Solution of eddy current testing problems for multilayer tubes with varying properties

Valentina Koliskina, and Inta Volodko

Abstract**—**Analytical solution for the change in impedance of a coil located inside or outside a multilayer conducting tube is obtained in the present paper. The electric conductivity and magnetic permeability of conducting cylindrical layers of the tube are assumed to be power functions of the radial coordinate. The change in impedance is expressed in terms of improper integral containing Bessel functions. Other analytical solutions are suggested in the paper. Three examples are discussed in detail: (a) a coil inside an infinite cylindrical layer, (b) a coil inside a two-layer tube, and (c) a coil outside a two-layer tube.

Keywords**—**eddy currents, electric conductivity, magnetic permeability, change in impedance

I. Introduction

Eddy current methods are widely used in practice in order to control properties of conducting materials. All problems where eddy current coils are used for inspection of materials can be divided into the following two categories: (a) estimation of properties and/or other parameters of conducting media (for example, electric conductivity of a conducting layer or thickness of metal coatings) and (b) detection of defects (or flaws) in a conducting medium (for example, estimation of the effect of corrosion or presence of voids or other non-metallic inclusions in the medium). In both cases theoretical models (with some unknown parameters) are usually compared with experimental data. Optimization methods (for example, the least squares method) are then used to estimate unknown parameters of the medium (see, for example, [1]-[4]).

Thus, a necessary step for the solution of an inverse problem (determination of unknown parameters of a medium) is the existence of a mathematical model describing the interaction of an alternating current in a coil with the conducting medium (direct problem).

Mathematical models for eddy current testing problems of conducting media with constant properties are well-known in the literature [5], [6]. Analytical solutions are presented in [7] for the case where a coil with alternating current is located above a multilayer medium. Similar problems for coils encircling multiple coaxial conductors or coils inside multiple coaxial conductors are analyzed in [8]. The properties of all media in [8] are assumed to be constant.

Some industrial applications (for example, surface hardening or decarbonization) modify the properties of a conducting medium (electric conductivity and/or magnetic permeability) which depend on geometrical coordinates. It is shown in [9], [10] that a thin layer of reduced magnetic permeability can exist in a medium which undergoes special treatment. In the case of a planar medium the magnetic permeability of the layer depends on the vertical coordinate.

Thus, in order to analyze such cases in practice one needs to develop mathematical models where electric conductivity and magnetic permeability of conducting layers depend on geometrical coordinates. There are at least two methods that can be used in order to take into account variability of the properties of the medium with respect to one geometrical coordinate. Regions where the properties of the medium are not constant can be divided into sufficiently large number of sub-layers where the electric conductivity and magnetic permeability are assumed to be constant. Therefore, in the whole region of interest the properties of the medium are piecewise constant functions of, say, depth. For example, up to 50 layers are used in [11] to model the change in electric conductivity with respect to a vertical coordinate.

Another approach is based on an attempt to use relatively simple electric conductivity and/or magnetic permeability profiles in order to model the variation of the properties of the medium with respect to one spatial coordinate. Some analytical solutions for the case where the properties of the medium depend on the vertical or radial coordinates are presented in [6].

In the present paper we follow the second approach. Analytical solutions are constructed for the case where a coil is located inside or outside a multilayer tube with varying properties. The electric conductivity and magnetic permeability are assumed to be power functions of the radial coordinate. The solution procedure is described for an arbitrary number of conducting layers with varying electric conductivity and magnetic permeability. Three cases are analyzed in detail: (a) the case of a coil inside one infinite outer layer, (b) the case of a coil inside a two-layer tube, and (c) the case of a coil outside a two-layer tube.

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V. Koliskina is with the Department of Engineering Mathematics of Riga Technical University, Riga, Latvia LV 1048 (phone: 371-6708-9528; fax: 371-6708-9694; e-mail: v.koliskina@ gmail.com).

I. Volodko is with the Department of Engineering Mathematics of Riga Technical University, Riga, Latvia LV 1048 (phone: 371-6708-9528; fax: 371-6708-9694; e-mail: inta.volodko@ rtu.lv).

II. A COIL INSIDE A MULTILAYER TUBE WITH VARYING PROPERTIES

Consider a single-turn coil of radius r_0 with alternating current located inside a multilayer tube where each coaxial layer (region R_i) is described by the inequalities: $R_i = \{r_i \le r \le r_{i+1}, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}, i = 1,2,...n$. Here r_i and r_{i+1} are the inner and outer radii of the cylindrical layer, respectively. Regions R_0 and R_{n+1} represent free space. In particular, $R_0 = \{0 \le r \le r_1, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$ and $R_{n+1} = \{r \ge r_{n+1}, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$. The coil is located in the plane $z = z_0$ perpendicular to the axes of the tubes.

