Modeling of flow of medium with homogeneous microstructure

V. I. Prosvetov, P.P. Sumets, N.D. Verveyko

***Abstract*—**The model of a medium, with a glance to its microstructure, is reviewed in this work. Accounting of the characteristic dimension of the medium in the main equations is fulfilled through specifying the strain velocity tensor. A model demonstrating liquid flow in a cylindrical pipe has been built. Dependencies describing distribution of the velocity in the pipe cross section have been presented. The rate of fluid flow, with a glance to the microstructure's influence, has been calculated.

***Keywords*—** microstructure, mathematical model, viscous fluid.

I. Introduction

DISCREPANCY between classical models of a continuous medium and models of real media with homogeneous microstructure arises when multiple practical tasks relating to movement of continuous media are to be solved. Absence of the microstructure's dimensionless characteristic parameter in the classical mathematical models of the continuous medium is the basic problem.

Studying of the microstructure properties through introducing micro-turns at a point of space together with the connection between this classic kinematics and the element's power characteristics is one of the main approaches to describing materials, while taking their homogeneous microstructure into account. In such approach the characteristic dimension of the microstructure is connected with the moment of inertia of the microstructure's characteristic volume, which is included into the dynamic momental equation. This approach has been developed in the works by A. Eringen [1], A. Aero and A. Bulygin [2], and others [3] - [6].

Another approach is connected with taking into account the influence on deformation at the point of space of not only close elements, but also - more distant elements, which is mathematically presented through introduction of not only movement gradients of the first order, but also - movement gradients of the second order into the strain-energy function W. This approach for accounting the microstructure's medium in kinematics is related to I. Kunin's works [7].

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Representative volume $\Delta V=h^3$ and microstructure characteristic parameter (h/L) are introduced in the suggested work to take the microstructure into consideration. Here h is characteristic linear dimension of the representative element, L is characteristic linear dimension of the studied phenomenon. Accounting of the medium characteristic dimension in the main equations of the continuous medium mechanics is fulfilled by specifying the strain velocity tensor.

We shall perceive representative element ΔV as such a volume of the material, which contains a large quantity of microstructure's elements, sufficient for providing coincidence between mechanical properties of this volume ΔV and mechanical properties of the material as a whole. Characteristic linear dimension h of the microstructure may have dimensions comparable with linear dimensions of the crystals and smaller, down to linear dimensions of the rock slabs, depending on the task scale. Choosing a suitable scale is a separate task [8], [9].

II. Kinematics of continuous medium, with microstructure taken into account

Let us choose rectangular coordinate system x_1,x_2,x_3 as the observer's reference frame and let us examine flow areas in point $M(x_1,x_2,x_3)$, representative volume V of the medium with characteristic dimension $2h$ at certain moment of time t.

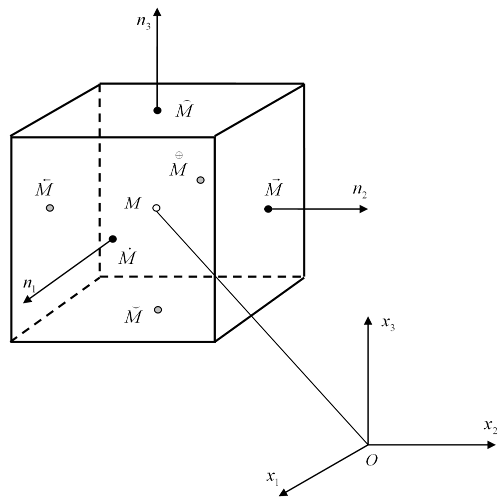

Fig. 1. Characteristic representative volume.

Coordinates of the points in elementary volume:

$$
\begin{aligned}
&M = (x_1; x_2; x_3),\\
&\dot{M} = (x_1 + hn_{11}; x_2 + hn_{12}; x_3 + hn_{13}),\\
&\overset{\oplus}{M} = (x_1 - hn_{11}; x_2 - hn_{12}; x_3 - hn_{13}),\\
&\vec{M} = (x_1 + hn_{21}; x_2 + hn_{22}; x_3 + hn_{23}),\\
&\overset{\leftarrow}{M} = (x_1 - hn_{21}; x_2 - hn_{22}; x_3 - hn_{23}),\\
&\hat{M} = (x_1 + hn_{31}; x_2 + hn_{32}; x_3 + hn_{33}),\\
&\overset{\vee}{M} = (x_1 - hn_{31}; x_2 - hn_{32}; x_3 - hn_{33}).
\end{aligned} \tag{1}
$$

Let us examine the Koshi strain velocity tensor at point M at the moment of time t

$$
\varepsilon^{c}_{i,j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{2}
$$

where v_j is components of the movement velocity vector. Let us take into account the studied values' character, which is spatially distributed through their series expansion by small parameter h in points of the representative volume:

The following developments with accuracy to the third summand shall be true for the whole strain velocity tensor at the k-ths opposite edges:

$$
\begin{aligned}
\varepsilon_{ij}\Big|_{\overset{+k}{M}} &= \varepsilon_{ij}\Big|_M + \frac{\partial \varepsilon_{ij}}{\partial x_l}\Bigg|_M \cdot hn_{kl} + \frac{1}{2}\frac{\partial^2 \varepsilon_{ij}}{\partial x_l \partial x_m}\Bigg|_M \cdot h^2 n_{kl} n_{km} +\\
&+ \frac{1}{6}\frac{\partial^3 \varepsilon_{ij}}{\partial x_l \partial x_m \partial x_n}\Bigg|_M \cdot h^3 n_{kl} n_{km} n_{kn},\\
\varepsilon_{ij}\Big|_{\overset{-k}{M}} &= \varepsilon_{ij}\Big|_M - \frac{\partial \varepsilon_{ij}}{\partial x_l}\Bigg|_M \cdot hn_{kl} + \frac{1}{2}\frac{\partial^2 \varepsilon_{ij}}{\partial x_l \partial x_m}\Bigg|_M \cdot h^2 n_{kl} n_{km} -\\
&- \frac{1}{6}\frac{\partial^3 \varepsilon_{ij}}{\partial x_l \partial x_m \partial x_n}\Bigg|_M \cdot h^3 n_{kl} n_{km} n_{kn}.
\end{aligned} \tag{3}
$$

Consequently, the mean value for the two opposite edges of the elementary representative volume is defined by the following formula:

$$
\overline{\varepsilon_{ij}}\Big|_{\overset{k}{M}} = \frac{\varepsilon_{ij}\Big|_{\overset{+k}{M}} + \varepsilon_{ij}\Big|_{\overset{-k}{M}}}{2} = \varepsilon_{ij}\Big|_M + \frac{1}{2}\frac{\partial^2 \varepsilon_{ij}}{\partial x_l \partial x_m}\Bigg|_M \cdot h^2 n_{kl} n_{km}. \tag{4}
$$

Let us average the whole strain velocity tensor over all the edges of the elementary representative cube:

$$
\overline{\varepsilon_{ij}}\Big|_M = \frac{\sum_{k=1}^{3} \overline{\varepsilon_{ij}}\Big|_{\overset{k}{M}}}{3} = \varepsilon_{ij}\Big|_M + \frac{1}{6}\sum_{k=1}^{3}\frac{\partial^2 \varepsilon_{ij}}{\partial x_l \partial x_m}\Bigg|_M \cdot h^2 n_{kl} n_{km}. \tag{5}
$$

In the rectangular coordinate system

$$
\begin{aligned}
&n_{11}n_{12} + n_{21}n_{22} + n_{31}n_{32} \equiv 0,\\
&n_{11}n_{13} + n_{21}n_{23} + n_{31}n_{33} \equiv 0,\\
&n_{12}n_{13} + n_{22}n_{23} + n_{32}n_{33} \equiv 0
\end{aligned} \tag{6}
$$

takes place.

Taking (2) and (6) into account, we obtain

$$
\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\ v_i}{\partial\ x_j} + \frac{\partial\ v_j}{\partial\ x_i}\right) + \frac{h^2}{12}\left(\frac{\partial^3 v_i}{\partial\ x_j^3} + \frac{\partial^3 v_j}{\partial\ x_i^3}\right). \tag{7}
$$

Let us mention that including microstructure parameter h into expressions describing deformation velocities makes it possible to make equations of motion, allowing for the material's microstructure for various rheological models [10]. For example, in the model of a linear viscous incompressible fluid, the rheological equations defining connection between the components of the strain tensor σ_{ij} and the components of the strain velocity tensor have the following appearance [11]:

$$
\sigma_{ij} = -p\delta_{ij} + \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{8}
$$

where p is pressure; λ, μ are constant fluid viscosity factors.

Equations of motion in stresses

$$
\frac{\partial \sigma_{ij}}{\partial x_j} + \rho\left(f_i - \frac{\partial v_i}{\partial t} - v_k\frac{\partial v_i}{\partial x_k}\right) = 0, \tag{9}
$$

where f_i denotes components of the density vector of the body forces.

Continuity equation:

$$
\frac{\partial \rho}{\partial t} + v_k\frac{\partial \rho}{\partial x_k} + \rho\frac{\partial v_k}{\partial x_k} = 0. \tag{10}
$$

III. Flow between Parallel Plates

Let us review solution of equations (8)-(10), describing movement of a linear viscous fluid, taking into account microstructure parameter h, with regard to the model task concerning the flow between parallel plates, when displacement of one of the plates leads to shearing of the fluid's layers parallel to the plates.

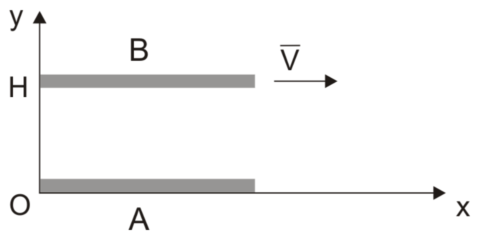

Fig. 2. Flow between parallel plates

Let us pass a plane of rectangular coordinate system *Oxy* orthogonally to the plates, so that axis *Ox* is directed at the replacement side, and coordinate *y* indicates the distance measured from the bottom plate *A* to the side of upper plate *B* (Fig. 2). Let *V=const* is the movement velocity of plate *B* relative to immovable plate *A*; *H* is the distance between the plates. Let us assume that the edges of the plates are sufficiently remote from the studied flow region, and the edges' influence may be neglected.

