A Novel and Precise Sixth-Order Method for Solving Nonlinear Equations

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Abstract - This study presents a novel and robust three-step sixth-order iterative scheme for solving nonlinear equations. The contributed without memory method includes two evaluations of the function and two evaluations of the first derivative per iteration which implies 1.565 as its efficiency index. Its theoretical proof is furnished to show the error equation. The most important merits of the novel method are as follows. First in numerical problems, the developed scheme mostly performs better or equal in contrast with the optimal eighth-order methods, such as [7] when the initial guesses are not so close to the sought zeros. Second, its convergence radius is more than the convergence radii of the optimal eighth-order methods. Third, its (extended) computational (operational) index is better in comparison with optimal eighth-order methods. That is, besides the high accuracy and bigger convergence radius in numerical examples for not so close starting points; our method has less computational complexity as well.

Keywords - Error equation, nonlinear equations, iterative methods, extended computational index, efficiency index, convergence radius, simple root, three-step methods, derivative approximation.

I. Introduction

In recent years, many iterative methods have been developed for solving one-variable nonlinear equations of the general form $f(x)=0$. In other words, nonlinear equations solving has a vast application in science and engineering, e.g., the analysis of geometrically non-linear structural problems has been a subject of interest for over three decades. The solution of a non-linear problem reduces to that of tracing a non-linear load–displacement path by solving a system of non-linear algebraic or differential equations. An abundance of procedures exists for attacking the non-linear equilibrium equations. These include the Newton–Raphson method, the modified Newton–Raphson procedure, multi-point methods, the perturbation method, the initial value approach and many more.

Let the scalar function $f(x)$ be sufficiently smooth in the real open domain D and α be its simple root. To develop the local convergence order of the known methods, such as the second-order Newton's method, the published papers have considered two, three or four steps in which we have a combination of some known methods.

This way increases the convergence order while more evaluations of the function or its derivatives are used per iteration. As a matter of fact, multi-point iterative methods for solving nonlinear equations are of great practical importance, since they overcome on the theoretical limits of one-point methods regarding the convergence order and computational efficiency.

Frequently in the literature, the efficiency index [14] is used to compare the obtained different methods. We here recall that this index is defined as $p^{1/\theta}$, where p is the order of convergence and θ is the total number of evaluations per iteration. In fact, two important features determine the choice of iterative method: the total number of iterations and the computational cost. The former is measured by the order of convergence and the latter by the necessary number of evaluations of the scalar function f and its derivatives at each step. In the scalar case, these two features are linked by the efficiency index. (For scalar equations, it is usually considered that the evaluation of f and its derivatives have a similar computational cost.)

Two-step methods were introduced to boost up the order of convergence. For example, Maheshwari in [6] presented the following closed-form optimal fourth-order method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left[\frac{\left[f\left(x_n-\frac{f(x_n)}{f'(x_n)}\right)\right]^2}{f(x_n)^2}-\frac{f(x_n)}{f\left(x_n-\frac{f(x_n)}{f'(x_n)}\right)-f(x_n)}\right], \tag{1}$$

in which we have two evaluations of the function and one of its first derivative.

Cordero et al. in [2] investigated another optimal fourth-order method by using the Potra-Pták's scheme as follows

$$\begin{cases} y_n=x_n-\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1}=x_n-\dfrac{f(x_n)+f(y_n)}{f'(x_n)}-\dfrac{2f(x_n)+f(y_n)}{f'(x_n)}\left(\dfrac{f(y_n)}{f(x_n)}\right)^2. \end{cases} \tag{2}$$

Higher order methods are developed by considering three-step methods in which we use three different points per iteration.