Due to axial symmetry the vector potential has only one nonzero component in each region $R_i, i = 0,1,2,..n+1$ which is the function of r and z only. We assume that the electric conductivity $\tilde{\sigma}_i$ and magnetic permeability $\tilde{\mu}_i$ in region R_i is modeled by the following relations

$$\tilde{\mu}_i = \mu_0 \mu_i r^{\alpha_i}, \quad \tilde{\sigma}_i = \sigma_i r^{\beta_i}, \quad i = 1,2,...n, \tag{1}$$

where α_i and β_i are given constants and μ_0 is the magnetic constant.

The system of equations for the amplitudes of the vector potential in regions $R_0, R_1, ..., R_{n+1}$ has the form (see, for example, [6]):

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0)\delta(z - z_0), \tag{2}$$

$$\frac{\partial^2 A_i}{\partial r^2} + \frac{(1-\alpha_i)}{r}\frac{\partial A_i}{\partial r} - \left(\frac{1+\alpha_i}{r^2} + p_i^2 r^{\alpha_i + \beta_i}\right) A_i + \frac{\partial^2 A_0}{\partial z^2} = 0, \tag{3}$$

$$\frac{\partial^2 A_{n+1}}{\partial r^2} + \frac{1}{r}\frac{\partial A_{n+1}}{\partial r} - \frac{A_{n+1}}{r^2} + \frac{\partial^2 A_{n+1}}{\partial z^2} = 0, \tag{4}$$

where $p_i = \sqrt{j\omega\sigma_i\mu_0\mu_i}$, j is the imaginary unit and ω is the frequency of the current in the coil.

Applying the Fourier transform of the form

$$\tilde{A}_i(r,\lambda) = \int_{-\infty}^{+\infty} A_i(r,z) e^{-i\lambda z} dz, \quad i = 0,1,..., n+1 \tag{5}$$

to (2)-(4) we obtain

$$\frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_0}{dr} - \frac{\tilde{A}_0}{r^2} - \lambda^2 \tilde{A}_0 = -\mu_0 I e^{-i\lambda z_0}\delta(r - r_0), \tag{6}$$

$$\frac{d^2 \tilde{A}_i}{dr^2} + \frac{(1-\alpha_i)}{r}\frac{d\tilde{A}_i}{dr} - \lambda^2 \tilde{A}_i - \left(\frac{1+\alpha_i}{r^2} + p_i^2 r^{\alpha_i+\beta_i}\right)\tilde{A}_i = 0, \quad i = 1,2,..., n, \tag{7}$$

$$\frac{d^2 \tilde{A}_{n+1}}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_{n+1}}{dr} - \frac{\tilde{A}_{n+1}}{r^2} - \lambda^2 \tilde{A}_{n+1} = 0. \tag{8}$$

The boundary conditions are (see [6]):

$$\tilde{A}_0 |_{r=r_1} = \tilde{A}_1 |_{r=r_1}, \quad \frac{d\tilde{A}_0}{dr}|_{r=r_1} = \frac{1}{\mu_1}\frac{d\tilde{A}_1}{dr}|_{r=r_1}, \tag{9}$$

$$\tilde{A}_i |_{r=r_i} = \tilde{A}_{i+1} |_{r=r_i}, \quad \frac{1}{\mu_i}\frac{d\tilde{A}_i}{dr}|_{r=r_{i1}} = \frac{1}{\mu_{i+1}}\frac{d\tilde{A}_{i+1}}{dr}|_{r=r_{i1}}, \quad i = 1,2,...n, \tag{10}$$

$$\tilde{A}_n |_{r=r_{n+1}} = \tilde{A}_{n+1} |_{r=r_{n+1}}, \quad \frac{1}{\mu_n}\frac{d\tilde{A}_n}{dr}|_{r=r_{n+1}} = \frac{d\tilde{A}_{n+1}}{dr}|_{r=r_{n+1}}. \tag{11}$$

In addition, $\tilde{A}_0$ is bounded at $r = 0$ and $\tilde{A}_{n+1}$ is bounded as $r \to \infty$.

