Stabilization of an Inverted Pendulum System via State-PI Feedback

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***Abstract*—** State feedback technique through a gain matrix has been a well-known method for pole assignment of a linear system. The technique could encounter a difficulty in eliminating the steady-state errors remained in some states. Introducing an integral element to work with the gain can effectively eliminate the errors. State-PI feedback is proposed by this article for pole placement of a delay-free linear time invariant system. The proposed method yields simple gain formulae. The article presents the derivation of the design formulae, the design steps and some simple numerical examples. The method is applied to stabilize an inherently unstable pneumatically actuated inverted pendulum. Simulation results show the effectiveness of the proposed method for disturbance dampening and stabilizing the system. Comparison with the results obtained from applying Ackermann's formula is also presented.

Keywords— state-PI feedback, gain formulae, linear system, pole placement, inverted pendulum.

I. INTRODUCTION

An inverted pendulum system is widely used for demonstration of a control method applied to an unstable plant. The dynamic is quite similar to two-wheeled mobile robots [1, 2], flexible link robot [3], biped robot limbs [4, 5], missiles [6], and so on. In [7], using a PD controller and state servo feedback was proposed to stabilize the pendulum in an inverted position. The approach employed linearized state feedback [7, 8], that could encounter a difficulty when the pendulum largely deviated from the equilibrium. To handle this situation, a robust control strategy was proposed [9] under an assumed ideal actuator. A pneumatic actuator was considered due to its low bandwidth characteristic, robustness, low maintenance and low cost.

Pole placement via state feedback has been known for many years [10, 11]. Recently, state-derivative feedback has been proposed [12, 13] due to considerably low gain and fast response. A linear quadratic regulator for state-derivative feedback has been proposed [14]. Moreover, state-PID feedback has also been proposed [15]. Regarding the method [15], the design must be conducted in 3 separate steps resulting in 2 intermediate closed-loop systems with some fictitious poles. Adaptive state-P feedback has been used with a fixed PI-controller to handle the variations in properties of the dynamic system [16]. Some recent researches have advanced the PID-control field in the s-domain direction with dead-time and optimality in consideration as in [17-19], for instance. Not many have been found in the state-PI, PD and PID feedback areas.

This article presents a new approach to obtain the gain matrices for state-PI feedback which is rather different from the issues in [20] applied specifically to control a communication network. Our approach provides a very simple set of PI gain matrices for delay-free LTI systems. The proposed method is applied to stabilize and reject disturbances in an inverted pendulum system. The results are compared with those obtained via the Ackermann's formula [10, 11].

This article contains 6 sections including the introduction. Section II reviews the concepts of pole placement by state-PI feedback, and transformation to the Frobenius canonical form. Section III presents the main theorem describing the derivation of the gain formulae with some numerical examples. Section IV explains the pneumatically actuated inverted pendulum system. Simulation results for stabilization of the pendulum system follow in Section V. Section VI provides the conclusion.

II. POLE PLACEMENT BY STATE-PI FEEDBACK

A. State-PI feedback pole placement problem formulation

Let's consider a delay-free completely controllable LTI system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad , \mathbf{x}(t_0) = \mathbf{x_0}, \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$ is the state vector, and $u \in R$ is the control input. $\mathbf{A}(n \times n)$ and $\mathbf{B}(n \times 1)$ are the system matrix and the control gain vector, respectively. From $\mathbf{A}$, the characteristic polynomial can be written as

$$\det(s\mathbf{I} - \mathbf{A}) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0, \tag{2}$$

where $\mathbf{a} = [a_0 \quad a_1 \quad \cdots \quad a_n]$, $\quad a_0 = \det(-\mathbf{A}) = (-1)^n \det(\mathbf{A})$ and $a_{n-1} = -trace(-\mathbf{A})$. The control u of the state-PI feedback is

$$u = \mathbf{K_P}\mathbf{x} + \mathbf{K_I}\int \mathbf{x}(\tau)d\tau, \tag{3}$$

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where $\mathbf{K_p}, \mathbf{K_I} \in \mathbf{R}^n$ are the designed gain matrices to achieve a desired closed-loop characteristic polynomial. The closed-loop system can be represented by Eqs. (4)-(5).

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}[\mathbf{K_p x} + \mathbf{K_I}\int_0^t \mathbf{x}(\tau)d\tau] \qquad (4)$$

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{BK_p})\mathbf{x} + \mathbf{BK_I}\int_0^t \mathbf{x}(\tau)d\tau \qquad (5)$$

Eq. (6) represents the closed-loop characteristic equation, while Eq. (7) represents the prescribed characteristic polynomial.

$$\det[s\mathbf{I} - (\mathbf{A} + \mathbf{BK_p}) - \frac{\mathbf{BK_I}}{s}] = 0 \qquad (6)$$

$$\Delta_d(s) = \alpha_0 + \alpha_1 s + ... + \alpha_{n-1} s^{n-1} + \alpha_n s^n + \alpha_{n+1} s^{n+1} \qquad (7)$$

It is noticed that the n-order of the open-loop system is increased by 1 due to the integral term.

B. Transformation into the Frobenius canonical form

The pole placement problem herein considers the Frobenius canonical form of a delay-free LTI system. Eq. (8) represents the state transformation

$$\xi = \mathbf{Tx}, \quad \mathbf{x} = \mathbf{T}^{-1}\xi, \qquad (8)$$

where $\mathbf{\xi(t)}(n \times 1)$ is the transformed state variable vector, and $\mathbf{T}(n \times n)$ is the transformation matrix. The matrices $\mathbf{A_c}(n \times n)$ and $\mathbf{B_c}(n \times 1)$ are the transformed system matrix and the control gain vector, respectively. Both matrices can be calculated as follows:

$$\mathbf{A_c} = \mathbf{TAT}^{-1}, \quad \mathbf{B_c} = \mathbf{TB}, \qquad (9)$$

Where

$$\mathbf{T} = \begin{bmatrix} \mathbf{q_1} \\ \mathbf{q_1 A} \\ \vdots \\ \mathbf{q_1 A}^{n-1} \end{bmatrix}. \qquad (10)$$