For the case of two-dimensional motion, the whole system of equations (8)-(10) is:

$$\begin{aligned} p_{xx} &= -p + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + 2\mu\varepsilon_{xx}, \\ p_{yy} &= -p + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + 2\mu\varepsilon_{yy}, \\ p_{xy} &= 2\mu\varepsilon_{xy}. \end{aligned} \tag{11}$$

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial v_x}{\partial x} + \frac{h^2}{6}\cdot\frac{\partial^3 v_x}{\partial x^3}, \\ \varepsilon_{yy} &= \frac{\partial v_y}{\partial y} + \frac{h^2}{6}\cdot\frac{\partial^3 v_y}{\partial y^3}, \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{h^2}{12}\left(\frac{\partial^3 v_x}{\partial y^3} + \frac{\partial^3 v_y}{\partial x^3}\right). \end{aligned} \tag{12}$$

$$\begin{aligned} &\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \rho\left(f_x - \frac{\partial v_x}{\partial t} - v_x\frac{\partial v_x}{\partial x} - v_y\frac{\partial v_x}{\partial y}\right) = 0, \\ &\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \rho\left(f_y - \frac{\partial v_y}{\partial t} - v_x\frac{\partial v_y}{\partial x} - v_y\frac{\partial v_y}{\partial y}\right) = 0. \end{aligned} \tag{13}$$

$$\frac{\partial \rho}{\partial t} + v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = 0. \tag{14}$$

$$\rho = \rho_0 = const. \tag{15}$$

Continuity equation (14), taking (15) into account, is:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{16}$$

Adhering to the classical approach to the solution of the task in question, let us assume that the fluid's movement is a shearing displacement of the layers parallel to the plates. In this case component v_y of the velocity is absent, and component v_x defines the velocity of any layer of the fluid:

$$v_x = v_x\left(y\right), v_y \equiv 0. \tag{17}$$

In assumption (17) continuity equation (16) is fulfilled automatically. Also assuming that the motion is stationary, and external body forces are absent, let us rewrite the remaining equations (11)-(15) as:

$$\begin{aligned} p_{xx} &= -p + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + 2\mu\varepsilon_{xx}, \\ p_{yy} &= -p + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + 2\mu\varepsilon_{yy}, \\ p_{xy} &= 2\mu\varepsilon_{xy}. \end{aligned} \tag{18}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = 0, \varepsilon_{xy} = \frac{1}{2}\cdot\frac{dv_x}{dy} + \frac{h^2}{12}\cdot\frac{d^3 v_x}{dy^3}. \tag{19}$$

$$\begin{aligned} &\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} = 0, \\ &\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} = 0. \end{aligned} \tag{20}$$

Substitution of deformation rates (19) into formulae (18) describing stresses gives us:

$$p_{xx} = p_{yy} = -p, p_{xy} = \mu\left(\frac{dv_x}{dy} + \frac{h^2}{6}\cdot\frac{d^3 v_x}{dy^3}\right). \tag{21}$$

Taking into account the fact that stress p_{xy}, according to (17) and (21), does not depend on *x*, let us find out of the second equation (20):

$$\frac{\partial p_{yy}}{\partial y} = -\frac{\partial p}{\partial y} = 0, \tag{22}$$

from which it follows that pressure *p* may be the function of (5.3) coordinate *x* only. The pressure gradient in the direction of movement is absent; we shall get from the first equation (20):

$$\frac{dp_{xy}}{dy} = 0. \tag{23}$$

Integrating equation (23) we shall obtain:

$$p_{xy} = \mu\left(\frac{dv_x}{dy} + \frac{h^2}{6}\cdot\frac{d^3 v_x}{dy^3}\right) = c, \tag{24}$$

where *c* - is constant of integration. Let us rewrite equation (24) as:

$$v'_x + \left(h^2/6\right)v'''_x = c_1, \text{ where } v'_x = \frac{dv_x}{dy}, c_1 = \frac{c}{\mu}. \tag{25}$$

Thus, to find velocity v_x, it is necessary to solve an ordinary differential equation of the third order with a non-nil right part.