Kou and Li in [5] presented a sixth-order method by taking into consideration the Chebyshev's iterate as follows

$$\begin{cases} L_f(x_n) = \dfrac{f''(x_n)f(x_n)}{[f'(x_n)]^2}, \\ z_n = x_n - \left(1 + \dfrac{1}{2}L_f(x_n)\right)\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \left(1 + L_f(x_n) + \dfrac{3f(z_n)}{f'(x_n)(x_n - z_n)}\right)\dfrac{f(z_n)}{f'(x_n)}. \end{cases} \tag{3}$$

[4] suggested the following sixth-order method by approximating the first derivative of the function in the third step using divided differences

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{f(x_n)}{f'(x_n)}\dfrac{f(x_n)}{f(x_n) - f(y_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}. \end{cases} \tag{4}$$

Soleymani in [10] proposed a sixth-order convergence method as follows

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{3\dfrac{a_1 - f(x_n)a_2}{\left(1 + a_2(y_n - x_n)\right)^2} + f'(x_n)}{6\dfrac{a_1 - f(x_n)a_2}{\left(1 + a_2(y_n - x_n)\right)^2} - 2f'(x_n)}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{\left(1 + b_2(z_n - x_n)\right)^2 f(z_n)}{f'(x_n) + b_1(z_n - x_n)\left(2 + b_2(z_n - x_n)\right)}, \end{cases}$$

where the parameters a_1, a_2, b_1 and b_2 are provided in the following way

$$\begin{cases} a_1 = -\dfrac{f'(x)f(y_n)}{f(x_n) - f(y_n)} + \dfrac{f(x_n)}{x_n - y_n}, \\ a_2 = \dfrac{f'(x)}{f(y_n) - f(x_n)} + \dfrac{1}{x_n - y_n}, \\ b_1 = \dfrac{f'(x)f[y_n, z_n] - f[x_n, y_n]f[x_n, z_n]}{x_n f[y_n, z_n] + \dfrac{y_n f(z_n) - z_n f(y_n)}{y_n - z_n} - f(x_n)}, \\ b_2 = \dfrac{b_1}{f[x_n, y_n]} + \dfrac{f'(x_n) - f[x_n, y_n]}{(y_n - x_n)f[x_n, y_n]}. \end{cases}$$

This scheme was given by considering a modification of Jarratt method in three steps in which there are three evaluations of the function and one evaluations of the first derivative per full cycle.

In 2010, Thukral and Petkovic proposed a family of three-step iterations [13] by using the King's optimal fourth-order family in the first and second steps and constructing a weight function to obtain the eighth-order convergence as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n) + bf(y_n)}{f(x_n) + (b-2)f(y_n)}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \left[\varphi(n) + \dfrac{f(z_n)}{f(y_n) - af(z_n)} + \dfrac{4f(z_n)}{f(x_n)}\right]\dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{5}$$

wherein $a, b \in \mathbb{R}$ and

$$\varphi(n) = 1 + 2\frac{f(y_n)}{f(x_n)} + (5 - 2b)\left[\frac{f(y_n)}{f(x_n)}\right]^2 + (12 - 12b + 2b^2)\left[\frac{f(y_n)}{f(x_n)}\right]^3.$$

Recently Neta and Petkovic in [7] investigated an accurate optimal eighth-order family in three steps with three evaluations of the function and one evaluation of the first derivative per iteration where the new-appeared first derivative of the function in the third step was approximated by an inverse interpolation polynomial of degree three in the following form

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n) + tf(y_n)}{f(x_n) + (t-2)f(y_n)}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = y_n + c[f(x_n)]^2 - d[f(x_n)]^3, \end{cases} \tag{6}$$

where $t \in \mathbb{R}$, and c, d are defined by

$$\begin{cases} d = \dfrac{1}{(f(y_n) - f(x_n))(f(y_n) - f(z_n))f[y_n, x_n]} - \dfrac{1}{(f(z_n) - f(x_n))(f(y_n) - f(z_n))f[z_n, x_n]} \\ \quad + \dfrac{1}{f'(x_n)(f(z_n) - f(x_n))(f(y_n) - f(z_n))} - \dfrac{1}{f'(x_n)(f(y_n) - f(x_n))(f(y_n) - f(z_n))}, \\ c = \dfrac{1}{(f(y_n) - f(x_n))f[y_n, x_n]} - \dfrac{1}{f'(x_n)(f(y_n) - f(x_n))} - d(f(y_n) - f(x_n)). \end{cases} \tag{7}$$

Among many indices for comparison of different methods such as index of efficiency, index of operational [3], radius of convergence, etc.; here we try to provide a novel method which fits better to the indices like (extended) operational index, as well as convergence radius, accuracy. For this reason in this paper, we build a sixth-order method in which we have two evaluations of the function and two evaluations of the first derivative.