The vector $\mathbf{q_1}(1 \times n)$ in (10) is

$$\mathbf{q_1} = \mathbf{e}_n^T \mathbf{w}_c^{-1}, \qquad (11)$$

in which $\mathbf{w_c}$ is the controllability matrix of the system (1)

$$\mathbf{w_c} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}], \qquad (12)$$

and the unit vector $\mathbf{e}_n = [0 \quad 0 \quad \cdots \quad 1]^T$. The Frobenius canonical form can be expressed as

$$\dot{\xi} = \mathbf{A_c}\xi + \mathbf{B_c}u$$

$$\dot{\xi} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \qquad (13)$$

III. POLE PLACEMENT FORMULAE FOR STATE-PI FEEDBACK

State feedback through a PI controller can be achieved via the gain matrices $\tilde{\mathbf{K}}_\mathbf{F}$ and $\bar{\mathbf{K}}_\mathbf{F}$, respectively. The control u for the state-P feedback is described by

$$u = \tilde{\mathbf{K}}_\mathbf{F}\, \xi, \qquad (14)$$

in which $\tilde{\mathbf{K}}_F = [\tilde{k}_1 \quad \tilde{k}_2 \quad \cdots \quad \tilde{k}_n]$. Let $\tilde{\Delta}_d(s) = \tilde{\alpha}_0 + \tilde{\alpha}_1 s + ... + \tilde{\alpha}_{n-1} s^{n-1} + \tilde{\alpha}_n s^n$ be the desired characteristic polynomial, where as $\tilde{\alpha}_n = a_n$. Hence, the canonical form of the system is

$$\dot{\xi} = \mathbf{A_c}\xi + \mathbf{B}\tilde{\mathbf{K}}_\mathbf{F}\, \xi,$$

having the closed-loop characteristic polynomial of

$$\tilde{\Delta}(s) = \det(s\mathbf{I} - \mathbf{A_c} - \mathbf{B_c}\tilde{\mathbf{K}}_\mathbf{F})$$

and for an n-order system

$$\tilde{\Delta}(s) = (a_0 - \tilde{k}_1) + (a_1 - \tilde{k}_2)s + \cdots + (a_{n-1} - \tilde{k}_n)s^{n-1} + s^n \quad (15)$$

The control u for the state-I feedback is

$$u = \bar{\mathbf{K}}_\mathbf{F}\int_0^t \xi(\tau)d\tau, \qquad (16)$$

where $\bar{\mathbf{K}}_\mathbf{F} = [\bar{k}_1 \quad \bar{k}_2 \quad \cdots \quad \bar{k}_n]$. The desired characteristic polynomial is

$$\bar{\Delta}_d(s) = \bar{\alpha}_0 + \bar{\alpha}_1 s + ... + \bar{\alpha}_{n-1}s^{n-1} + \bar{\alpha}_n s^n + \bar{\alpha}_{n+1}s^{n+1}, \bar{\alpha}_{n+1} = 1.$$

Therefore, the system in canonical form is

$$\dot{\xi} = \mathbf{A_c}\xi + \mathbf{B}\bar{\mathbf{K}}_\mathbf{F}\int_0^t \xi(\tau)d\tau$$

$$\dot{\xi} - \mathbf{A_c}\xi - \mathbf{B_c}\bar{\mathbf{K}}_\mathbf{F}\int_0^t \xi(\tau)d\tau = 0. \qquad (17)$$

The closed-loop characteristic polynomial is

$$\overline{\Delta}(s) = \det[s\mathbf{I} - \mathbf{A_c} - \frac{\mathbf{B_c}\overline{\mathbf{K}}_\mathbf{F}}{s}]$$

and for an n-order system

$$\overline{\Delta}(s) = -\overline{k}_1 + (a_0 - \overline{k}_2)s + \cdots + (a_{n-2} - \overline{k}_n)s^{n-1} + a_{n-1}s^n + s^{n+1} \quad (18)$$

The following is the main theorem.

Theorem 1. A delay-free LTI system according to (1) or (13) having the control input $u = \mathbf{K_p x} + \mathbf{K_I}\int_0^t \mathbf{x}(\tau)d\tau$ or $u = \tilde{\mathbf{K}}_\mathbf{F}\,\xi + \overline{\mathbf{K}}_\mathbf{F}\int_0^t \xi(\tau)d\tau$, where $[\mathbf{K_p}, \mathbf{K_I}] = [\tilde{\mathbf{K}}_\mathbf{F}, \overline{\mathbf{K}}_\mathbf{F}]\mathbf{T}$, and having the desired characteristic polynomial .

$$\Delta_d(s) = \alpha_0 + \alpha_1 s + ... + \alpha_{n-1}s^{n-1} + \alpha_n s^n + \alpha_{n+1}s^{n+1}$$

Placement of stable closed-loop poles can be achieved by using the gains

$$\mathbf{K_P} = \left[a_0 \;\vdots\; a_1 \;\vdots\; a_2 \;\vdots\; \cdots \;\vdots\; a_{n-1} - \alpha_n\right]\mathbf{T} \text{ and}$$
$$\mathbf{K_I} = \left[-\alpha_0 \;\vdots\; -\alpha_1 \;\vdots\; -\alpha_2 \;\vdots\; \cdots \;\vdots\; -\alpha_{n-1}\right]\mathbf{T}. \quad (19)$$

<u>Proof</u> The closed-loop system in the canonical form can be expressed as

$$\dot{\xi} = \mathbf{A_c}\xi + \mathbf{B_c}[\tilde{\mathbf{K}}_\mathbf{F}\,\xi + \overline{\mathbf{K}}_\mathbf{F}\int_0^t \xi(\tau)d\tau]$$

Its characteristic polynomial can be obtained from

$$\Delta_{PI}(s) = \det[s\mathbf{I} - \mathbf{A_c} - \mathbf{B_c}\tilde{\mathbf{K}}_\mathbf{F} - \frac{\mathbf{B_c}\overline{\mathbf{K}}_\mathbf{F}}{s}]$$

, and hence for an n-order system

$$\Delta_{PI}(s) = -\overline{k}_1 + (a_0 - \tilde{k}_1 - \overline{k}_2)s + \cdots + (a_{n-2} - \tilde{k}_{n-1} - \overline{k}_n)s^{n-1} + (a_{n-1} - \tilde{k}_n)s^n + s^{n+1} \quad (20)$$

By equating (20) to the desired characteristic polynomial, the following relations can be obtained

$$\begin{aligned} -\overline{k}_1 &= \alpha_0 \\ a_0 - \tilde{k}_1 - \overline{k}_2 &= \alpha_1 \\ &\vdots \\ a_{n-2} - \tilde{k}_{n-1} - \overline{k}_n &= \alpha_{n-1} \\ a_{n-1} - \tilde{k}_n &= \alpha_n \\ 1 &= \alpha_{n+1} \end{aligned} \quad (21)$$

Therefore, the gain matrices according to (19) can be concluded. This completes the proof.