In order to find the solution to (6) we consider two sub-regions, R_{00c} and R_{01c}, of region R_0, namely, $R_{00} = \{0 \le r < r_0, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$ and $R_{01} = \{r_0 < r \le r_1, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$, respectively. The solutions in regions R_{00} and R_{01} are denoted by $\tilde{A}_{00}$ and $\tilde{A}_{01}$, respectively. The bounded solution to (6) in region R_{00} is

$$\tilde{A}_{00}(r,\lambda) = B_1 I_1(\lambda r). \tag{12}$$

The general solution to (6) in R_{01} has the form

$$\tilde{A}_{01}(r,\lambda) = B_2 I_1(\lambda r) + B_3 K_1(\lambda r). \tag{13}$$

Here $I_1(\lambda r)$ and $K_1(\lambda r)$ are the modified Bessel functions of order 1 of the first and second kind, respectively.

The functions $\tilde{A}_{00}(r,\lambda)$ and $\tilde{A}_{01}(r,\lambda)$ satisfy the following conditions at $r = r_0$:

$$\tilde{A}_{00} |_{r=r_0} = \tilde{A}_{01} |_{r=r_0}, \quad \frac{d\tilde{A}_{01}}{dr}|_{r=r_0} - \frac{d\tilde{A}_{00}}{dr}|_{r=r_0} = -\mu_0 I e^{-i\lambda z_0}. \tag{14}$$

The first condition in (14) reflects the fact that the function $\widetilde{A}_0(r,\lambda)$ is continuous at $r = r_0$ while the second condition in (14) is obtained integrating (6) with respect to r from $r_0 - \varepsilon$ to $r_0 + \varepsilon$ and considering the limit in the resulting expression as $\varepsilon \to +0$.

The bounded solution to (8) is

$$\widetilde{A}_{n+1}(r,\lambda) = D_{n+1} K_1(\lambda r). \tag{15}$$

The solution to (7) can be expressed in terms of different special functions. For example, the solution to (7) for the case $\alpha_i = -1,\ \beta_i = -1$ is

$$\widetilde{A}_i(r,\lambda) = C_i \frac{I_\nu(\lambda r)}{\sqrt{r}} + D_i \frac{K_\nu(\lambda r)}{\sqrt{r}}, \tag{16}$$

where $\nu = \sqrt{p_i^2 + 1/4}$ (see [12]).

Arbitrary constants $B_1, B_2, B_3, C_i, D_i\ (i = 1,2,.., n)$ and D_{n+1} can be found from conditions (9)-(11) and (14). Then the solution in each region $R_i, i = 0,1,..., n+1$ can be found by means of the inverse Fourier transform of the form

$$A_i(r,z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widetilde{A}_i(r,\lambda) e^{i\lambda z} d\lambda. \tag{17}$$

It can be shown that the induced vector potential in free space due to multilayer conducting tubes has the form

$$A_0^{ind}(r,z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} B_2 I_1(\lambda r) e^{i\lambda z} d\lambda. \tag{18}$$

Three applications of the theory presented in this section are considered in detail in the next two sections: (a) a coil inside an unbounded cylindrical layer, (b) a coil inside a two-layer tube and (c) a coil outside a two-layer tube.

III. A COIL INSIDE A CYLINDRICAL REGION

Suppose that a single-turn coil with alternating current is located inside a cylindrical region $R_1 = \{r \ge r_1, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$ (see [13]). The radius of the coil, r_0, is chosen as the measure of length ($r_0 < r_1$). The solution in region R_0 is given by (12) and (13). The solution to (7) in region R_1 that is bounded at infinity has the form

$$\widetilde{A}_1(r,\lambda) = D_1 \frac{K_\nu(\lambda r)}{\sqrt{r}}. \tag{19}$$

The unknown constants B_1, B_2, B_3 and D_1 in (12), (13) and (19) are determined using conditions (9) and (14). In particular,

$$B_2 = \frac{\mu_0 I r_c^2 I_1(\lambda)}{2} \frac{\gamma_1}{\gamma_2}, \tag{20}$$

where

$$\gamma_1 = 2r_1\lambda K_1(\lambda r_1) K_\nu'(\lambda r_1) - K_\nu(\lambda r_1)[K_1(\lambda r_1) + 2r_1\mu_1\lambda K_1'(\lambda r_1)],$$

$$\gamma_2 = 2r_1\lambda\mu_1 K_\nu(\lambda r_1) I_1'(\lambda r_1) + I_1(\lambda r_1)[K_1(\lambda r_1) - 2r_1\lambda K_1'(\lambda r_1)].$$