It is shown that the proposed three-step scheme mostly performs better or equal than optimal eighth-order iterative methods when the starting points are not so close to the exact roots. This shows that our contribution is so competent wherein less computational complexity is used per iteration in comparison with high order methods such as (5) and (6) for such initial guesses.

In the next section, we build our scheme and prove its convergence order. Then in Section III, we provide a comparison among the methods available in literature to put on show the efficacy of the new contributed method. Last section comprises a short conclusion of this study.

II. Main Result

We know that the convergence behavior of the multi-point methods strongly depends on the structure of tested functions and the accuracy of starting points. It is also known that multi-point iterative methods without memory of the same order and the same computational cost show a similar convergence behavior and produce results of roughly same accuracy especially when the compared schemes use Newton-like or Steffensen-like methods in their first steps.

In this paper, we consider Jarratt-type methods, i.e. methods in which the first two steps is optimal fourth-order iteration with two evaluations of the first derivative and one evaluation of the function. Such schemes are better predictors for predictor-corrector methods when the initial guesses are in the vicinity of the roots but not so close [9, 11]. Hence, let us consider the proposed method by Basu in [1] as the predictor

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n \\ -\left[\dfrac{9\left(f'(y_n)\right)^2 - 2f'(y_n)f'(x_n) + 9\left(f'(x_n)\right)^2}{12\left(f'(y_n)\right)^2 + 4f'(y_n)f'(x_n)}\right]\dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{8}$$

wherein we have one evaluation of the function and two evaluations of the first derivative per iteration to obtain the fourth-order convergence. Now we add a third step by Newton's iteration (the corrector).

Clearly, the considered iterative scheme is of order eight with 1.515 as its efficiency index. In order to boost up the efficiency index, the new-appeared first derivative of the function in the third step is approximated by the known values. To build a powerful estimation of $f'(z_n)$, we use all the four known values, i.e., $f(x_n), f'(x_n), f'(y_n)$ and $f(z_n)$. Hence, the degree two Taylor polynomial of $f(z_n)$ around y_n is written in the following form

$$f(z_n) = f(y_n) + f'(y_n)(z_n - y_n) + \frac{1}{2}f''(y_n)(z_n - y_n)^2, \tag{9}$$

and also for the second derivative of the function $f''(y_n)$, we have

$$f''(y_n) \approx 2\frac{f[z_n, x_n] - f'(x_n)}{(z_n - x_n)} = 2f[z_n, x_n, x_n]. \tag{10}$$

Note that $f'(y_n)$, is available from the second step of our iterative scheme. Accordingly, by considering (9) and (10), a new approximation of the first derivative of the function in the third step (wherein all of the four known values are used) is obtained as follows

$$f'(z_n) \approx f'(y_n) + 2f[z_n, x_n, x_n](z_n - y_n), \tag{11}$$

and consequently our contributed iterative method in which we have two evaluations of the function and two evaluations of the first derivative is presented in the following three-step view

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n \\ -\left[\dfrac{9\left(f'(y_n)\right)^2 - 2f'(y_n)f'(x_n) + 9\left(f'(x_n)\right)^2}{12\left(f'(y_n)\right)^2 + 4f'(y_n)f'(x_n)}\right]\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(y_n) + 2f[z_n, x_n, x_n](z_n - y_n)}. \end{cases} \tag{12}$$