General solution of equation (25) looks like:

$$v_x(y) = c_1 y + c_2 + c_3 \cos\left(\sqrt{6}y/h\right) + c_4 \sin\left(\sqrt{6}y/h\right). \tag{26}$$

Let us compare expression (26) with an analogous expression, which was deduced without taking microstructure parameter h into account. Assuming that h equals to zero, we shall derive a classical equation for velocity, integration of which gives us:

$$v_x^0(y) = c_1 y + c_2. \tag{27}$$

Formula (27) contains two constants of integration c_1 and c_2, which may be calculated from the conditions defining adherence between the fluid and the bottom and upper plates:

$$v_x^0\Big|_{y=0} = 0, v_x^0\Big|_{y=H} = V. \tag{28}$$

Satisfying conditions (28), expression for velocity (27) shall be:

$$v_x^0(y) = (V/H)\,y. \tag{29}$$

General solution (26), in contrast to solution (27), contains four unknown constants c_1, c_2, c_3, c_4, which we shall define out of adherence conditions:

$$v_x\big|_{y=0} = 0, v_x\big|_{y=H} = V, \tag{30}$$

and rolling motion conditions at the bottom and upper boundaries:

$$\begin{aligned} &v_x''\big|_{y=0} + \frac{d}{3} v_x'''\big|_{y=0} = 0, \\ &v_x''\big|_{y=H} - \frac{d}{3} v_x'''\big|_{y=H} = 0, \end{aligned} \tag{31}$$

where d is characteristic diameter of microparticles weighed in the fluid. Since constants c_1 and c_2 are included into expression for velocity (26) as coefficients of a first-degree polynomial, and the rolling motion conditions (31) include derivatives from velocity of not less than the second order, then substituting (26) into (31) gives us two equations, which may be used for determining constants c_3 and c_4:

$$\begin{aligned} &c_3 + \left(\sqrt{2/3}\, d/h\right)c_4 = 0, \\ &\left[c_3 - \left(\sqrt{2/3}\, d/h\right)c_4\right]\cos\left(\sqrt{6}\, H/h\right) + \\ &+\left[c_4 + \left(\sqrt{2/3}\, d/h\right)c_3\right]\sin\left(\sqrt{6}\, H/h\right) = 0. \end{aligned} \tag{32}$$

Solving linear algebraic combined equations (32), we shall find:

$$c_3 = c_4 = 0. \tag{33}$$

Equality of constants c_3 and c_4 to zero means that the velocity component, arising when the microstructure is taken into account, equals to zero; and distribution $v_x(y)$ coincides with the classical distribution:

$$v_x(y) = v_x^0(y) = (V/H)\,y. \tag{34}$$

Thus, within the built model, the microstructure does not influence the flow velocity in case of simple shearing strain.

Next, we shall study the way the microstructure influences fluid movement in a cylindrical pipe. This flow also refers to the class of shear flows, though this flow occurs not by reason of shearing of one of the channel's walls, but is caused by the pressure gradient along its axis.

IV. Flow in a Cylindrical Pipe

With reference to the above, let us consider model movement of the fluid inside a pipe with a circular cross-section of radius $R=const$. Let us introduce cylindrical coordinate system $Or\varphi z$ to describe this movement with the fluid's microstructure taken into account.

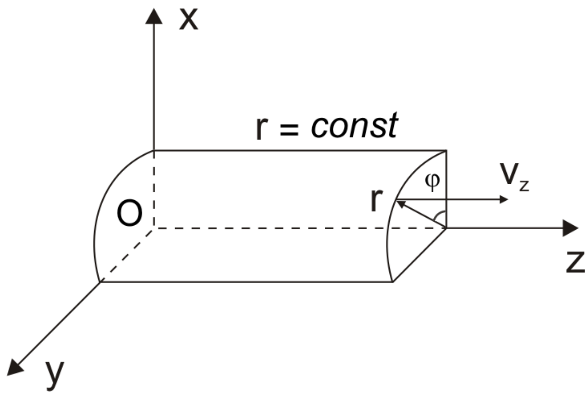

Fig. 3. Flow in the cylindrical pipe.

Let us specify the pressure gradient in the line of pipe's axis Oz:

$$\frac{\partial p}{\partial z} = -\frac{\Delta p}{L} = const < 0\,. \tag{35}$$

Assuming that in this case the fluid motion is an out-of-plane shear of the cylindrical layers *r=const*, we shall have:

$$v_r \equiv v_\varphi \equiv 0, v_z = v_z(r). \quad (36)$$

Let us write condition $\rho=\rho_0=const$ for a uniform incompressible fluid. This equation holds true, taking (36) into account, and the expressions for deformation rates are as follows:

$$\varepsilon_{rr} = \varepsilon_{\varphi\varphi} = \varepsilon_{zz} = \varepsilon_{r\varphi} = \varepsilon_{\varphi z} = 0, \varepsilon_{rz} = \frac{1}{2}\frac{dv_z}{dr} + \frac{h^2}{12}\frac{d^3 v_z}{dr^3}. \quad (37)$$

Inserting deformation rates (37) into rheological relationships, we shall have:

$$\begin{gathered} p_{rr} = p_{\varphi\varphi} = p_{zz} = -p, p_{r\varphi} = p_{\varphi z} = 0, \\ p_{rz} = 2\mu\varepsilon_{rz} = \mu\left(\frac{dv_z}{dr} + \frac{h^2}{6}\frac{d^3 v_z}{dr^3}\right), \end{gathered} \quad (38)$$

where μ – is fluid viscosity. Let us neglect the summands allowing for the external body forces ($f_\varphi=f_r=f_z=0$) in the momentum equations, and also use the motion stationary condition ($\partial v_r/\partial t=\partial v_\varphi/\partial t=\partial v_z/\partial t=0$) and condition (37), we shall have:

$$\begin{gathered} \frac{\partial p_{rr}}{\partial r} + \frac{1}{r}\frac{\partial p_{r\varphi}}{\partial \varphi} + \frac{\partial p_{rz}}{\partial z} + \frac{p_{rr} - p_{\varphi\varphi}}{r} = 0, \\ \frac{\partial p_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial p_{\varphi\varphi}}{\partial \varphi} + \frac{\partial p_{\varphi z}}{\partial z} + 2\frac{p_{r\varphi}}{r} = 0, \\ \frac{\partial p_{rz}}{\partial r} + \frac{1}{r}\frac{\partial p_{\varphi z}}{\partial \varphi} + \frac{\partial p_{zz}}{\partial z} + \frac{p_{rz}}{r} = 0. \end{gathered} \quad (39)$$

Substitution of tensions (38) into (39) shall lead to the following flow equations in the tensions:

$$\frac{\partial p}{\partial r} = 0, \frac{\partial p}{\partial \varphi} = 0, \frac{dp_{rz}}{dr} + \frac{p_{rz}}{r} = -\frac{\Delta p}{L}. \quad (40)$$

The first two equalities (40) demonstrate that pressure p changes only along coordinate z, that is along the pipe. Integrating the first equation of equations (40) we shall have:

$$p_{rz} = -\frac{\Delta p}{2L}r + \frac{c}{r}, \quad (41)$$

where $c=0$ owing to limitation of tangential stress p_{rz} along the pipe axis ($r=0$). Substituting the left part of (41) with equation (38), we shall have a differential equation for velocity:

$$v_z' + \left(h^2/6\right)v_z''' = -\frac{\Delta p}{2\mu L}r, v_z' = dv_z/dr \quad (42)$$

The general solution for a homogeneous equation:

$$v_{g.h.}' + \left(h^2/6\right)v_{g.h.}''' = 0, \quad (43)$$

has the form:

$$v_{g.h.}(r) = c_1 + c_2\cos\left(\sqrt{6}\,r/h\right) + c_3\sin\left(\sqrt{6}\,r/h\right), \quad (44)$$

where c_1, c_2, c_3 - are constants of integration. Let us present partial solution for nonhomogeneous equation (42) as:

$$v_{p.n.}(r) = -\frac{\Delta p}{4\mu L}r^2. \quad (45)$$

Let us find particular solution for velocity adding equations (44) and (45):

$$v_z(r) = -\frac{\Delta p}{4\mu L}r^2 + c_1 + c_2\cos\left(\sqrt{6}r/h\right) + c_3\sin\left(\sqrt{6}r/h\right). \quad (46)$$

Let us define integration constants c_1, c_2 and c_3 out of boundary conditions:

$$v_z\big|_{r=R} = 0, \quad (47)$$

$$v_z''\big|_{r=R} - (d/3)v_z'''\big|_{r=R} = 0, \quad (48)$$

$$v_z'\big|_{r=0} = 0. \quad (49)$$

Here (47), (48) – are correspondingly conditions for adhesion and rolling of the particles along the pipe wall, and equation (49) takes place in the assumption allowing for axial symmetry of the movement. Value d – is particle diameter. We define from (49) that $c_3=0$. We define from (47), (48):

$$\begin{gathered} c_1 = \frac{\Delta p}{4\mu L}\left[R^2 + \frac{h^2}{3\left[1+\left(\sqrt{2/3}d/h\right)tg\left(\sqrt{6}R/h\right)\right]}\right], \\ c_2 = -\frac{\Delta p h^2}{12\mu L\left[\cos\left(\sqrt{6}R/h\right)+\left(\sqrt{2/3}d/h\right)\sin\left(\sqrt{6}R/h\right)\right]}. \end{gathered} \quad (50)$$

Therefore, the equation describing flow velocity in a cylindrical pipe, with a glance to fluid microstructure, has the form:

$$v_z(r) = \frac{\Delta p}{4\mu L}\cdot$$
$$\cdot\left[R^2 - r^2 + \frac{h^2}{3}\cdot\frac{\cos\left(\sqrt{6}R/h\right) - \cos\left(\sqrt{6}r/h\right)}{\cos\left(\sqrt{6}R/h\right) + \left(\sqrt{2/3}d/h\right)\sin\left(\sqrt{6}R/h\right)}\right]. \quad (51)$$

In practice the assumption that $h\to 0$ corresponds to the fact that the representative volume is arbitrary small if compared with the scale of the movement region, and then for the studied flow there may be introduced small parameter $k_1=h/H$, equal to ratio of characteristic dimension of the representative volume to diameter $2R$. Let us also review relation $2h/d$, demonstrating by how many times the linear dimension of the representative volume exceeds the inclusion diameter. Taking into account the fact that the representative volume must contain a rather large of microparticles, we shall assume that $d/2h=k_2<<1$. Let us represent equations (51) in a dimensionless form, with a glance to numerical parameters k_1 and k_2:

$$\bar{v}_z(\bar{r}) = 1 - \bar{r}^2 + \frac{k_1^2}{3}\cdot\frac{\cos\left(\sqrt{6}/k_1\right) - \cos\left(\sqrt{6}\bar{r}/k_1\right)}{\cos\left(\sqrt{6}/k_1\right) + 2\sqrt{2/3}k_2\sin\left(\sqrt{6}/k_1\right)},$$
$$\bar{r} = \frac{r}{R}, \bar{r}\in[0..1], \bar{v}_z = \frac{4\mu L v_z}{\Delta p R^2}, k_1 = \frac{h}{R}, k_2 = \frac{d}{2h}. \quad (52)$$