To design a controller for pole placement, one can follow the steps listed below.

Step 1. Calculate the characteristic polynomial of the original system according to (2).

Step 2. Obtain the Frobenius canonical form of the system using (9)-(12).

Step 3. Specify n+1 numbers of desired poles. One of the poles must be a fast real pole.

Step 4. Calculate the desired characteristic polynomial

$$\Delta_d(s) = \alpha_0 + \alpha_1 s + ... + \alpha_{n-1}s^{n-1} + \alpha_n s^n + \alpha_{n+1}s^{n+1}.$$

Step 5. Calculate the gain matrices using (19).

Consider the following single-input controllable systems:

Example 1.

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The system in example 1 is originally unstable with its poles at 0 and -2. It is desirable to have the closed-loop poles at -4.1002±3.8486j. As a result of transformation, the canonical form of the system model is

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

To achieve the prescribed pole locations for step 3, an additional pole at -20 is considered. The desired characteristic polynomial is

$$\Delta_d(s) = 632.46 + 195.63s + 28.20s^2 + s^3$$

The obtained gain matrices are

$$\mathbf{K_P} = \begin{bmatrix} -26.2 & 0 \end{bmatrix}$$
$$\mathbf{K_I} = \begin{bmatrix} -195.63 & -632.41 \end{bmatrix}.$$

Example 2.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 980 & 0 & -2.8 \\ 0 & 0 & -100 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

The system is originally unstable with its poles at ±31.3050 and -100. It is desirable to have the closed-loop poles at -10±10j and -20. As a result of transformation, the canonical form of the system model is

$$\dot{\xi} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 98000 & 980 & -100 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

To achieve the prescribed pole locations for step 3, an additional pole at -100 is considered. The desired characteristic polynomial is

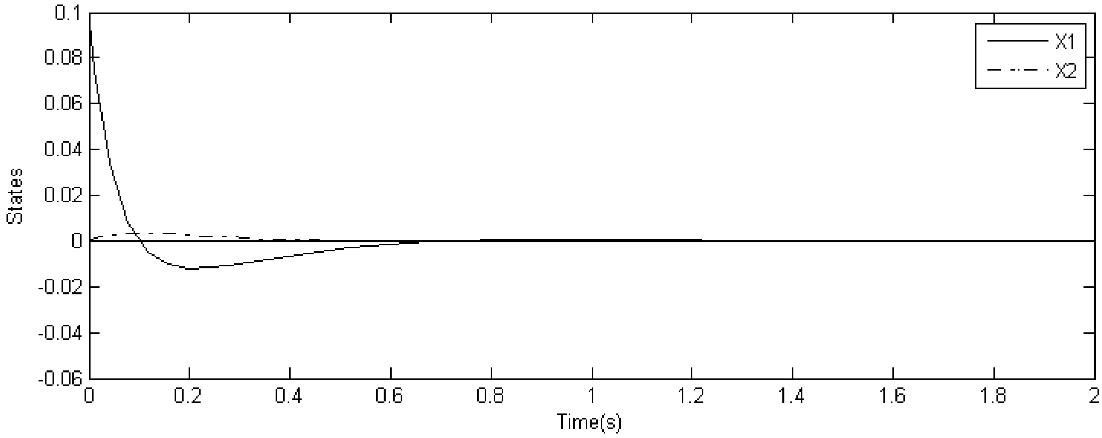
$$\Delta_d(s) = 200000 + 34000s + 2600s^2 + 90s^3 + s^4$$

The obtained gain matrices are

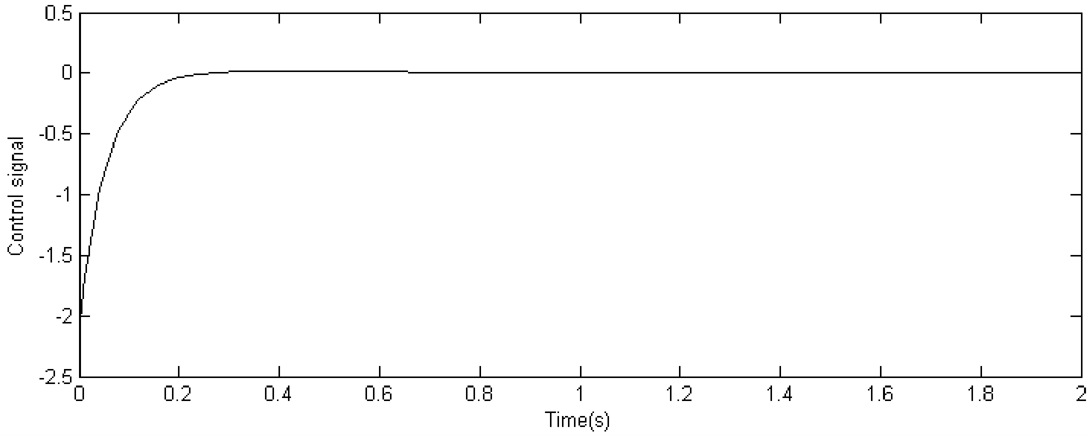
$$\mathbf{K_P} = \begin{bmatrix} 317.71 & -7.79 & -0.30 \end{bmatrix}$$

$$\mathbf{K_I} = \begin{bmatrix} 9820 & 122.4 & -26 \end{bmatrix}.$$

The results shown in Figs. 1 and 2 have the initial states of $\mathbf{x}(t_0) = \begin{bmatrix} 0.1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0.005 & 0 & 0 \end{bmatrix}^T$.