Theorem 1. *Assume $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and has a simple root α in D. Then the three-step without memory iterative scheme denoted by (12) is of local order of convergence six and it consists of two evaluations of the function and two evaluations of the first derivative per iteration.*

Proof. Let that $e_n = x_n - \alpha$ be the error of the iterative scheme in the *n*th iterate. We formally expand $f(x_n)$ and the other required parts of the iterative scheme about the simple root α. We should remark that for simplicity, we let

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k \geq 2.$$

Therefore, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)].$$

Furthermore, we obtain

$$f'(x_n)=f'(\alpha)[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+O(e_n^6)].$$

Dividing the new two Taylor expansions to each other and by considering the first step of (12), we attain

$$x_n-\frac{2}{3}\frac{f(x_n)}{f'(x_n)}-\alpha=\frac{e_n}{3}+\frac{2c_2e_n^2}{3}-\frac{4}{3}(c_2^2-c_3)e_n^3+\frac{2}{3}(4c_2^3-7c_2c_3+3c_4)e_n^4-\frac{4}{3}(4c_2^4-10c_2^2c_3+3c_3^2+5c_2c_4-2c_5)e_n^5+O(e_n^6). \quad (13)$$

To keep on, the Taylor expansion of the second step of (12) is needed. Thus by writing the Taylor expansion of $f'(y_n)$ and the second step of (12) we have

$$z_n-\alpha=\frac{1}{9}(15c_2^3-9c_2c_3+c_4)e_n^4+\left(-\frac{70c_2^4}{9}+12c_2^2c_3-2c_3^2-\frac{20c_2c_4}{9}+\frac{8c_5}{27}\right)e_n^5+\frac{2}{27}\Big(301c_2^5-759c_2^3c_3+243c_2^2c_4-99c_3c_4+9c_2(39c_3^2-5c_5+7c_6)\Big)e_n^6+O(e_n^7).$$

Subsequently, the Taylor expansion of $f(z_n)$ has the following form

$$f(z_n)=\frac{1}{9}(15c_2^3-9c_2c_3+c_4)f'(\alpha)e_n^4-\frac{2}{27}\Big((105c_2^4-162c_2^2c_3+27c_3^2+30c_2c_4-4c_5)f'(\alpha)\Big)e_n^5+\frac{2}{27}(301c_2^5-759c_2^3c_3+243c_2^2c_4-99c_3c_4+9c_2(39c_3^2-5c_5)+7c_6)f'(\alpha)e_n^6+O(e_n^7).$$

For the approximation function, we also obtain

$$f'(y_n)+2f[z_n,x_n,x_n](z_n-y_n)=f'(\alpha)-c_3f'(\alpha)e_n^2-\frac{2}{27}\Big((18c_2c_3+25c_4)f'(\alpha)\Big)e_n^3+\frac{1}{81}(18(15c_2^4+9c_2^2c_3-12c_3^2-13c_2c_4)-211c_5)f'(\alpha)e_n^4+\frac{2}{81}(-630c_2^5+765c_2^3c_3+297c_2c_3^2+144c_2^2c_4-399c_3c_4-172c_2c_5+134c_6)f'(\alpha)e_n^5+O(e_n^6).$$

Now by dividing the last two relations to one another and taking into consideration the last step of (12), we have

$$e_{n+1}=z_n-\frac{f(z_n)}{f'(y_n)+2f[z_n,x_n,x_n](z_n-y_n)}-\alpha=\frac{1}{9}c_3(-15c_2^3+9c_2c_3-c_4)e_n^6+O(e_n^7). \quad (14)$$

This completes the proof and shows that the order of convergence for our contributed method is six. ■

Remark 1. The efficiency index of our scheme is 1.565 which is bigger than 1.414 of Newton's method, 1.442 of methods in [8, 15], and is equal to the sixth-order methods such as (3), (4) and the method in [12]. Although the presented scheme has lower efficiency index in contrast with 1.682 of optimal eighth-order methods, its convergence radius and its accuracy *for not so close starting points* are better than the accuracy and convergence radii of the optimal eighth-order methods (See Section III).