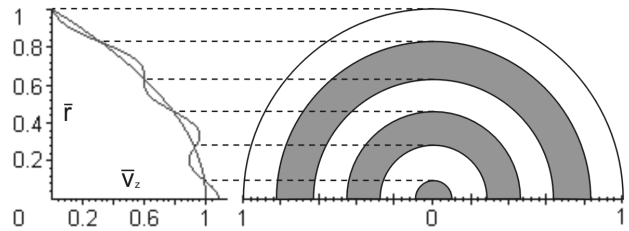

Fig. 4. Influence of microstructure on the fluid velocity inside a cylindrical pipe.

Fig. 4 shows velocity profile $\bar{v}_z(\bar{r})$ with annular zones of velocity's increase and decrease, distinguished in the pipe cross-section, due to the fluid microstructure. In practice this result may be compared with, for example, blood behavior in small vessels, which is stipulated by presence of erythrocytes and other microparticles in the blood plasm.

Integrating equation (51) describing the velocity along the pipe cross-section, let us calculate rate of fluid flow Q, taking microstructure's influence into account. We shall have:

$$Q = 2\pi\int_0^R r v_z(r)\,dr = Q_0\left[1 + \frac{2k_1^2}{3}\cdot\frac{1}{1 + 2\sqrt{2/3}k_2 tg\left(\sqrt{6}/k_1\right)}\right],$$
$$Q_0 = \frac{\pi\Delta p R^4}{8\mu L}. \quad (53)$$

Here Q_0 - is rate of fluid flow in the pipe cross-section, without regard for the microstructure. If $k_2\to 0$, then the formal relative augmentation in the fluid flow due to the microstructure is $2k_1^2/3$:

$$Q = Q_0\left[1 + 2k_1^2/3\right]. \quad (54)$$

Therefore, the built model allows for the following inferences. Influence of the microstructure on the present type of movement consists in alternate increase and decrease in the velocity, as compared with the classic Puazeil distribution. Taking the microstructure into account leads to an increase in the fluid flow-rate.

V. FLOW BETWEEN COAXIAL CYLIDERS

As the region for fluid motion, let us consider the cavity between the two coaxial cylinders, which correspondingly have radii $R\_$ and R_+, where $R\_ < R_+$ (Fig. 5).

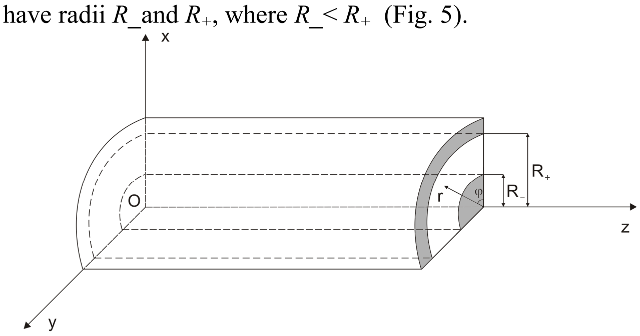

Fig. 5. Flow between coaxial cylinders.

Describing longitudinal flow between the cylinders, let us assume the shearing displacement of fluid's surfaces $r=const$ along axis Oz. In this case velocity components are:

$$v_r \equiv v_\varphi \equiv 0, v_z = v_z(r). \quad (55)$$

Let us reduce the combined equations describing movement by analogy with the case of movement in a cylindrical pipe:

$$p_{rz} = \mu\left(\frac{d\ \ v_z}{d\ \ r} + \frac{h^2}{6}\ \frac{d^3 v_z}{d\ \ r^3}\right),$$
$$\frac{\partial p}{\partial r} = 0, \frac{\partial p}{\partial \varphi} = 0, \quad (56)$$
$$\frac{dp_{rz}}{dr} + \frac{p_{rz}}{r} = \frac{\partial p}{\partial z},$$

where $\partial p/\partial z = 0$ in case when motion is caused by displacement of one of the cylinders, and $\partial p/\partial z = -\Delta p/L = const$, if motion of the fluid occurs due to the pressure gradient. Integration of equation (56) gives us:

$$p_{rz} = \frac{1}{2}\frac{\partial p}{\partial z}r + \frac{c}{r}. \qquad (57)$$

In case of the cylinder pipe where $0 \le r \le R$, integration constant c was assumed to be equal to zero, so that shearing stress p_{rz} would be limited with $r=0$. However, in the studied motion $0 < R_- \le r \le R_+$, and c may be nonzero. Thus, the differential equation for velocity is:

$$\frac{h^2}{6}v_z''' + v_z' = \frac{1}{2\mu}\cdot\frac{\partial p}{\partial z}r + \frac{c}{\mu r}. \qquad (58)$$

