\Omega$. The functions $X(\cdot,\omega)$ are called the trajectories of $X(n)$. We use $E(\xi(n))$ for the mean value and $E(\xi(n)^2)$ the square mean value of $\xi(n)=\xi_n$.

The discrete-time stochastic model is the stochastic perturbed of (6) and it is given by:

$$\begin{aligned}
&q_1(n+1) = q_1(n) + \alpha q_1(n)(a - c_1 - 2bq_1(n) - \\
&-bq_2(n) - bq_3(n)) + b_{11}(q_1(n) - q_{10})\xi_n \\
&q_2(n+1) = (1-\beta)q_2(n) + \frac{\beta}{2b}(a - c_2 - bq_1(n) - \\
&-bq_3(n)) + b_{22}(q_2(n) - q_{20})\xi_n \\
&q_3(n+1) = \frac{1}{2b}(a - c_3 - bq_1(n) - bq_2(n)) + \\
&+b_{33}(q_3(n) - q_{30})\xi_n, \\
&n = 0, 1, 2, ...
\end{aligned} \tag{24}$$

where ξ_n is a random variable with $E(\xi_n) = 0$, $E(\xi_n^2) = \sigma < \infty$ and (q_{10}, q_{20}, q_{30}) are the coordinates of a fixed point given by E_1 or E_2.

The linear stochastic dynamical system with discrete time associated to (24) in the fixed point (10) is:

$$u(n+1) = Au(n) + \xi_n Bu(n) \tag{25}$$

where

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}, B = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$

where $a_{11} = 1 + \alpha\frac{a + c_2 - 2c_1}{3}$, $a_{21} = -\frac{\beta}{2}$, $a_{22} = 1 - \beta$, $a_{23} = -\frac{\beta}{2}$, $a_{31} = -\frac{1}{2}$, $a_{32} = -\frac{1}{2}$ and $u(n) = (u_1(n), u_2(n), u_3(n))^T$.

For (25) consider $E(u(n)) = E_n$ the mean values of the variables and $E(u(n)u(n)^T) = V_n$ the matrix of the square mean values.

Using $E(\xi_n) = 0$ and $E(\xi_n^2) = \sigma < \infty$ by straight calculation we obtain:

Proposition 5. *(i) The mean values satisfy the discrete-time system of equations:*

$$E_{n+1} = AE_n, \quad n \in I\!N; \tag{26}$$

(ii) The square mean values satisfy the discrete-time system of equations:

$$V_{n+1} = AV_nA^T + \sigma^2 BV_nB^T, \quad n \in I\!N; \tag{27}$$

(iii) The characteristic polynomial of (26) is:

$$P_1(\lambda) = det(\lambda I - A) = (\lambda - a_{11})(\lambda^2 - a_{22}\lambda - a_{23}a_{32}); \tag{28}$$

(iv) The characteristic polynomial of (27) is:

$$\begin{aligned}
&P_2(\lambda) = (\lambda - a_{11}^2 - \sigma b_{11}^2)(\lambda - a_{23}a_{32} - \sigma b_{22}b_{33}) \cdot \\
&\cdot (\lambda^2 - (a_{22}^2 + \sigma(b_{22}^2 + b_{33}^2))\lambda + (a_{22}^2 + \sigma b_{22}^2)\sigma b_{33}^2 - \\
&- a_{32}^2 a_{23}^2)(\lambda^2 - (a_{11}a_{22} + b_{11}b_{33} + \sigma b_{11}(b_{11} + \\
&+ b_{33}))\lambda + (a_{11}a_{22} + \sigma^2 b_{11}b_{22})b_{11}b_{33}\sigma - \\
&- a_{11}^2 a_{23}a_{32}).
\end{aligned} \tag{29}$$

The analysis of the mean values and square mean values of the variables can be done analyzing the roots of the equation $P_1(\lambda) = 0$ and $P_2(\lambda) = 0$ respectively.