Remark 2. Let r be the total operations (including additions, subtractions, divisions, multiplications and so on) of an iterative method per iteration, then the (extended) computational index (also known as extended operational index) is defined by $p^{1/\mathrm{r}}$, where p is the order of convergence. Now, we can compare the computational index of some well-known high-order methods with our scheme. The computational index of our method is $6^{1/33}\approx 1.055$ which is bigger than $8^{1/58}\approx 1.036$ of (6) and $8^{1/39}\approx 1.054$ of (5).

III. Computational Experiments

In this section, we check the effectiveness of our contributed method (12) by solving some nonlinear equations with different initial guesses for each given test functions. The test problems and their roots are as follows.

- $f_1(x)=\sqrt{x^4+8}\sin\left(\frac{\pi}{x^2+2}\right)+\frac{x^3}{x^4+1}-\sqrt{6}+\frac{8}{17}$, $\quad\alpha=-2$,
- $f_2(x)=\sin x-\frac{1}{2}$, $\quad\alpha\approx 0.523598775598299$,
- $f_3(x)=xe^{x^2}-(\sin(x))^2+3\cos(x)+5$, $\quad\alpha\approx -1.207647827130919$,
- $f_4(x)=\left(\sin(x)-\frac{\sqrt{2}}{2}\right)^2(x+1)$, $\quad\alpha=-1$,
- $f_5(x)=x^5-8x^4+24x^3-34x^2+23x-6$, $\quad\alpha=1$,

- $f_6(x) = \cos(x) - x$, $\alpha \approx 0.739085133215161$.

To show the reliability of (12), we compare the results with the fourth-order method of Maheshwari (1), the fourth-order method of Cordero et al. (2), the sixth-order of Cordero et al. (4), the optimal efficient eighth-order method of Thukral and Petkovic (5) with $a = b = 0$ and the optimal novel eighth-order method of Neta and Petkovic (6) with $t = 0$. The results are summarized in Table I in terms of required number of iterations to obtain the root which is correct up to 15 decimal places.

Note that for (5), we should pull the attention toward this, which it has, a very low convergence radius and that is why in most of the cases; it turns out to divergence *when the starting points are in the vicinity of the zero but not so close*. The applications of such methods whose convergence radii, are low, are indeed restricted in practice.

All computations were performed in MATLAB 7.6. We use the following stopping criterion in our computations: $|x_n| < \varepsilon$ where α is the exact solution of the considered one variable nonlinear equations. For numerical illustrations in Tables I and II, we used the fixed stopping criterion $\varepsilon = 10^{-15}$. Note that termination is the ending criteria of a process which depends on the level of acceptability of the allowable error. Since a numerical method gives only the approximation of the result, so it is a critical step in deciding the accuracy of any method and reliability of the result.

One of the frequently occurring problems in root-finding is that one (user) cannot easily understand that a zero of a nonlinear function is multiple or simple. In such cases although simple zero-finders have lower convergence order for multiple roots, the users again refer to them. For this reason, we have given the nonlinear function f_5 with multiple roots. As can be seen also in this case, our proposed method has definite superiority.

We also provide the Total Number of Evaluations (TNE) for each method to obtain the root up to 15 decimal places in Table II. As we can see, the contributed method is robust and accurate in comparison with other efficient schemes. By comparisons with (6), we could claim that the method compete any optimal eighth-order scheme in [7] while its computational complexity is less and its convergence radius is bigger too.

