# A subclass of quasi self adjoint lubrication equations: conservations laws

M.L. Gandarias and M. S. Bruzón

***Abstract*—In [20] a general theorem on conservation laws for arbitrary differential equation has been proved. This new theorem is based on the concept of adjoint equations for non-linear equations. The notion of self-adjoint equations and quasi self adjoint has been also extended to non-linear equations. In this paper we consider a generalized fourth-order nonlinear partial differential equation which arises in modelling the dynamics of thin liquid films. We use the free software MAXIMA program symmgrp2009.max derived by W. Heremann to calculate the determining equations for the classical symmetries of the modified lubrication equation. We determine the subclasses of this equations which are self-adjoint and quasi-self adjoint and we find conservation laws for some of these partial differential equations without classical Lagrangians.**

***Index Terms*—Symmetries, partial differential equation, exact solutions,Self-adjointness,Conservation laws**

## I. Introduction

We consider the fourth order degenerate diffusion equation

$$u_t = -\nabla \cdot (f(u)\nabla \cdot (\Delta u)) \tag{1}$$

in one space dimension. This equation, derived from a 'lubrication approximation', models surface tension dominated motion of thin viscous films and spreading droplets. The equation with $f(u) = |u|$ also models a thin neck of fluid in the Hele-Shaw cell. The thin-film dynamics if the liquid is uniform in one direction can be modeled by the one-dimensional equation

$$u_t = -(f(u)u_{xxx})_x \tag{2}$$

$u$ stands by the thickness of the film, the fourth order term reflects surface tension effects

$$u_t = -(f(u)u_{xxx})_x. \tag{3}$$

In previous papers [2],[12] we have classified the classical symmetries admitted by the generalized equation (3) and a modified version given by

$$u_t = -f(u)u_{xxxx}. \tag{4}$$

By using symmetry reductions we found that for some particular functional forms of $f$ the one-dimensional lubrication model admits some solutions of physical interest as similarity solutions, travelling-wave solutions, source and sink solutions, waiting time solutions and blow-up solutions. We were also able to characterize those solutions as solutions for some lower-order ordinary differential equations (ODEs) and moreover we obtained some particular solutions. In a previous paper [8] we have derived the subclasses of equations which are self-adjoint. For these classes of self-adjoint equations we apply Lie classical method and determine the functions for which equations (4) have additional symmetries. We also determine, by using the notation and techniques of [20], some nontrivial conservation laws for (4).

Many equations having remarkable symmetry properties and physical significance are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from conservation laws of these equations by setting $v = u$, [21] generalized the concept of self-adjoint equations by introducing the definition of quasi-self-adjoint equations.

The aim of this paper is to determine, for the generalized modified equation (4) the subclasses of equations which are quasi-self-adjoint. For these classes of quasi-self-adjoint equations we apply Lie classical method and determine the functions for which Eqs. (4) have additional symmetries.

We show how the free software MAXIMA program symmgrp2009.max, derived by W. Heremann, can be used to calculate the determining equations for the classical symmetries of the generalized modified equation (4). We also determine, by using the notation and techniques of [20] and [21] some nontrivial conservation laws for Eqs. (4).

### A. Classical symmetries

In a previous work, we have studied equation (4) from the point of view of the theory of symmetry reductions in partial differential equations. We have obtained the classical symmetries admitted by (4) for arbitrary $f$ and the functional forms of $f$ for which equation (4) admits extra classical symmetries. We have used the transformations groups to reduce the equations to ODEs.

To apply the classical method to equation (4), one looks for infinitesimal generators of the form

$$\mathbf{v} = \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_t + \psi(x,t,u)\partial_u,$$

that leave invariant these equations.

### B. Symbolic manipulation programs

In this section we first show how the free software MAXIMA program symmgrp2009.max derived by W. Heremann can be used to calculate the determining equations for the classical symmetries of the modified lubrication equation (4). To use symmgrp2009.max, we have to convert (4) into the appropriate MACSYMA and MAXIMA syntax: $x[1]$ and $x[2]$ represent the independent variables $x$ and $t$, respectively, $u[1]$

M.L. Gandarias M. S. Bruzón are with Department of Mathematics, University of Cádiz, Spain. E-mails: m.bruzon@uca.es, marialuz.gandarias@uca.es
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represents the dependent variable $u$, $u[1,[1,0]]$ represents $u_x$, $u[1,[2,0]]$ represents $u_{xx}$, $u[1,[4,0]]$ represents $u_{xxxx}$, and $u[1,[0,2]]$ represents $u_t$. Hence (4) is rewritten as

$$u[1,[0,1]] + f * u[1,[4,0]]$$

with $f = f(u)$. The infinitesimals $\xi$, $\tau$ and $\phi$ are represented by $eta1$, $eta2$ and $phi1$, respectively. The program symmgrp2009. max automatically computes the determining equations for the infinitesimals. The batchfile batch containing the MAXIMA commands to implement the program symmgrp2009.max, which we have called lubrimo.mac is

```
kill(all);
batchload("c:\\cla
\\symmgrp2009.max");
/* u_t = f(u)u_xxx*/
batch("c:\\camb\\lubrimo.dat");
symmetry(1,0,0);
printeqn(lode);
for j thru q do
(x[j]:=concat(x,j));
for j thru q do
(u[j]:=concat(u,j));
ev(lode)$
gnlhode:ev(%,x1=x,x2=t,u1=u);
grind:true$
stringout("gnlhode",gnlhode);
derivabbrev:true;
```