Thus, we get:

Proposition 6. *If the model's parameters satisfy conditions (9) then the roots of equation $P_1(\lambda) = 0$ are $|\lambda_1| > 1$, $|\lambda_2, \lambda_3| < 1$. The equilibrium point is a saiddle point. The mean values are unstable.*

The analysis of $P_2(\lambda)$ is done using Maple 12, for $a = 10$, $b = 0.5$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $\beta = 0.33$.

V. Discrete-time deterministic and stochastic triopoly game with heterogeneous players and delay

In this case, the profit function of the first player, Π_1, is related to the rationality of the first player. It takes into account the quantities $q_2(n-1)$ and $q_3(n-1)$. A similar consideration can be found in [6].

The profit function of the first firm is $\Pi_1 : I\!R_+^3 \to I\!R$ defined by:

$$\begin{aligned}
&\Pi_1(q_1(n), q_2(n), q_3(n), q_2(n-1), q_3(n-1)) = \\
&= q_1(n)P(q_1(n) + \omega_2 q_2(n) + (1-\omega_2)q_2(n-1) + \\
&+ \omega_3 q_3(n) + (1-\omega_3)q_3(n-1))q_1(n) - C_1(q_1(n)),
\end{aligned} \tag{30}$$

$\omega_2, \omega_3 \in [0, 1]$.

The profit functions for the other players are given by:

$$\begin{aligned}
&\Pi_i(q_1(n), q_2(n), q_3(n)) = q_i(n)P(q_1(n) + q_2(n) + \\
&+ q_3(n)) - C_i(q_i(n)), i = 2, 3.
\end{aligned}$$

The discrete-time dynamical system with delay is given by:

$$\begin{aligned}
&q_1(n+1) = q_1(n) + \alpha q_1(n)\frac{\partial \Pi_1(n, n-1)}{\partial q_1}, \\
&q_2(n+1) = (1-\beta)q_2(n) + \beta r_2(q_1(n), q_3(n)), \\
&q_3(n+1) = r_3(q_1(n) + q_2(n)), \quad n = 0, 1, 2, ...
\end{aligned} \tag{31}$$

In what follows we consider the linear case for the functions that define the system (31). Consider the particular case (5).

We obtain the discrete-time dynamical system with delay:

$$\begin{aligned}
&q_1(n+1) = q_1(n) + \alpha q_1(n)(a - c_1 - 2bq_1(n) - \\
&-b\omega_2 q_2(n) - b(1-\omega_2)q_4(n)) - b\omega_3 q_3(n) - \\
&-b(1-\omega_3)q_5(n)) \\
&q_2(n+1) = (1-\beta)q_2(n) + \frac{\beta}{2b}(a - c_2 - bq_1(n) - \\
&-bq_3(n)) \\
&q_3(n+1) = \frac{1}{2b}(a - c_3 - bq_1(n) - bq_2(n)), \\
&q_4(n+1) = q_2(n), \\
&q_5(n+1) = q_3(n), \\
&n = 0, 1, 2, ...
\end{aligned} \tag{32}$$

Application $F : I\!R_+^5 \to I\!R_+^3$ associated to system (32) is:

$$\begin{pmatrix} x \\ y \\ z \\ u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} x+\alpha x(a-c_1-2bx-b\omega_2 y-b\omega_3 z- \\ -b(1-\omega_2)u_1-b(1-\omega_3)u_2) \\ (1-\beta)y+\frac{\beta}{2b}(a-c_2-bx-bz) \\ \frac{1}{2b}(a-c_3-bx-bz) \\ y \\ z \end{pmatrix}. \tag{33}$$

The fixed points with positive coordinates of (33) are given by

$$E_{10}(0, q_{20}, q_{30}, q_{20}, q_{30}), E_{21} = (q_{11}, q_{21}, q_{31}, q_{21}, q_{31}), \tag{34}$$

where q_{20}, q_{30} and q_{i1}, $i = 1, 2, 3$ are given by (10).