The induced vector potential in free space due to the presence of a cylindrical region is given by (18) and (20). The induced change in impedance of the coil is given by the formula (the case $z_0 = 0$ is considered below)

$$Z^{ind} = \frac{j\omega}{I} \oint_L A_0^{ind}(r,z) dl, \tag{21}$$

where L is the contour of the coil. Substituting (18) and (20) into (21) we obtain

$$Z^{ind} = 2\omega r_0^2 \mu_0 Z, \tag{22}$$

where

$$Z = j \int_0^\infty \frac{\gamma_1}{\gamma_2} I_1^2(\lambda) d\lambda. \tag{23}$$

Formula (23) is used to compute the change in impedance of the coil. Calculations are performed with "Mathematica" since it has built-in functions to evaluate modified Bessel functions of complex order. "Mathematica" program which is used to compute the change in impedance (23) is shown in Fig. 1. The results of computations are shown in Fig. 2. The calculated points (from top to bottom) for each curve correspond to the following values of $\eta : 1,2,...,10$. The parameter η is defined by $\eta = r_0\sqrt{\omega\sigma_1\mu_0\mu_1}$ (in this case $p_1 = \eta\sqrt{j}$). The three curves in Fig. 2 (from right to left) correspond to the cases $r_1 = 1.2; 1.4$ and 1.6, respectively.

```
R□1.6; mu1 □1 Sqrt R ;
Do Do p□eta Sqrt  ; nu □Sqrt p^2 1 4 ;
    I1 x_ :□BesselI 1, x R ;
    K1 x_ :□BesselK 1, x R ;
    Knu x_ :□BesselK nu, x R ;
    I2 x_ :□BesselI 2, x R ;
    K2 x_ :□BesselK 2, x R ;
    Knup x_ :□BesselK nu 1, x R ;
    I1pr x_ :□I2 x  1 x R  I1 x ;
    K1pr x_ :□ K2 x  1 x R  K1 x ;
    Knupr x_ :□nu  x R  Knu x  Knup x ;
    f1 x_ :□2 R x K1 x  Knupr x
       Knu x  K1 x  2 R mu1 x K1pr x ;
    f2 x_ :□2 R x mu1 Knu x  I1pr x
       I1 x  Knu x  2 R x Knupr x ;
    f x_ :□f1 x  f2 x  BesselI 1, x ^2;
    Z□□ NIntegrate f x ,
        x, 0.0001, Infinity , MaxRecursion□50 ;
    data1 eta □Re Z ;
    data2 eta □Im Z  , eta, 1, 10 ;
  data□Table If m□1, data1 k ,
     data2 k  , k, 1, 10 , m, 1, 2 ;
  gr1 n □ListPlot data, PlotStyle□
      PointSize 0.02 , AxesLabel□
      Re z , Im z  , DisplayFunction□Identity ;
  gr2 n □ListPlot data, AxesLabel□
      Re z , Im z  , PlotJoined□True,
    DisplayFunction□Identity ; R □R 0.2;
  mu1□1 Sqrt R  , n, 1, 3
Show gr1 1 , gr2 1 , gr1 2 , gr2 2 , gr1 3 ,
  gr2 3 , DisplayFunction□$DisplayFunction
Remove "Global`  "
```

Fig. 1. "Mathematica" program for numerical calculation of the change in impedance given by formula (23).

It is seen that the modulus of the change in impedance increases as the parameter η increases (that is, if the frequency increases). In addition, the change in impedance is larger when the coil is closer to the cylindrical region.

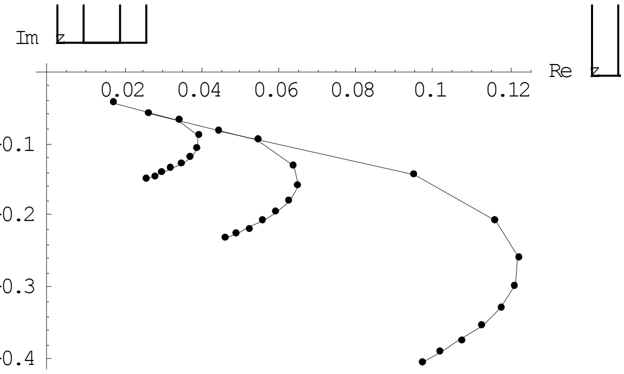

Fig. 2. The change in impedance of a coil for different values of r_1.