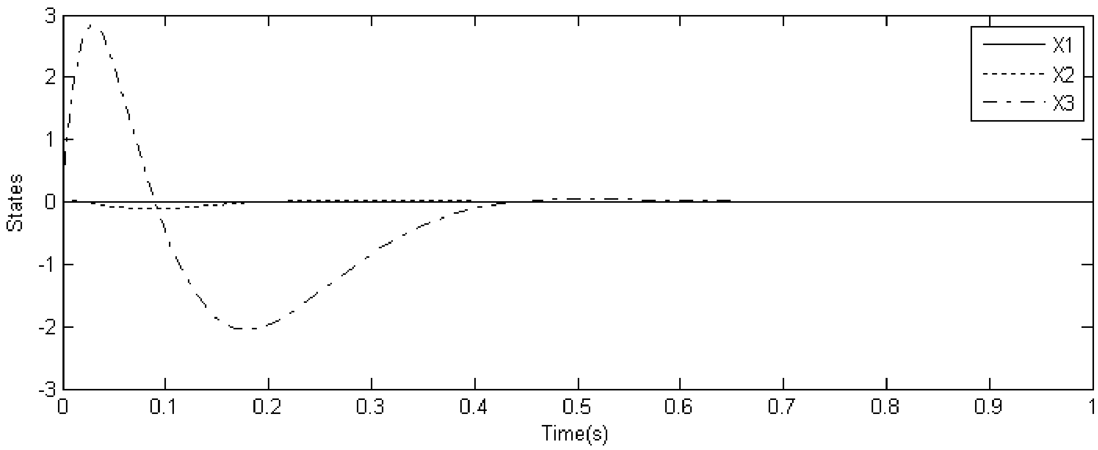


(a) Responses of states: x_1, x_2.

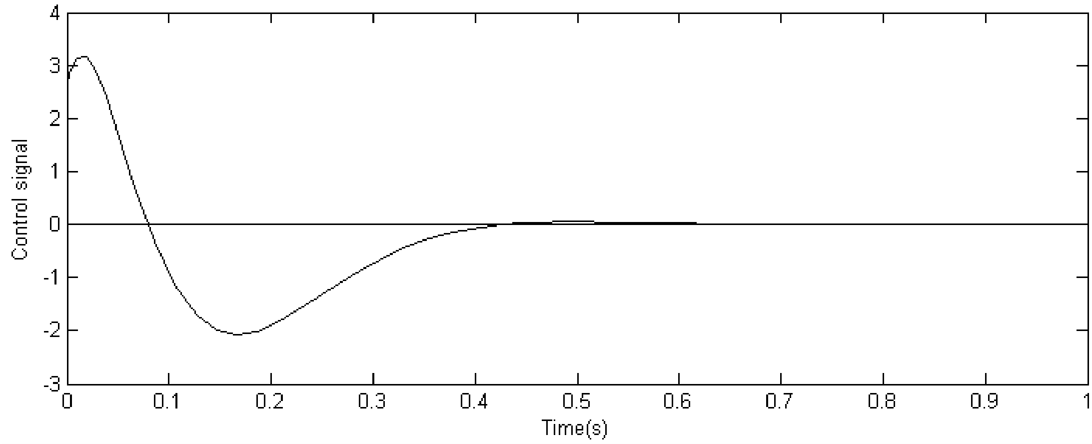


(b) Control signal u

Fig. 1 Time responses and control signal of the numerical example 1 with state-PI feedback from the proposed method



(a) Responses of states: x_1, x_2 and x_3



(b) Control signal u

Fig. 2 Time responses and control signal of the numerical example 2 with state-PI feedback from the proposed method

IV. INVERTED PENDULUM SYSTEM

The inverted pendulum system considered by this article has a low bandwidth since it is driven by a pneumatic motor. The diagram representing the system is shown in Fig. 3. Fig. 4 is the diagram representing the rod-less pneumatic actuator that is driven by a proportional control valve.

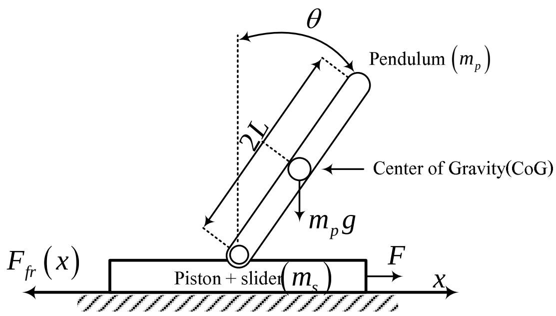


Fig. 3 Simplified diagram representing inverted pendulum.

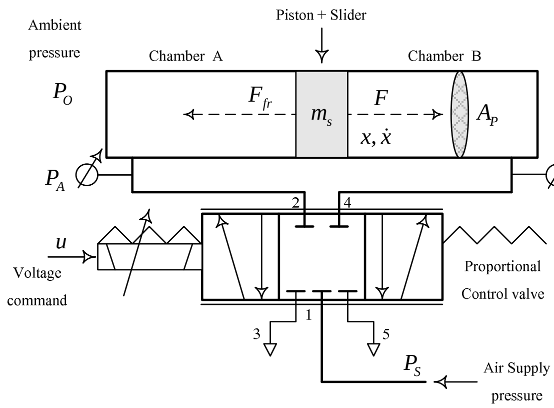


Fig. 4 Diagram representing rod-less pneumatic actuator.