The Jacobian matrix of application F_2 is:

$$J(x, y, z, u_1, u_2) = \\ = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \\ -\frac{\beta}{2} & 1-\beta & -\frac{\beta}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{35}$$

where $d_1 = 1+\alpha(a - c_1 - 4bq_1 - b\omega_2 q_2 - b(1-\omega_2)u_1 - b\omega_3 q_3 - b(1-\omega_3)u_2$, $d_2 = -\alpha b\omega_2 q_1$, $d_3 = -\alpha b\omega_3 q_1$, $d_4 = -b(1-\omega_2)u_1$, $d_5 = -b(1-\omega_3)u_2$.

From (34) and (35) we obtain:

Proposition 7. *(i) The characteristic function of (35) in the point E_{10} is given by:*

$$P_{10}(\lambda) = \lambda^2(\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4)$$

where

$$\begin{aligned} a_1 &= \beta - 2 - \alpha(a - c_1 - bq_{20} - bq_{30}) \\ a_2 &= -\frac{3\beta}{4} - \frac{\alpha\beta}{2}(a - c_1 - bq_{20} - bq_{30}) - \frac{b(1-\omega_3)}{2}q_{30} \\ a_3 &= \frac{\beta}{2} + \alpha(a - c_1 - bq_{20} - bq_{30}) - \\ &- \frac{\beta b}{2}(1-\omega_3)q_{20} + \frac{b(1-\omega_3)}{4}(2-\beta)q_{30} \\ a_4 &= \frac{b\beta}{4}(1-\omega_3)q_{20}. \end{aligned} \tag{36}$$

(ii) The characteristic function of (35) in the point E_{21} is given by:

$$P_{20}(\lambda) = \lambda(\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4),$$

where

$$\begin{aligned} b_1 &= \beta - 2 - \alpha(a - c_1 - 4bq_{11} - bq_{21} - bq_{31}) \\ b_2 &= -\frac{3\beta}{4} - \frac{\alpha\beta}{2}(a - c_1 - 4bq_{11} - bq_{21} - bq_{31}) - \\ &- \frac{b(1-\omega_3)}{2}q_{31} \\ b_3 &= \frac{\beta}{2} + \alpha(a - c_1 - 4bq_{11} - bq_{21} - bq_{31}) - \\ &- \frac{\beta b(1-\omega_3)}{2}q_{21} + \frac{b(1-\omega_3)}{4}(2-\beta)q_{31} \\ b_4 &= \frac{b\beta}{4}(1-\omega_3)q_{21}. \end{aligned} \tag{37}$$

From Proposition 7, using Schur criteria we obtain:

Proposition 8. *(i). If the coefficients a_1, a_2, a_3, a_4 satisfy the inequalities:*

$$\begin{aligned} & a_4 < 1, 1 + a_4 > \frac{a_2}{3}, 1 + a_1 + a_2 + a_3 + a_4 > 0, \\ & 1 - a_1 + a_2 - a_3 + a_4 > 0, \\ & (1-a_4)(1-a_4^2) - a_2(1-a_4)^2 + (a_1-a_3)(a_3-a_1a_4) > 0 \end{aligned} \tag{38}$$

then the absolute values of the roots of equation $P_{10}(\lambda) = 0$ are less than 1;

(ii) If the coefficients b_1, b_2, b_3, b_4 satisfy the inequalities:

$$\begin{aligned} & b_4 < 1, 1 + b_4 > \frac{b_2}{3}, 1 + b_1 + b_2 + b_3 + b_4 > 0, \\ & 1 - b_1 + b_2 - b_3 + b_4 > 0, \\ & (1-b_4)(1-b_4^2) - b_2(1-b_4)^2 + (b_1-b_3)(b_3-b_1b_4) > 0 \end{aligned} \tag{39}$$

then the roots of equation $P_{20}(\lambda) = 0$ has the roots in module less than one.

In according with Jury criteria, if the roots of equations $P_{10}(\lambda) = 0$, $P_{20}(\lambda)$ have the absolute values of the real parts less than 1, then the equilibrium points E_{10}, E_{20} are locally asymptotically stable.