IV. A COIL INSIDE A TWO-LAYER TUBE

Consider a single-turn coil of radius r_0 located inside a two-layer tube (see [14]). The electric conductivity and magnetic permeability of the inner layer are given by formula (1) while the electric conductivity σ_2 and magnetic permeability μ_2 of the outer layer are constants. The outer layer is unbounded in the radial direction. The solution in region R_0 is given by (12) and (13). The solution to (7) in region R_1 is

$$\widetilde{A}_1(r,\lambda) = C_1 \frac{I_\nu(\lambda r)}{\sqrt{r}} + D_1 \frac{K_\nu(\lambda r)}{\sqrt{r}}. \qquad (24)$$

Finally, the bounded solution to (8) in region R_2 is

$$\widetilde{A}_2(r,\lambda) = D_2 K_1(qr), \qquad (25)$$

where $q = \sqrt{\lambda^2 + p_2^2}$.

It is convenient to choose the radius of the inner coil, r_1, as the measure of length. In this case r_2 represents the ratio of the radii of the tube. The unknown constants B_1, B_2, B_3, C_1, D_1 and D_2 in (12), (13), (24) and (25) are determined from the boundary conditions (14) and (9)-(11) with $n = 1$. In particular,

$$B_2 = -\gamma_3 \mu_0 I r_1^2 r_0 I_1(\lambda r_0), \qquad (26)$$

where $\gamma_3 = \gamma_4 / \gamma_5$ and

$$\gamma_4 = K_1(\lambda)[\lambda I_\nu'(\lambda) - I_\nu(\lambda)/2 - \gamma_6(\lambda K_\nu'(\lambda) - K_\nu(\lambda)/2] - \mu_1 \lambda K_1'(\lambda)[I_\nu(\lambda) - \gamma_6 K_\nu(\lambda)], \qquad (27)$$

$$\gamma_5 = I_1(\lambda)[\lambda I_\nu'(\lambda) - I_\nu(\lambda)/2 - \gamma_6(\lambda K_\nu'(\lambda) - K_\nu(\lambda)/2] - \mu_1 \lambda I_1'(\lambda)[I_\nu(\lambda) - \gamma_6 K_\nu(\lambda)]. \qquad (28)$$

The parameter γ_6 in (27) and (28) is defined by

$$\gamma_6 = \gamma_7 / \gamma_8,$$

where

$$\gamma_7 = \tilde{\mu} q K_1'(qr_2) I_\nu(\lambda r_2) - \mu_2 K_1(qr_2)[\lambda I_\nu'(\lambda r_2) - I_\nu(\lambda r_2)/(2r_2)],$$

$$\gamma_8 = \tilde{\mu} q K_1'(qr_2) K_\nu(\lambda r_2) - \mu_2 K_1(qr_2)[\lambda K_\nu'(\lambda r_2) - K_\nu(\lambda r_2)/(2r_2)].$$

Using formulas (18), (21) and (26) we obtain the change in impedance of the coil in the form

$$Z^{ind} = -2\omega\mu_0 r_1^2 r_0 Z,$$

where

$$Z = j\int_0^\infty \gamma_3 I_1^2(\lambda r_0)\,d\lambda. \qquad (29)$$

Formula (29) is used to compute the change in impedance of the coil for different values of the parameters of the problem. In order to minimize the number of the parameters of the problem we introduce the following dimensionless variables: $\eta_1 = r_1\sqrt{\omega\sigma_1\mu_1\mu_0}$ and $s = \sqrt{\sigma_2\mu_2/(\sigma_1\mu_1)}$ so that $p_1 = \eta_1\sqrt{j}$, $p_2 = \eta_2\sqrt{j}$ and $\eta_2 = s\eta_1$. Fig. 3 plots the real and imaginary parts of the change in impedance for three different values of r_2 :1.1,1.2 and 1.3 (from right to left). The curves correspond to the following values of $\eta_1 = 3,4,...,10$ (from top to bottom). The other parameters are $\mu_1 = 1, \mu_2 = 4, r_0 = 0.9, s = 1.5$.