The piston-slider component is subjected to nonlinear friction force $F_{fr}(\dot{x})$ causing a stick-slip motion. The friction force can thus be approximated by Stribeck model [21] consisting of static, Coulomb and viscous frictions, respectively. In [9], a control design model is proposed in state-variable form, whereas the states are

$$\begin{aligned}\mathbf{x} &= [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6]^T \\ &= [x \quad \dot{x} \quad \theta \quad \dot{\theta} \quad F \quad \dot{F}]^T\end{aligned} \tag{22}$$

The system model in the form $\dot{\mathbf{x}} = F(\mathbf{x}, u)$ is as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{x_5 + T_D x_6 - F_{fr}(x_2)}{m_s + m_p(1-\frac{3}{4}\cos^2 x_3)} - \dfrac{\frac{3}{4} m_p g \sin x_3 \cos x_3 + m_p L x_4^2 \sin x_3}{m_s + m_p(1-\frac{3}{4}\cos^2 x_3)} \\ x_4 \\ \dfrac{\frac{3g}{4L}\sin(x_3)}{1-\frac{3}{4}\frac{m_p}{m_s+m_p}\cos^2 x_3} - \dfrac{3\cos x_3}{4L(m_s+m_p)} \times \dfrac{x_5 + x_4^2 \sin x_3 - F_{fr}(x_2)}{1-\frac{3}{4}\frac{m_p}{m_s+m_p}\cos^2 x_3} \\ x_6 \\ -\Omega_0^2 x_5 - 2\varsigma\Omega_0 x_6 + K_0 \Omega_0^2 u \end{bmatrix} \tag{23}$$

For the purpose of state feedback control design, the saturation effect of the proportional valve is less critical. The nonlinear model in (23) can be linearized around the unstable equilibrium, i.e. upright position or $\theta = 0$. Considering the angle θ with small variations, $\sin x_3 \approx x_3$, $\cos x_3 \approx 1$ and $x_4 \approx 0$. Furthermore, the slider is assumed to be in continuous motion. The friction force is thus reduced to $F_{fr} = b\dot{x} = bx_2$. Therefore, the linearized model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ can be obtained, in which

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{-4b}{4m_s + m_p} & \dfrac{-3m_p g}{4m_s + m_p} & 0 & \dfrac{4}{4m_s + m_p} & \dfrac{4T_D}{4m_s + m_p} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \dfrac{3b}{L(4m_s + m_p)} & \dfrac{3g(m_s + m_p)}{L(4m_s + m_p)} & 0 & \dfrac{-3}{L(4m_s + m_p)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\Omega_0^2 & -2\varsigma\Omega_0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & K_0\Omega_0^2 \end{bmatrix}^T \tag{24}$$

$$\mathbf{y} = \mathbf{Cx} = \begin{bmatrix} x_1 & x_3 & x_5 \end{bmatrix}^T$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (25)$$

Eq. (25) is the output equation taking the slider position (x_1), pendulum deflection angle (x_3) and cylinder force (x_5) as output variables.

V. SIMULATION RESULTS

The proposed method is applied to the stabilization and disturbance rejection problems of the inverted pendulum system. The block diagram in Fig. 5 represents a LTI system with state-PI feedback. For comparison purposes, the method based on Ackermann's formula [10,11] is also used.

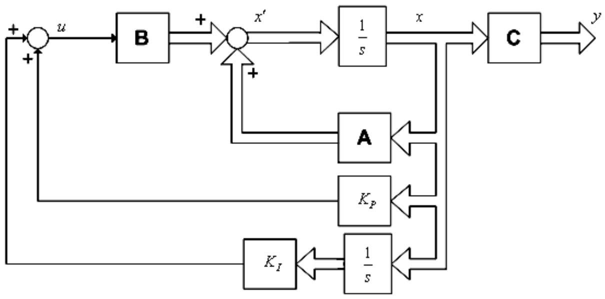


Fig. 5 Block diagram representation of a linear system with state-PI feedback.

The inverted pendulum system is described by the following state-variable models:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -28.1081 & -0.4772 & 0 & 1.0811 & 0.0400 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 105.4054 & 38.5772 & 0 & -4.0541 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -784 & -38.8000 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 48686.4000 \end{bmatrix} u$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}$$

The system is inherently unstable since it has open-loop poles at $0,\ -6.0249,\ 6.0915,\ -19.0400 \pm 20.5299j$ and -28.1747. To stabilize this system, the system poles are to be placed at $-3 \pm 2j, -1 \pm 1.5j, -5$ and -5.

As a result of transformation, the canonical form of the system model is

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 810.6814\times10^3 & 69.6205\times10^3 & -19.5337\times10^3 & -1.8158\times10^3 & -0.0662\times10^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

To achieve the prescribed pole locations, an additional pole at -20 and -50 are considered for 2 cases. Using the proposed method, the following gain matrices are obtained:

- (for adding poles at -20)

$\mathbf{K_p}$ = [0 3.198 -0.125 -0.179 -0.107 0.001]

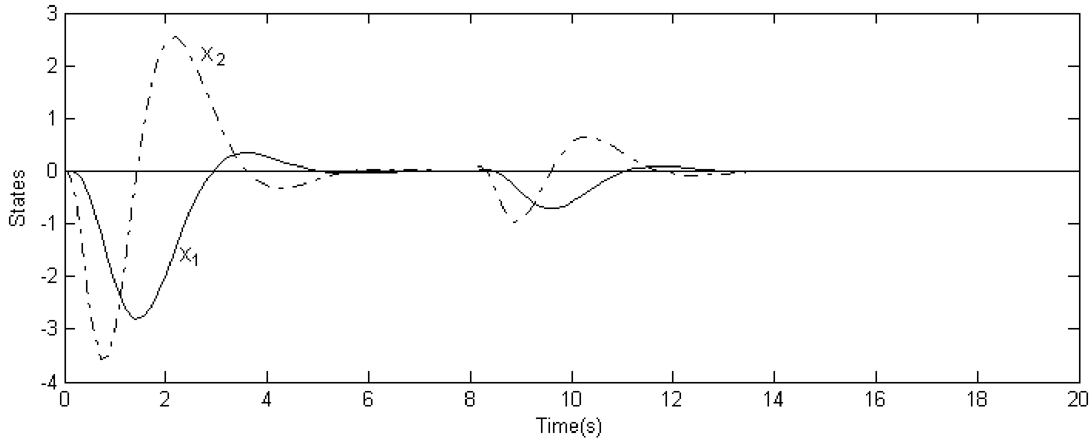
$\mathbf{K_I}$ = [0.011 142.136 -15.765 -4.011 -5.466 -0.011]

- (for adding poles at -50)