The analysis of the characteristic equations can be done by considering different values for the parameters of the model.

The stochastic system with time delay is given by:

$$\begin{aligned} & q_1(n+1) = q_1(n) + \alpha q_1(n)(a - c_1 - 2bq_1(n) - \\ & -b\omega_2 q_2(n) - b(1-\omega)q_4(n) - b\omega_3 q_3(n) - \\ & -b(1-\omega_3)q_5(n)) + \sigma_1(q_1(n) - q_{10})\xi_n \\ & q_2(n+1) = (1-\beta)q_2(n) + \frac{\beta}{2b}(a - c_2 - bq_1(n) - \\ & -bq_3(n)) + \sigma_2(q_2(n) - q_{20})\xi_n \\ & q_3(n+1) = \frac{1}{2b}(a - c_3 - bq_1(n) - bq_2(n)) + \\ & +\sigma_3(q_3(n) - q_{30})\xi_n, \\ & q_4(n+1) = q_2(n) + \sigma_4(q_4(n) - q_{20})\xi_n, \\ & q_5(n+1) = q_3(n) + \sigma_5(q_5(n) - q_{30})\xi_n, \\ & n = 0, 1, 2, ... \end{aligned} \tag{40}$$

where ξ_n is a random variable with $E(\xi_n) = 0$, $E(\xi_n^2) = \sigma < \infty$ and q_{10}, q_{20}, q_{30}, q_{20}, q_{30} are the coordinates of the fixed point given by E_{10} or E_{21}.

The linear stochastic dynamical system with time delay associated to (40) in the fixed point (34) is:

$$U(n+1) = A_1 U(n) + \xi_n B_1 U(n), \qquad (41)$$

where $U(n) = (q_1(n), q_2(n), q_3(n), q_4(n), q_5(n))^T$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \end{pmatrix}.$$

For E_{10} the elements of the matrices A and B are given by: $a_{11} = 1 + \alpha(a - c_1 - bq_{20} - b_{30})$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = -b(1 - \omega_3)q_{20}$, $a_{15} = -b(1 - \omega_3)q_{30}$, $a_{21} = -\frac{\beta}{2}$, $a_{22} = 1 - \beta$, $a_{23} = -\frac{\beta}{2}$, $a_{31} = -\frac{1}{2}$, $a_{32} = \frac{1}{2}$.

For E_{21} the elements of the matrices A and B are given by: $a_{11} = 1 + \alpha(a - c_1 - 4bq_{11} - bq_{21} - bq_{31})$, $a_{12} = -\alpha b\omega_2 q_{11}$, $a_{13} = -\alpha b\omega_3 q_{11}$, $a_{14} = -b(1 - \omega_3)q_{21}$, $a_{15} = -b(1 - \omega_3)q_{31}$, $a_{21} = -\frac{\beta}{2}$, $a_{22} = 1 - \beta$, $a_{23} = -\frac{\beta}{2}$, $a_{31} = -\frac{1}{2}$, $a_{32} = \frac{1}{2}$.

For (41) we consider $E(u_i(n)) = E_i(n)$, $i = 1, 2, 3, 4, 5$ where $E_n = (E_1(n), E_2(n), E_3(n), E_4(n))^T$ are the mean values of the variables and $E(U(n)U(n)^T) = V_n$ is the matrix of the square mean values.

Using $E(\xi_n) = 0$ and $E(\xi_n^2) = \sigma < \infty$ by the straight calculation we obtain:

Proposition 9. *(i) The mean values satisfy the following discrete time system:*

$$E_{n+1} = A_1 E_n, n \in N; \qquad (42)$$

(ii) The square mean values satisfy the discrete time system of equation

$$V_{n+1} = A_1 V_n A_1^T + \sigma^2 B V_n B, n \in N; \qquad (43)$$

(iii) The characteristic function of (42) is $P_{10}(\lambda)$, $P_{20}(\lambda)$ respectively, given by Proposition 7.

The analysis of the roots for the characteristic equation of (43) can be done using Maple 12.