The first lines of this file are standard to symmgrp.max and explained in [9]. The last lines are in order to create an output suitable for solving the determining equations. This changes $x[1]$, $x[2]$ and $u[1]$ to $x$, $t$ and $u$, respectively. The file lubri.mac in turn batches the file lubrimo.dat which contains the requisite data about (4).

```
p:2$
q:1$
m:1$
parameters:[a,b]$
warnings:true$
sublisteqs:[all]$
subst_deriv_of_vi:true$
info_given:true$
highest_derivatives:all$
depends([eta1,eta2,phi1],
[x[1],x[2],u[1]]);
depends([f],[u[1]]);
e1:u[1,[0,1]]+f*u[1,[4,0]];
v1:u[1,[0,1]];
```

The program symmgrp2009.max generates the system of twenty eight determining equations. From this system we get

$$\begin{aligned}&\xi = \xi(x,t),\\ &\tau = \tau(t),\\ &\phi = \alpha(x,t)u + \beta(x,t)\end{aligned}$$

and the following five determining equations

$$\begin{aligned}(\alpha_{xxxx}\, f + \alpha_t)\, u + \beta_{xxxx}\, f + \beta_t &= 0\\ -4\, f\, \xi_x + \alpha\, f_u\, u + f\, \tau_t + \beta\, f_u &= 0\\ -f\, (3\, \xi_{xx} - 2\, \alpha_x) &= 0\\ -f\, (2\, \xi_{xxx} - 3\, \alpha_{xx}) &= 0\\ -f\, \xi_{xxxx} - \xi_t + 4\, \alpha_{xxx}\, f &= 0\end{aligned} \quad (5)$$

Solving these equations we find that if $f$ is an arbitrary function, *the only symmetries that are admitted* by (3) are

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t.$$

$$\mathbf{v}_3 = x\partial_x + 4t\partial_t$$

The functional forms of $f$ which have extra symmetries and the corresponding generators are:

$$f(u) = c(u+b)^a, \quad f(u) = \gamma e^{\alpha u} \quad (6)$$

We can take in (6) $c = 1$, $b = 0$, $\alpha = -1$, $\gamma = 1$

**Case 1:** $f(u) = u^a$, $a \neq \frac{8}{3}$

$$\begin{aligned}&\mathbf{v}_1 = \partial_x,\\ &\mathbf{v}_2 = \partial_t,\\ &\mathbf{v}_3 = x\partial_x + 4t\partial_t\\ &\mathbf{v}_4 = -at\partial_t + u\partial_u,\\ &\mathbf{v}_5 = x\partial_x + (4 - \frac{3}{2}a)t\partial_t + \frac{3}{2}u\partial_u\end{aligned}$$

**Case 2:** $f(u) = u^{8/3}$,

$$\begin{aligned}&\mathbf{v}_1 = \partial_x,\\ &\mathbf{v}_2 = \partial_t,\\ &\mathbf{v}_3 = x\partial_x + 4t\partial_t\\ &\mathbf{v}_4 = -at\partial_t + u\partial_u,\\ &\mathbf{v}_5 = x^2\partial_x + 3xu\partial_u\end{aligned}$$

**Case 3:** $f(u) = e^{-u}$

$$\begin{aligned}&\mathbf{v}_1 = \partial_x,\\ &\mathbf{v}_2 = \partial_t,\\ &\mathbf{v}_3 = x\partial_x + 4t\partial_t\\ &\mathbf{v}_4 = x\partial_x - 4\partial_u\\ &\mathbf{v}_5 = t\partial_t + \partial_u.\end{aligned}$$

## II. Optimal systems and reductions

In order to construct the one-dimensional optimal system, following Olver, we construct the commutator table and the adjoint table which shows the separate adjoint actions of each element in $\mathbf{v}_i$, $i = 1\ldots 5$, as it acts on all other elements. This construction is done easily by summing the Lie series. An example of these tables, corresponding to $f(u) = u^a$, appear in the Appendix.

In [13], reductions of the equation (4) to ODEs were obtained using the generators of the optimal system.