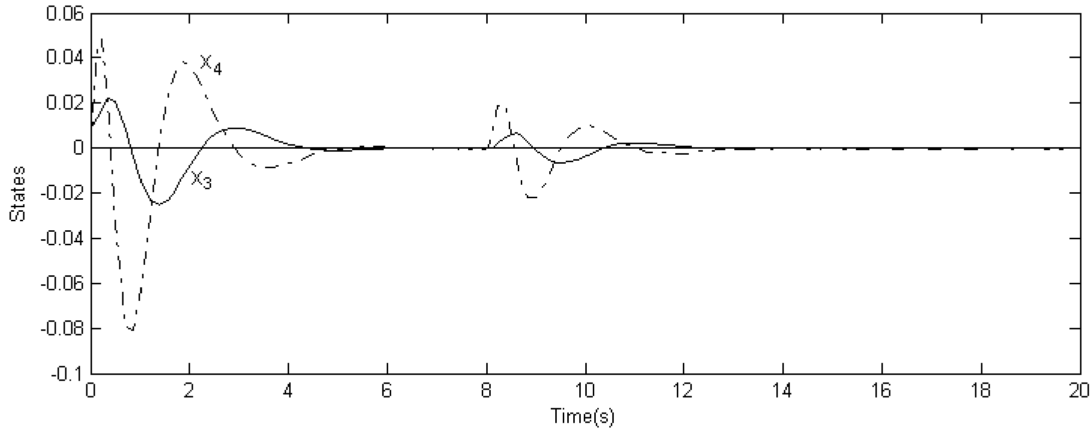
$\mathbf{K_p}$ = [0 14.461 0.067 -0.326 -0.541 -3.726]

$\mathbf{K_I}$ = [0.027 290.897 -36.526 -8.814 -11.187 -0.021]

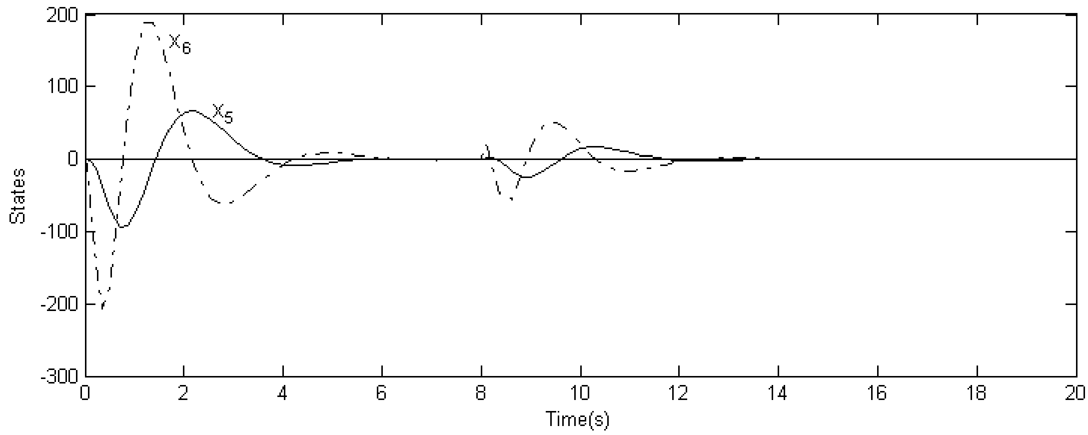
For a comparison, using the Ackermann's formula one can obtain the gain matrix $\mathbf{K}$ = [-0.001 6.109 1.059 0.154 -0.251 -0.001]. The results shown in Figs. 6 and 7 have the initial states of $\mathbf{x}(t_0) = \begin{bmatrix} 0 & 0 & 0.01 & 0 & 0 & 0 \end{bmatrix}^T$ meaning that an external disturbance of 1 unit occurs to the 3rd state variable at the time of 8 s. It can be clearly seen from Fig. 6 that with our proposed method the system responses faster and recovers completely from the external disturbance within a short duration. Some steady-state errors still remain in the system with the conventional design as shown in Fig. 7. Further results are illustrated in Fig. 8 to show the effects of the additional real pole due to the design step 3 on the dynamic responses. It is found that an additional fast real pole results in better transient responses in an exchange of high gains. Moreover, the system is more robusted to external disturbances.



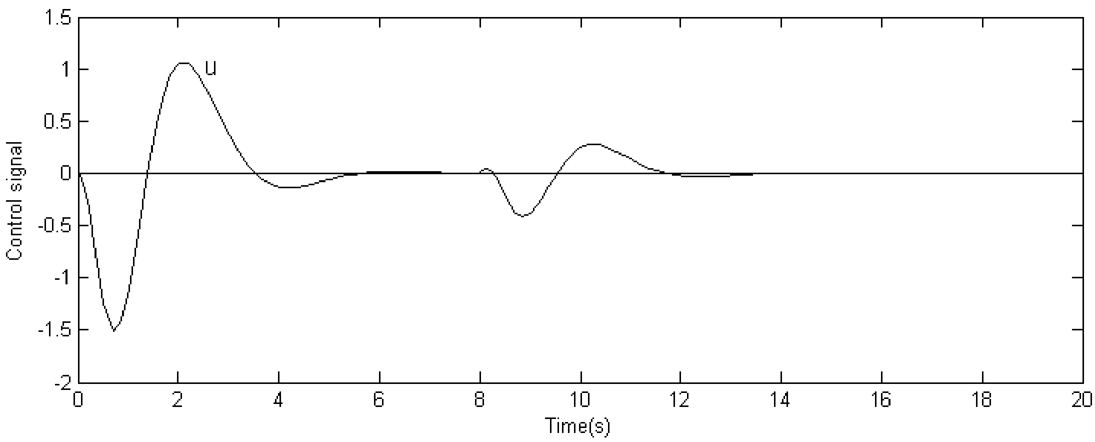
(a) Responses of states: x_1, x_2.



(b) Responses of states: x_3, x_4.