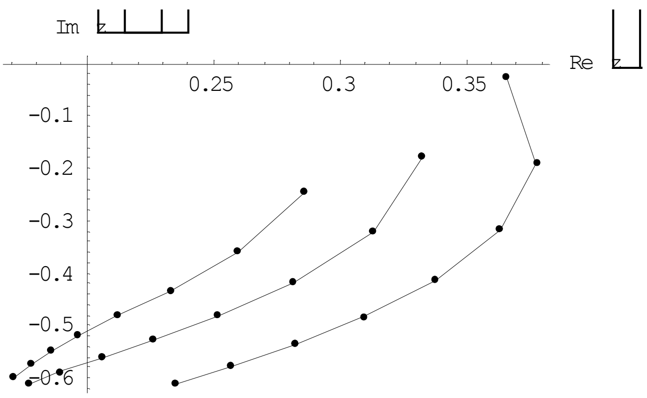

Fig. 3. The change in impedance computed by formula (29) for three different values of r_2.

It is seen from Fig. 3 that for higher frequencies (larger values of η_1) the modulus of the change in impedance decreases. The decrease is related to smaller values of the real part of the change in impedance.

The values of Z for three different values of μ_2, namely, $\mu_2 = 2;4;6$ (from left to right) are plotted in Fig. 4. The points on each curve correspond to the following values of $\eta_1 = 3,4,...,10$ (from top to bottom). The other parameters are set at $\mu_1 = 1, r_2 = 1.1, r_0 = 0.9, s = 1.5$.

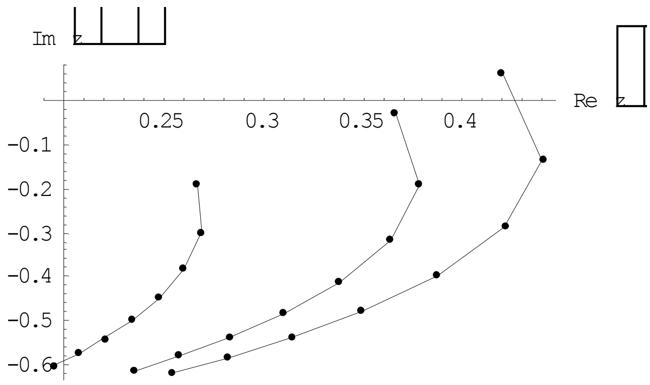

Fig. 4. The change in impedance computed by formula (29) for three different values of μ_2.

It follows from Fig. 4 that for high frequencies (large values of η_1) the calculated points are very close to one another as the parameter μ_2 increases.

V. A COIL OUTSIDE A MULTILAYER TUBE

There are many different types of eddy current probes that are used to control the properties of objects with cylindrical symmetry. Encircling coils are widely used for inspecting cylinders, rods or tubes. Mathematical models described in the previous sections can also be applied for the case of encircling coils. Consider a multilayer tube described in Section II. We assume that a single-turn coil of radius r_0 is located outside the tube ($r_0 > r_1$). The axis of the coil coincides with the axis of the tube.

The system of equations for the amplitudes of the vector potential in each region $R_i = \{r_i \le r \le r_{i+1}, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}, i = 1,2,...n$ is given by (2)-(4) where r_0 (the radius of the coil) is larger than r_1 (the radius of the inner cylinder of the tube). The solution procedure is essentially the same as in Section II with minor modifications. Applying the Fourier transform to (2)-(4) we obtain the system of equations in the form (6)-(8). The boundary conditions (9)-(11) are the same. For the case of an encircling coil we have to assume that $\widetilde{A}_0$ is bounded as $r \to \infty$ and $\widetilde{A}_{n+1}$ is bounded at $r = 0$. As a result, the solution in regions R_0 and R_{n+1} has to be modified. We consider two sub-regions of region R_0, namely, R_{00} and R_{01}, which are defined as follows: $R_{00} = \{r_1 \le r < r_0, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$, $R_{01} = \{r > r_0, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$. The solutions in R_{00} and R_{01} are denoted, as before, by $\widetilde{A}_{00}$ and $\widetilde{A}_{01}$, respectively. Solving (6) in regions R_{00} and R_{01} we obtain

$$\widetilde{A}_{00}(r,\lambda) = B_1 I_1(\lambda r) + B_2 K_1(\lambda r) \tag{30}$$

and

$$\widetilde{A}_{01}(r,\lambda) = B_3 K_1(\lambda r). \tag{31}$$

The bounded solution to (8) in region R_{n+1} is

$$\widetilde{A}_{n+1}(r,\lambda) = D_{n+1} I_1(,r). \tag{32}$$

where $q = \sqrt{\lambda^2 + p_{n+1}^2}$.

It is assumed here that the electric conductivity and magnetic permeability of the layer that contains the point $r = 0$ are constant.