Having integrated equation (58) by r once, we shall have:

$$\frac{h^2}{6}v_z'' + v_z = -\frac{q}{2}r^2 + \frac{c_1^*}{\mu}\ln r + c_2^*, \qquad (59)$$

where $q = (-\partial p/\partial z)/(2\mu)$; c_2^* is integration constant. We shall notice that having omitted summand $\left(h^2/6\right)v_z''$ allowing for influence of the fluid's microstructure in the left part of (59), we shall have a general classical solution for velocity in the following form:

$$v_z^0(r) = -\frac{q}{2}r^2 + \frac{c_1^*}{\mu_0}\ln r + c_2^*. \qquad (60)$$

The general solution for inhomogeneous equation (59) is made up from the general solution for homogeneous equation:

$$v_{g.h.}(r) = c_3\cos\left(\sqrt{6}r/h\right) + c_4\sin\left(\sqrt{6}r/h\right), \qquad (61)$$

and particular solution $v_{p.n.}(r)$, which cannot be expressed through elementary functions since the right part of (59) contains a logarithmic function of integration variable r. Let us substitute function ln$r$ for an approximate expression, using the following expansion:

$$\ln\frac{r}{R_+} = \frac{r}{R_+} - 1 - \frac{1}{2}\left(\frac{r}{R_+} - 1\right)^2 + \frac{1}{3}\left(\frac{r}{R_+} - 1\right)^3 - \ldots \qquad (62)$$

where $0 < r/R_+ \le 1$. Expansion (62) shall be limited by a linear term. We shall have:

$$\ln r = \ln\left(R_+\frac{r}{R_+}\right) = \ln R_+ + \ln\frac{r}{R_+} \approx \ln R_+ + \frac{r}{R_+} - 1. \qquad (63)$$

Let us rewrite (59) taking (63) into account:

$$\frac{h^2}{6}v_z'' + v_z = -\frac{q}{2}r^2 + c_1^{**}r + c_2^{**},$$
$$c_1^{**} = \frac{c_1^*}{\mu R_+}, c_2^{**} = c_2^* + c_1^*\frac{\ln R_+ - 1}{\mu}. \qquad (64)$$

Using the right part of equation (65), we may deduce a partial solution:

$$v_{p.n.}(r) = (-q/2)r^2 + c_1^{**}r + c_2^{**} + h^2 q/6. \qquad (65)$$

Let us find the general solution for velocity summing (61) and (65):

$$v_z(r) = -\frac{q}{2}r^2 + c_1 r + c_2 + c_3\cos\frac{\sqrt{6}r}{h} + c_4\sin\frac{\sqrt{6}r}{h}, \qquad (66)$$

where $c_1 = c_1^{**}$, $c_2 = c_2^{**} + h^2 q/6$.

Formulating boundary conditions for adherence, it is necessary to separate the cases when the fluid moves as a result from displacement of one of the cylinders (simple shearing strain):

$$v_z\big|_{r=R_+} = V, v_z\big|_{r=R_-} = 0, \qquad (67)$$

and, when the movement is caused by pressure gradient $\partial p/\partial z$:

$$v_z\big|_{r=R_+} = 0, v_z\big|_{r=R_-} = 0. \qquad (68)$$

where V is velocity of the upper cylinder's movement. For both specified cases, the conditions defining rolling motion on the upper and lower boundaries are:

$$v_z''\big|_{r=R_+} - \frac{d}{3}v_z'''\Big|_{r=R_+} = 0,$$
$$v_z''\big|_{r=R_-} + \frac{d}{3}v_z'''\big|_{r=R_-} = 0. \qquad (69)$$

Substitution of general solution (66) into relations (69) gives us a linear algebraic system relative to unknown quantities c_3, c_4, after solving which we shall find:

$$c_3 = -\frac{h^2}{6} \frac{q \cos\left(\sqrt{\frac{3}{2}} \frac{R_+ + R_-}{h}\right)}{\cos\left(\sqrt{\frac{3}{2}} \frac{R_+ - R_-}{h}\right) + \sqrt{\frac{2}{3}} \frac{d}{h} \sin\left(\sqrt{\frac{3}{2}} \frac{R_+ - R_-}{h}\right)},$$
$$c_4 = -\frac{h^2}{6} \frac{q \sin\left(\sqrt{\frac{3}{2}} \frac{R_+ + R_-}{h}\right)}{\cos\left(\sqrt{\frac{3}{2}} \frac{R_+ - R_-}{h}\right) + \sqrt{\frac{2}{3}} \frac{d}{h} \sin\left(\sqrt{\frac{3}{2}} \frac{R_+ - R_-}{h}\right)}. \quad (70)$$