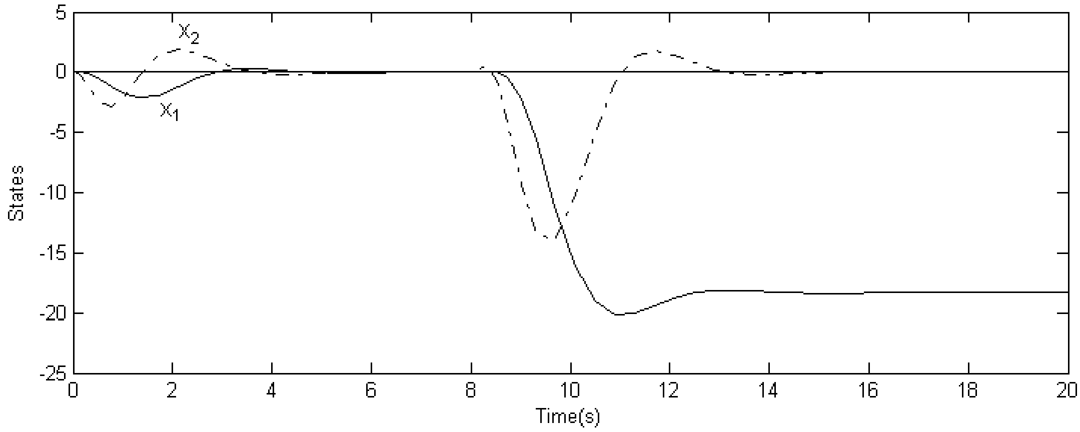


(c) Responses of states: x_5, x_6.

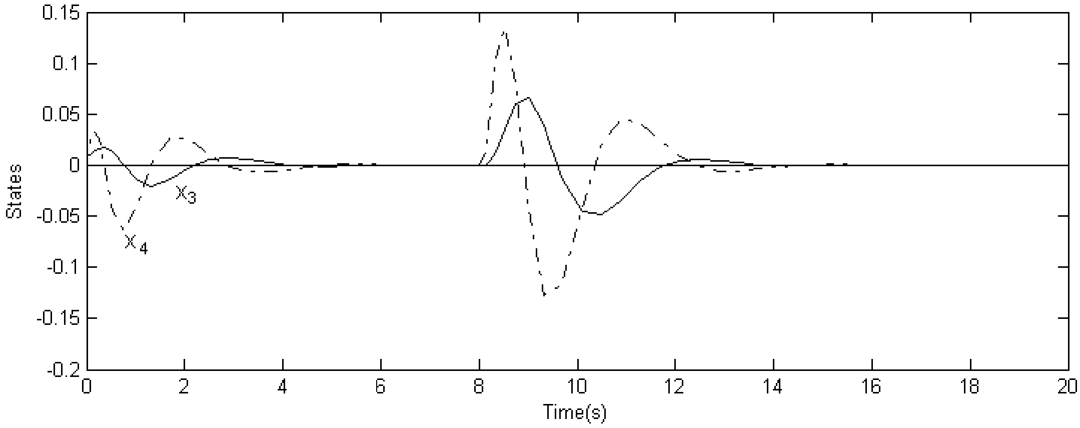


(d) Control signal u

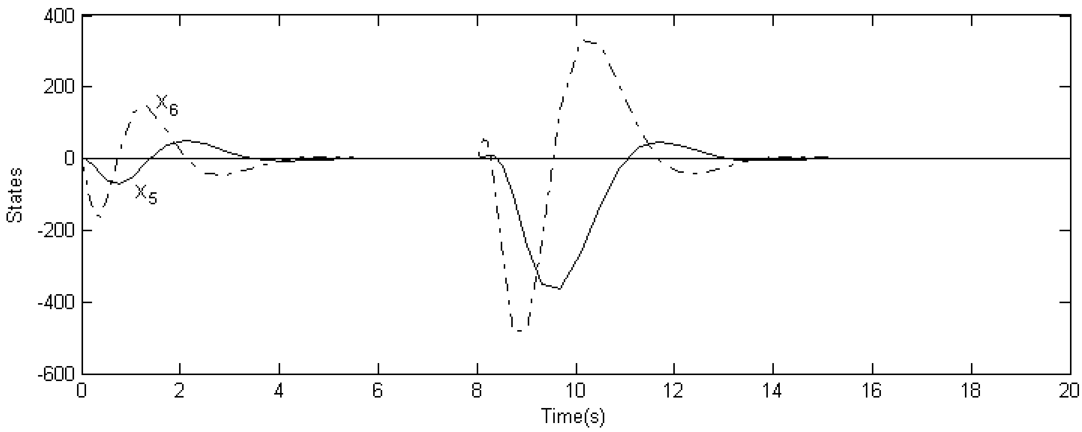
Fig. 6 Time responses and control signal of the system with state-PI feedback from the proposed method (added pole at -20).



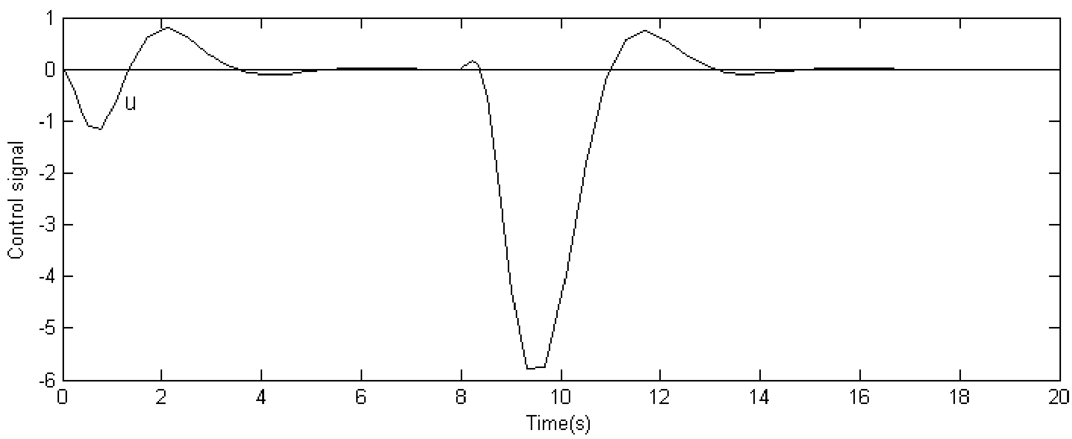
(a) Responses of states: x_1, x_2.



(b) Responses of states: x_3, x_4.