The solution in other conducting layers can be constructed as in Section II. For example, if $\alpha_i = -1$ and $\beta_i = -1$, the solution is

$$\widetilde{A}_i(r,\lambda) = C_i \frac{I_\nu(\lambda r)}{\sqrt{r}} + D_i \frac{K_\nu(\lambda r)}{\sqrt{r}}, \tag{33}$$

where $\nu = \sqrt{p_i^2 + 1/4}$.

The constants $B_1, B_2, B_3, C_i, D_i, i = 1,2,..n$ and D_{n+1} can be determined from the boundary conditions (9)-(11) and (14). It can be shown that the induced vector potential in this case is given by

$$\widetilde{A}_0^{ind}(r,\lambda) = B_2 K_1(\lambda r). \tag{34}$$

Applying the inverse Fourier transform (17) to (34) we obtain the induced vector potential due to a multilayer tube in the following form

$$A_0^{ind}(r,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B_2 K_1(\lambda r) e^{i\lambda z} d\lambda. \tag{35}$$

As an example we consider the case where a single-turn coil of radius r_0 is located outside a two-layer coaxial conducting tube. The inner radius of the tube, r_1, is chosen as the measure of length. The conducting layer (region R_1) is defined by the inequalities: $R_1 = \{1 \le r \le r_2, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$. The electric conductivity and magnetic permeability of region R_1 are given by (1). The properties of region $R_2 = \{0 \le r \le r_2, 0 \le \varphi \le 2\pi, -\infty < z < +\infty\}$ (σ_2 and μ_2) are assumed to be constant.

It can be shown that the change in impedance of the coil due to a two-layer tube is given by

$$Z^{ind} = 2\omega r_1^2 r_0^2 \mu_0 Z, \tag{36}$$

where

$$Z = j\int_0^\infty \frac{\gamma_9}{\gamma_{10}} K_1^2(\lambda r_0) d\lambda \tag{37}$$

and

$$\gamma_9 = -\mu_* \lambda I_1'(\lambda)[K_\nu(\lambda) + \gamma_{11} I_\nu(\lambda)] + I_1(\lambda)[\lambda K_\nu'(\lambda) - 0.5 \cdot K_\nu(\lambda) + \gamma_3 \lambda I_\nu'(\lambda) - 0.5 \cdot \gamma_{11} I_\nu(\lambda)],$$

$$\gamma_{10} = -\mu_* \lambda K_1'(\lambda)[K_\nu(\lambda) + \gamma_{11} I_\nu(\lambda)] + K_1(\lambda)[\lambda K_\nu'(\lambda) - 0.5 \cdot K_\nu(\lambda) + \gamma_3 \lambda I_\nu'(\lambda) - 0.5 \cdot \gamma_{11} I_\nu(\lambda)].$$

Here

$$\gamma_{11} = -\frac{\gamma_{12}}{\gamma_{13}},$$

$$\gamma_{12} = 2r_2 \tilde{\mu} q I_1'(qr_2) K_\nu(\lambda r_2) - \mu_2 I_1(qr_2)[2r_2 \lambda K_\nu'(\lambda r_2) - K_\nu(\lambda r_2)],$$

$$\lambda_{13} = 2r_2 \tilde{\mu} q I_1'(qr_2) I_\nu(\lambda r_2) - \mu_2 I_1(qr_2)[2r_2 \lambda I_\nu'(\lambda r_2) - I_\nu(\lambda r_2)].$$

Results of numerical calculations using formula (37) are shown in Fig. 5. The following values of the parameters are used for calculations: $\mu_1 = 1, \mu_2 = 5, r_2 = 0.9, s = 1.3$. The points on each curve correspond to the values of $\eta_1 = 3,4,...,10$ (from top to bottom). The graphs in Fig. 5 are shown for the following three values of r_0: 1.1, 1.3 and 1.5 (from right to left). It is seen from Fig. 5 that for larger values of r_0 the induced change in impedance is weaker: the modulus of Z decreases as the distance from the coil to the tube increases.