Table I.
Comparison of different methods in terms of needed iterations to obtain the root

Test Functions	Guess	(1)	(2)	(4)	(5)	(6)	(12)
f_1	-1.6	Div.	Div.	3	Div.	3	3
	-4	Div.	Div.	Div.	Div.	Div.	3
	-3.8	Div.	Div.	Div.	Div.	Div.	3
f_2	1.4	Div.	Div.	5	Div.	3	3
	0.1	3	3	3	2	2	2
	0.2	3	3	2	2	2	2
f_3	-0.1	Div.	Div.	Div.	Div.	12	3
	-0.2	Div.	Div.	Div.	Div.	7	6
	-0.42	Div.	Div.	Div.	Div.	3	4
f_4	-0.5	Div.	Div.	3	Div.	3	3
	-1.9	5	4	3	3	3	3
	-0.6	Div.	Div.	3	3	3	3
f_5	1.5	48	49	31	33	30	14
	0.7	52	52	33	35	32	14
	1.0001	4	4	3	3	3	1
f_6	-0.5	9	4	3	Div.	3	3
	3	4	4	3	3	3	3
	2.2	3	3	2	2	2	2

In fact, there is no way to guarantee the convergence of the high-order methods without implementing so much complexity effort on the root solvers. In other hand, the closeness of the initial guess to the sought zero is of great importance in using such techniques.

Remark 3. If the initial guesses be enough close (very close) to the sought zeros, then the optimal eighth-order methods will perform better than (12).

Although if one chooses a guess close to the root then the optimal eighth-order methods will be more accurate than (12), the question is: "is a completely close starting point to the sought zero always at hand?" Definitely, it is of grave significance to have iterative methods of high order of convergence such as (12) that have bigger convergence radius in comparison to optimal three-step three-point methods.

Another point that should be mentioned on (12) is that it consists of two function and two first derivative evaluations per full cycle to reach the local order six. As a matter of fact, one of the *open problems* in root-finding topic is that if one considers an optimal fourth-order method without memory consuming one evaluation of the function and two evaluations of the first derivative in the first two steps of a three-step cycle, and then approximate the new-appeared first derivative of the function such that an optimal method without memory of order eight using two function and two first derivative evaluations be attained!

Although structures like (12) in which the new-appeared first derivative of the function in the third step is approximated by a combination of all known values are totally convenient, they are not optimal and it is still an *open problem* to make them optimal.

Tables I and II, based on 15 decimal places are satisfactory *to reveal the importance of the distance between the starting points and the sought root*. They also show the bigger convergence radius of (12). But we give the results of comparisons by taking into consideration the stopping criterion $|f(x_n)| < 10^{-100}$, for such initial guesses.

The attained results are provided in Tables III and IV to also manifests that when the starting points are not so close to the sough zeros then the other high order optimal eighth-order techniques diverge or they mostly give less number of correct decimal places of the sough zero, while (12) mostly includes more even decimal places and has bigger convergence radius.

Numerical experiments have been performed in Tables III and IV with the minimum number of precision digits chosen as 100, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants requiring small number divisions. Under the same order of convergence, one should note that the speed of local convergence of $|x_n - \alpha|$ is dependent on c_j, namely $f(x)$ and α.

In general, computational accuracy strongly depends on the structures of the iterative methods, the sought zeros and the test functions as well as good initial approximations. One should be aware that no iterative method always shows best accuracy for all the test functions.

However, a natural question of practical interest arises: does the construction of faster and faster multipoint methods always have a justification? Certainly not if initial approximations are not sufficiently close to the sought zeros. In those cases it is not possible, in practice, to attain the expected convergence speed (determined in a theoretical analysis).

Practical experiments showed that multipoint methods can converge very slowly at the beginning of iterative process *for not so close initial guesses*. It is often reasonable to put an effort into a localization procedure, including the determination of a good initial approximation, instead of using a very fast algorithm with poor starting guesses.