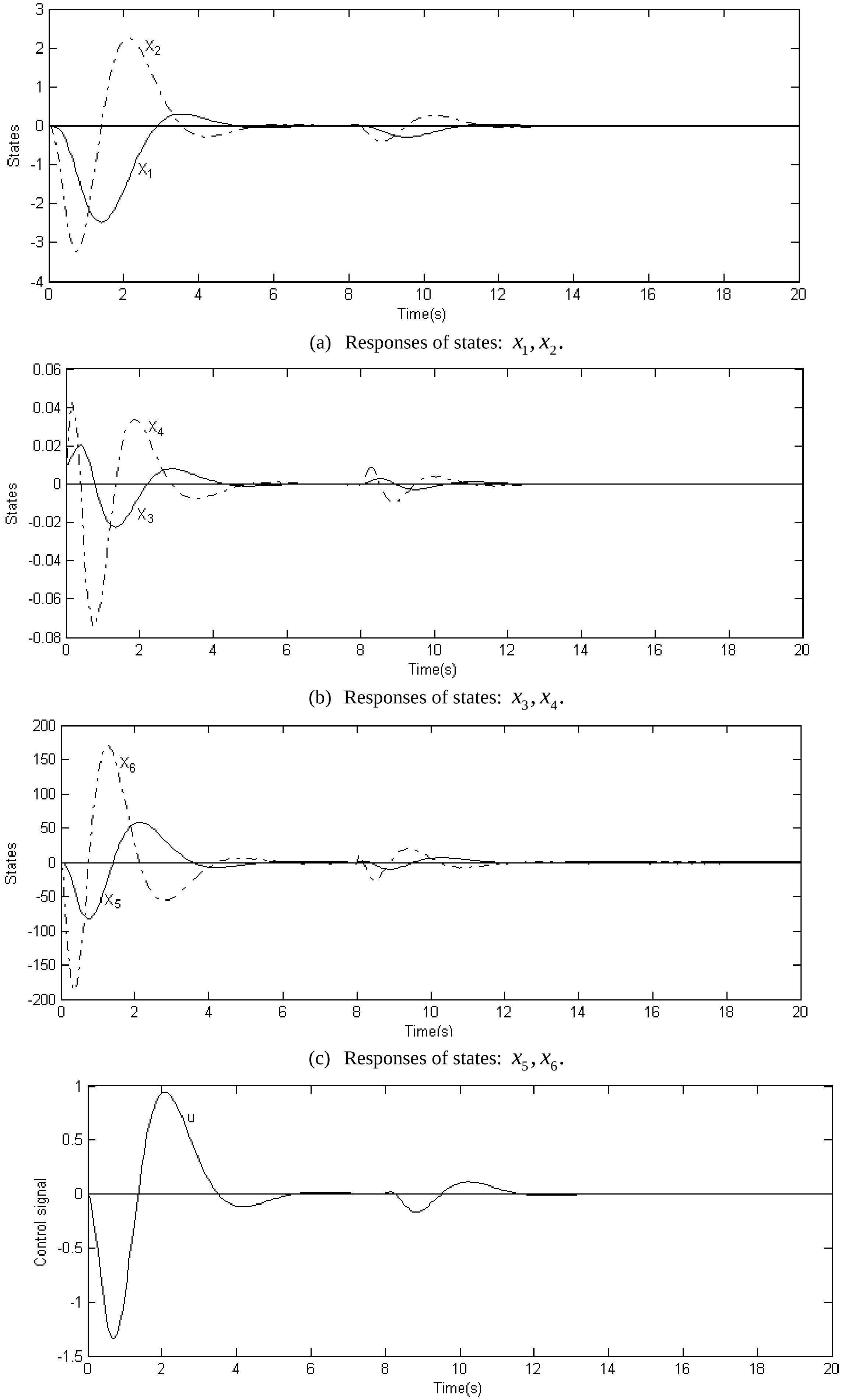


(c) Responses of states: x_5, x_6.



(d) Control signal u

Fig.7 Time responses and control signal of the system with the Ackermann's method.



(a) Responses of states: x_1, x_2.

(b) Responses of states: x_3, x_4.

(c) Responses of states: x_5, x_6.

(d) Control signal u

Fig. 8 Time responses and control signal of the system with state-PI feedback from the proposed method (added pole at -50).

VI. CONCLUSION

The gain formulae for state-PI feedback have been proposed by this article. One could follow the simple 5 design steps listed in Section III. The proposed approach is applied to the stabilization problem and disturbance rejection of an inverted pendulum system with a pneumatic motor. The system is explained in Section IV. Simulation results are illustrated in Section V including a comparison between the approach of Ackermann's formula and the proposed method. As a result, the states of the inverted pendulum system with the state-PI feedback controller obtained from the proposed method have considerably fast responses, and can recover from the disturbances quickly without any residue errors. This is not the case with using the Ackermann's formula in which some residue errors remain after the disturbances. Also, the effects of the position of one additional pole required according to the integral term are investigated. It is recommended that a fast real pole be added to achieve fast transient response bearing in mind on the increase in the feedback gains.

APPENDIX

- Slider + piston mass: $m_s = 0.91\ kg$
- Pendulum mass: $m_p = 0.06\ kg$
- Pendulum total length: $2L = 0.4\ m$
- Piston cross-section: $A_p = 1.8 \times 10^{-4}\ m^2$
- Total air volume: $V_0 = 9 \times 10^{-5}\ m^3$
- Air supply pressure: $p_s = 10\ bar$
- Atmospheric pressure: $p_0 = 1\ bar$
- Proportional valve gain parameter:
  $K_p = 3.9 \times 10^{-6}\ N.kg.s.V^{-1}$
- Applied force model gain parameter:
  $K_0 = 62.1\ N/V$
- Applied force model natural frequency:
  $\Omega_0 = 28\ rad/s$
- Applied force model damping ratio: $\varsigma = 0.68$
- Applied force model lead time constant: $T_D = 37\ ms$
- Breakaway friction: $F_s = 31.2\ N$
- Coulomb friction: $F_c = 14.5\ N$
- Viscous friction coefficient: $b = 26\ Ns/m$
- Striebeck coefficient and Striebeck speed:
  $\delta = 1,\ v_s = 0.02\ m/s.$

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