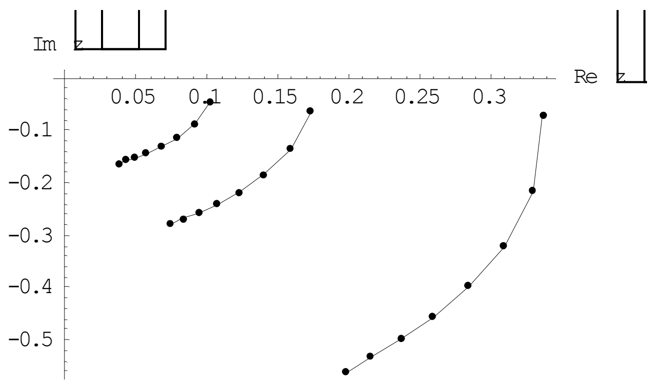

Fig. 5. The change in impedance computed by formula (37) for three different values of r_0.

VI. Other analytical solutions

Equation (7) is the second order ordinary differential equation with variable coefficients that depend on two parameters α_i and β_i. As it is shown in the previous sections the solution to (7) for the case $\alpha_i = -1,\ \beta_i = -1$ is expressed in terms of the modified Bessel functions (see (16)). It is possible to construct closed-form solutions to (7) for other combinations of the parameters α_i and β_i. If $\alpha_i = -2,\ \beta_i = 0$ the solution to (7) can be written in the form (see [12]):

$$\widetilde{A}_i(r,\lambda) = C_i \frac{I_{p_i}(\lambda r)}{r} + D_i \frac{K_{p_i}(\lambda r)}{r}.$$

A one-parameter family of analytical solutions is obtained for the case $\alpha_i \neq 1,\ \alpha_i + \beta_i = -2$. The solution to (7) in this case is (see [12])

$$\widetilde{A}_i(r,\lambda) = C_i r^{\alpha_i/2} I_\varsigma(\lambda r) + D_i r^{\alpha_i/2} K_\varsigma(\lambda r),$$

where $\varsigma = \sqrt{1 + \alpha_i + \dfrac{\alpha_i^2}{4} + p_i^2}$.

In addition, if $\alpha_i \neq 1,\ \alpha_i + \beta_i = -1$ equation (7) reduces to

$$\frac{d^2\widetilde{A}_i}{dr^2} + \frac{(1-\alpha_i)}{r}\frac{d\widetilde{A}_i}{dr} - \lambda^2 \widetilde{A}_i - \left(\frac{1+\alpha_i}{r^2} + \frac{p_i^2}{r}\right)\widetilde{A}_i = 0, \quad i = 1,2,\ldots,n,$$

which is a particular form of the confluent hypergeometric equation

$$\frac{d^2 y}{dx^2} + \left(a + \frac{b}{x}\right)\frac{dy}{dx} + \left(c + \frac{d}{x} + \frac{e}{x^2}\right)y = 0$$

with $a = 0, b = 1 - \alpha_i, c = -\lambda^2, d = -p_i^2, e = 1 + \alpha_i$ (see [15]). In this case the solution to (7) can be expressed in terms of Whittaker functions (see [15]).

Probably, other analytical solutions of equation (7) can be constructed for other combinations of the parameters α_i and β_i.

In summary, the idea of using relatively simple model one-parameter electric conductivity and magnetic permeability profiles allows one to obtain different analytical solutions that can be used in eddy current testing of objects of cylindrical shapes with varying electric conductivity and magnetic permeability.

VII. Conclusion

The change in impedance of a single-turn coil with alternating current located inside or outside a multilayer tube with arbitrary number of conducting layers is obtained in the present paper. The electric conductivity and/or magnetic permeability of each conducting layer are assumed to be power functions of the radial coordinate. The closed-form solution is expressed in terms of improper integral containing Bessel functions. It is shown that for some combinations of the parameters the solution in a conducting layer with variable properties can be expressed in terms of different special functions (Bessel functions and Whittaker functions). Theoretical model is developed for an arbitrary number of concentric conducting layers. Three examples are considered in detail. The first two examples correspond to the case where a coil is located inside a multilayer tube: (a) the case of an infinite outer conducting layer with varying properties and (b) the case of a two-layer tube where the electric conductivity and magnetic permeability depend on the radial coordinate. In addition, the case of a coil located outside a two-layer tube with varying properties considered as well.

Results of numerical calculations for all three examples are presented. Calculations are performed with “Mathematica”.

There are at least two cases where analytical solutions for eddy current testing problems can be helpful. First, analytical solutions suggested in the present paper can be used to solve inverse problems in cylindrical geometry where the electric conductivity and magnetic permeability of each conducting layer depend on the radial coordinate. Second, analytical solutions are often used to test numerical algorithms developed for more complicated cases (examples include equations with variable coefficients where the coefficients depend on more than one variable or nonlinear equations).

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