Table II.
Comparison of TNE to obtain the roots for different methods

Test Functions	Guess	(1)	(2)	(4)	(5)	(6)	(12)
f_1	-1.6	-	-	12	-	12	12
	-4	-	-	-	-	-	12
	-3.8	-	-	-	-	-	
f_2	1.4	-	-	20	-	12	12
	0.1	9	9	12	8	8	8
	0.2	9	9	8	8	8	8
f_3	-0.1	-	-	-	-	48	12
	-0.2	-	-	-	-	28	24
	-0.42	-	-	-	-	12	16
f_4	-0.5	-	-	12	-	12	12
	-1.9	15	12	12	12	12	12
	-0.6	-	-	12	12	12	12
f_5	1.5	144	147	124	132	120	56
	0.7	156	156	132	140	128	56
	1.0001	12	12	12	12	12	12
f_6	-0.5	27	12	12	-	12	12
	3	12	12	12	12	12	12
	2.2	9	9	8	8	8	8

IV. CONCLUSIONS

In order to solve nonlinear equations of one variable, we should refer to numerical methods due to failure of analytical procedures to find the solutions. There exists extensive literature which investigates the quadratic convergent behavior of Newton's method. By considering the sufficiently smooth function f in an open domain D, we have proposed in this paper a novel sixth-order method for solving single-valued nonlinear equations in which there are two evaluations of the function and two evaluations of the first derivative per iteration. The usual expectation from a mathematical method of such type is to obtain the fast result by using minimum order derivatives, which is particularly beneficial for the cases where the higher derivatives are difficult to evaluate. Present work found these expectations by converting the given nonlinear problems to a well defined numerical iterative scheme through the use of Taylor's expansion; thereby obtaining a well efficient, highly convergent method that not only works faster than conventional methods but also takes lesser iteration steps than those of recently proposed iterative methods. The analytical proof of the main contribution was given in Section II. Numerical results in Section III have revealed the efficacy of the method in contrast with the most efficient optimal three-step method of Neta and Petkovic (6) when the initial guesses are in the vicinity of the roots but not so close. And subsequently we could conclude that the method is more accurate than any optimal eighth-order method which is quoted in [7] *for such initial guesses*. The presented method has less computational burden and its convergence radius is bigger than the convergence radii of the well-known high-order existing methods. Note that in general, in applying iterative zero-finding methods, special attention should be paid to find good starting points. Accordingly, the contribution in this article can be viewed as a novel and precise iterative method for solving nonlinear equations especially for real-world applications when *the method users have no starting point very close to the root* and they need fast root solvers.

Table III.
Comparison of various methods to find the root with the same Total Number of Evaluation (TNE=12)

Function	Guess	(1)	(2)	(4)	(5)	(6)	(12)
$\lvert f_1\rvert$	-3.6	0.2e-10	0.2e-14	Div.	0.3e-85	Div.	0.2e-60
$\lvert f_2\rvert$	1.4	Div.	Div.	0.5e-3	Div.	0.7e-58	0.2e-77
$\lvert f_3\rvert$	-0.4	Div.	Div.	6.5	Div.	0.1e-14	0.3e-8
$\lvert f_4\rvert$	-0.5	Div.	Div.	0.9e-18	Div.	0.2e-28	0.1e-38
$\lvert f_5\rvert$	1.01	0.4e-9	0.4e-9	0.8e-10	0.1e-9	0.5e-10	0.3e-16
$\lvert f_6\rvert$	3	0.3e-76	0.1e-21	0.3e-37	0.1e-40	0.8e-90	0.2e-50

Table IV.
Comparison of various methods to find the root with the same Total Number of Evaluation (TNE=12)

Function	Guess	(1)	(2)	(4)	(5)	(6)	(12)
$\lvert f_1\rvert$	-3.5	0.1e-26	0.5e-28	0.3e-37	0.1e-142	0.1e-61	0.2e-66
$\lvert f_2\rvert$	1.5	Div.	Div.	Div.	Div.	Div.	0.9e-51
$\lvert f_3\rvert$	-0.37	Div.	Div.	Div.	Div.	0.2e-8	0.5e-4
$\lvert f_4\rvert$	-0.55	Div.	Div.	0.4e-28	Div.	0.2e-48	0.6e-51
$\lvert f_5\rvert$	1.001	0.4e-12	0.4e-12	0.8e-13	0.1e-12	0.5e-13	0.2e-19
$\lvert f_6\rvert$	3.4	Div.	Div.	0.1e-2	Div.	Div.	0.6e-46

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