VI. Numerical simulation

For $a = 10$, $b = 0.5$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $\beta = 0.33$ using a program in Maple 12, the bifurcation diagrams with respect to α (the speed of adjustment of boundedly rational player) are displayed in the figures 1, 2, 3:

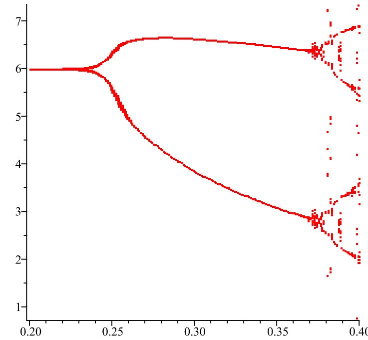

Figure 1. $q_1(n)$ bifurcation, for $\alpha \in (0.2, 0.4)$

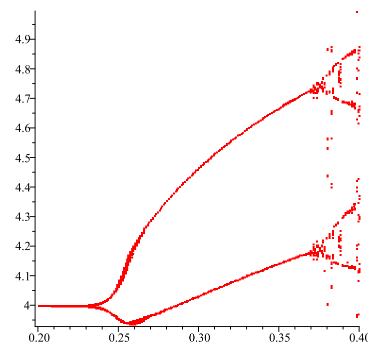

Figure 2. $q_2(n)$ bifurcation, for $\alpha \in (0.2, 0.4)$

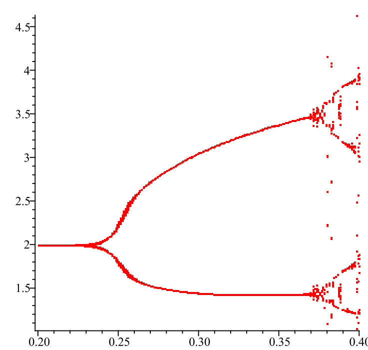

Figure 3. $q_3(n)$ bifurcation, for $\alpha \in (0.2, 0.4)$

Also, the bifurcation diagrams with respect to the speed of adjustment of the adaptive player can be shown.

Using Maple 12, for $a = 10$, $b = 0.5$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $\alpha = 0.41$, $\beta = 0.33$ the greater Lyapunov coefficient is $L_1 = 0.2202540126$. Due to its positive value, the system is chaotic. A strange attractor is given in figure 4 and it exhibits a fractal structure:

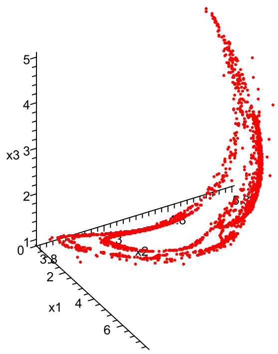

Figure 4. The strange attractor for $\alpha = 0.41$, $\beta = 0.33$

In what follows consider the stochastic system (24). For $a = 0.05$, $b = 0.5$, $c_1 = 0.001$, $c_2 = 0.002$, $c_3 = 0.003$, $\alpha = 0.3$, $\beta = 0.33$, $\sigma = 2$, $b_1 = 0.1$, $b_2 = 0.2$, $b_3 = 0.1$ we obtain two fixed points: $E_1 = (0,$ $0.00327,$ $0.03067)$ and $E_2 = (0.026,$ $0.024,$ $0.022)$. For the first one the mean values and the square mean values of the variables are not asymptotically stable, because the absolute values for all roots of the characteristic equation are not less than 1. In the stochastic case, the orbits of the variables can be visualized:

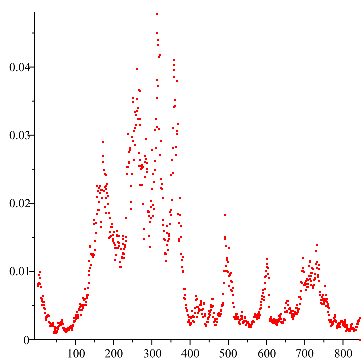

Figure 5. $(n, q_1(n, \omega))$

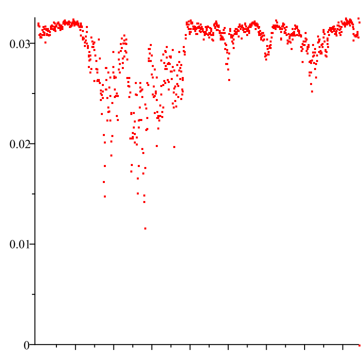

Figure 6. $(n, q_2(n, \omega))$

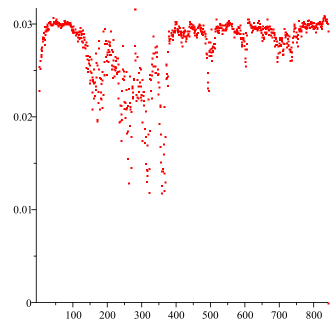

Figure 7. $(n, q_3(n, \omega))$

For the deterministic system with delay and the values: $a = 0.10$, $b = 0.5$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, we obtain two fixed points: $E_1 = (0,$ $0.00327,$ $0.03067,$ $0.00327,$ $0.03067)$ and $E_2 = (6,$ $4,$ $2,$ $4,$ $2)$.

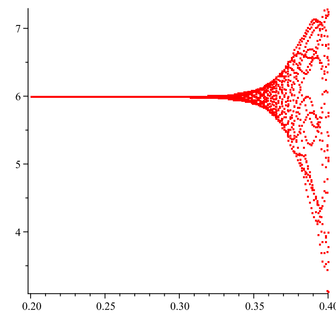

Figure 8. $(n, q_1(n))$ for $\alpha \in (0.2, 0.4)$

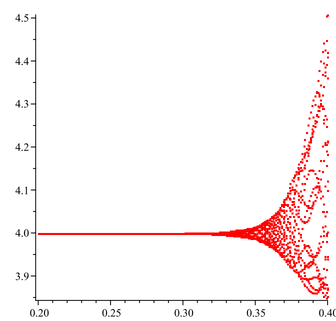

Figure 9. $(n, q_2(n))$ for $\alpha \in (0.2, 0.4)$

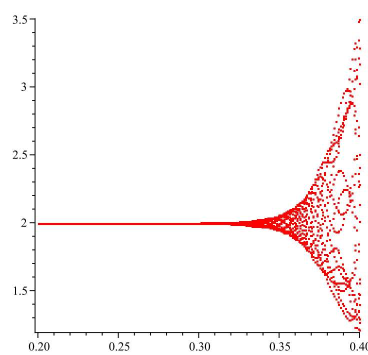

Figure 10. $(n, q_3(n))$ for $\alpha \in (0.2, 0.4)$

For $\alpha = 0.41$, $\beta = 0.33$ the greater Lyapunov coefficient is $L_1 = 0.02370345236$. Due to its positive value, the system is chaotic. A strange attractor is given in figure 11 and it exhibits a fractal structure:

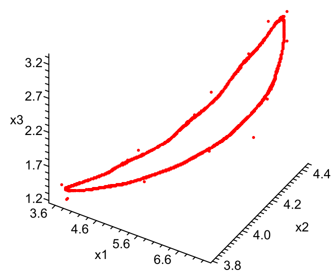

Figure 11. The strange attractor for $\alpha = 0.41$ and $\beta = 0.33$

VII. Conclusions

In the present paper, we consider the triopoly game where players take different strategies for computing their expected output. Such choices make the triopoly game to provide complex dynamics. For the linear case of the price and cost functions, the fixed points and their stability, bifurcation diagrams, strange attractor and chaotic behavior were analyzed. Also, the stochastic approach is considered. We use numerical simulations in order to observe the locally asymptotic stability of the solution. Moreover, we analyze the triopoly game with delay in the deterministic and stochastic cases. The findings of the present paper can be extended in the oligopoly case.

ACKNOWLEDGMENT

The research was supported by the project with the title ”Real access to the real market by the simulated firm”, POSDRU 90/2.1/S/63442.

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