Let us assume that $\partial p/\partial z = 0$ for the simple shearing strain, then $q=0$, and expression (66) for velocity, taking (70) into account, shall be:

$$v_z(r) = c_1 r + c_2. \quad (71)$$

Having calculated c_1 and c_2 out of boundary conditions (67) we shall have:

$$v_z(r) = \frac{V}{R_+ - R_-}\,(r - R_-). \quad (72)$$

Let us compare solution (72) with classical solution:

$$v_z^0(r) = \frac{V}{\ln(R_-/R_+)} \ln(R_-/r), \quad (73)$$

which can be obtained from (60) with $q=0$ and values of constants c_1^*, c_2^* meeting boundary conditions (67). Let us introduce dimensionless quantities $\Delta = R_-/R_+$, $\bar{r} = r/R_+$, $\bar{v}_z = v_z/V$, $\bar{v}_z^0 = v_z^0/V$, and represent expressions (72) and (73) correspondingly as:

$$\bar{v}_z(\bar{r}) = \frac{\bar{r} - \Delta}{1 - \Delta},$$
$$\bar{v}_z^0(\bar{r}) = \frac{\ln\Delta - \ln\bar{r}}{\ln\Delta}, \quad (74)$$

where $\Delta \in (0..1)$, $\bar{r} \in [0..1]$. Changing of values Δ and $\bar{r}$ within the specified limits makes it possible to present expressions $\ln\Delta$ and $\ln\bar{r}$ as Taylor convergent series:

$$\ln\Delta = \Delta - 1 - (\Delta - 1)^2/2 + (\Delta - 1)^3/3 - \dots$$
$$\ln\bar{r} = \bar{r} - 1 - (\bar{r} - 1)^2/2 + (\bar{r} - 1)^3/3 - \dots \quad (75)$$

Substituting expansions (75), taken in linear approximation, into expression (74), we shall have:

$$\bar{v}_z^0(\bar{r}) \approx \frac{\bar{r} - \Delta}{1 - \Delta} = \bar{v}_z(\bar{r}). \quad (76)$$

Thus, in case of simple shearing strain between coaxial cylinders, the solution for velocity $v_z(r)$ is an approximate example of classical solution $v_z^0(r)$, and the harmonic component of velocity in general solution (66) turns to zero under boundary rolling conditions, as well as in the case of simple shearing strain between parallel plates.

Next, let us examine fluid motion between stationary coaxial cylinders under pressure gradient $(q \neq 0)$. Transforming equations into dimensionless form, we shall have, correspondingly:

$$\bar{v}_z(\bar{r}) = (1+\Delta)\bar{r} - \bar{r}^2 - \Delta + \frac{2k_1^2}{3} \cdot \frac{\sin\left(\sqrt{\frac{3}{2}} \frac{1-\bar{r}}{k_1}\right) \sin\left(\sqrt{\frac{3}{2}} \frac{\Delta-\bar{r}}{k_1}\right)}{\cos\left(\sqrt{\frac{3}{2}} \frac{1-\Delta}{k_1}\right) + 2k_2\sqrt{\frac{2}{3}} \sin\left(\sqrt{\frac{3}{2}} \frac{1-\Delta}{k_1}\right)}, \quad (77)$$
$$\bar{v}_z^0(\bar{r}) = \frac{(\Delta^2 - 1)\ln\bar{r} + \ln\Delta}{\ln\Delta} - \bar{r}^2,$$

where

$$\bar{v}_z = v_z/(qR_+^2/2), \bar{v}_z^0 = v_z^0/(qR_+^2/2),$$
$$k_1 = h/R_+, k_2 = d/(2h).$$

In (77) let us use expansions (75) retaining only linear terms in them; we shall have:

$$\bar{v}_z^0(\bar{r}) \approx (1+\Delta)\bar{r} - \bar{r}^2 - \Delta, \quad (78)$$

which corresponds to the main part of expression (77). Thus, influence of the microstructure on fluid velocity $\bar{v}_z(\bar{r})$ is presented in formula (78) by a fraction with a low coefficient of second order by k_1. The character of the influence may be followed up if we compare the schematics for expressions (77) and (78) built for values: $\Delta = 0.6$, $k_1 = 0.1$, $k_2 = 0.1$ on Fig. 6.

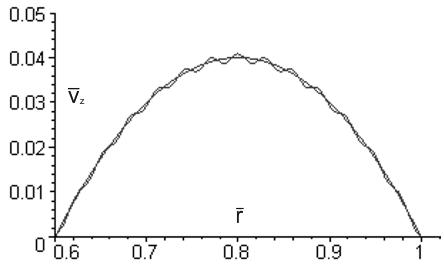

Fig. 6. Influence of fluid's microstructure on the velocity of longitudinal flow between coaxial cylinders

According to the schematics, the changing of the fluid's velocity caused by the microstructure, for the case of a longitudinal flow between coaxial cylinders, presents the same picture as in the case of enforced flows in a flat-bed channel and a cylindrical pipe. Notwithstanding some difference in the mathematical expression, the disturbance imposed on the square velocity distribution by microstructure preserves harmonic character. Disturbance period $T = 2\sqrt{2/3}\pi k_1$ is proportional to small number k_1 defining the relation between the typical structural scale h and radius R_+ of the biggest cylinder. It may also be seen out of the presented schematics that the maximum value of the velocity is not achieved at the midpoint between the upper and the lower cylinders, but it is achieved nearer to the external boundary, as in the classical solution for the task.

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