### A. Reductions for f arbitrary

**1** Reduction with the generator $\mu\mathbf{v_1} + \mathbf{v_2}$

$$z = x - \mu t, \quad u = \omega, \quad (7)$$

and the ODE

$$f(w)\omega'''' + \mu\omega' = 0 \quad (8)$$

**2** Reduction with the generator $\mathbf{v_1}$

$$z = t, \quad u = \omega, \tag{9}$$

and the ODE

$$\omega' = 0 \tag{10}$$

**3** Reduction with the generator $\mathbf{v_3}$

$$z = \frac{x^{1/4}}{t}, \quad u = \omega, \tag{11}$$

and the ODE

$$4f(w)\omega'''' + z\omega' = 0 \tag{12}$$

*B. Reductions for* $f = u^a$ $\alpha = -\frac{1}{\lambda\alpha}$, $\beta = -\frac{4+\lambda\alpha}{\lambda\alpha^2}$

**1** Reduction with $(\lambda + \frac{4}{a} - \frac{3}{2})\mathbf{v}_3 + \mathbf{v}_4$

$$z = t^{-\alpha}x, \quad u = t^{\beta}\omega, \tag{13}$$

and the ODE

$$w^a\omega'''' + \alpha z\omega' - \beta\omega = 0 \tag{14}$$

**2** Reduction with $\lambda\mathbf{v}_2 + (\frac{4}{a} - \frac{3}{2})\mathbf{v}_3 + \mathbf{v}_4$

$$z = xe^{-\frac{t}{\lambda}}, \quad u = e^{\frac{4t}{a\lambda}}\omega, \tag{15}$$

and the ODE

$$a\lambda w^{\alpha}\omega'''' + az\omega' - 4\omega = 0 \tag{16}$$

**3** Reduction with $(\frac{4}{a} - \frac{3}{2})\mathbf{v}_3 + \mathbf{v}_4$

$$z = t, \quad u = x^{\frac{4}{a\lambda}}\omega, \tag{17}$$

and the ODE

$$\omega' - \frac{4}{a}(\frac{4}{a} - 1)(\frac{4}{a} - 2)(\frac{4}{a} - 3)\omega^{a+1} = 0 \tag{18}$$

**4** Reduction with $\mu\mathbf{v}_1 + \mathbf{v}_3$

$$z = x + \tfrac{\mu}{a}ln|t|, \quad u = t^{-\frac{1}{a\lambda}}\omega, \tag{19}$$

and the ODE

$$w^{\alpha}\omega'''' - \mu\omega' + \omega = 0 \tag{20}$$

**5** Reduction with the generator $\mu\mathbf{v_1} + \mathbf{v_2}$

$$z = x - \mu t, \quad u = \omega, \tag{21}$$

and the ODE

$$\omega^a\omega'''' + \mu\omega' = 0 \tag{22}$$

**6** Reduction with the generator $\mathbf{v_1}$

$$z = t, \quad u = \omega, \tag{23}$$

and the ODE

$$\omega' = 0 \tag{24}$$

*C. Reductions for* $f = u^{\frac{8}{3}}$

Besides the previous reductions we get

**7** Reduction with $\lambda_1^2\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \mathbf{v}_5$

$$\begin{aligned} z &= \frac{1}{\lambda_1}\text{atan}\,\frac{x}{\lambda_1} - \frac{t}{\lambda_2}, \\ u &= (x^2 + \lambda_1^2)^{\frac{3}{2}}\omega, \end{aligned} \tag{25}$$

and the ODE

$$\lambda_2\omega^{\frac{8}{3}}\omega'''' + 10\lambda_1^2\lambda_2\omega^{\frac{8}{3}}\omega'' + \omega' + 9\lambda_1^4\lambda_2\omega^{\frac{11}{3}} = 0 \tag{26}$$

**8** Reduction with $\lambda_1^2\mathbf{v}_1 + \lambda_2\mathbf{v}_3 + \mathbf{v}_5$

$$\begin{aligned} z &= \frac{1}{\lambda_1}\text{atan}\frac{x}{\lambda_1} + \frac{3}{8\lambda_2}\ln|t|, \\ u &= (x^2 + \lambda_1^2)^{\frac{3}{2}}e^{\frac{\lambda_2}{\lambda_1}}\text{atan}\frac{x}{\lambda_1}\omega, \end{aligned} \tag{27}$$

and the ODE

$$\begin{aligned} &8k\omega^{\frac{8}{3}}\left(\omega'''' + 4k\omega''' + 2(5\lambda_1^2 + 3k^2)\omega'' +\right. \\ &4k(5\lambda_1^2 + k^2)\omega'\left.\right) - 3e^{-\frac{8k}{3}}z\omega' + \\ &8k(k^4 + 10\lambda_1^2k^2 + 9\lambda_1^4)\,\omega^{\frac{11}{3}} = 0 \end{aligned} \tag{28}$$

**9** Reduction with $\lambda\mathbf{v}_2 + \mathbf{v}_5$

$$\begin{aligned} z &= \frac{1}{x} + \frac{t}{\lambda}, \\ u &= x^3\omega, \end{aligned} \tag{29}$$

and the ODE

$$\lambda\omega^{\frac{8}{3}}\omega'''' - \omega' = 0 \tag{30}$$

**10** Reduction with $\lambda\mathbf{v}_3 + \mathbf{v}_5$

$$\begin{aligned} z &= \frac{1}{x} + \frac{3}{8\lambda}\ln|t|, \\ u &= x^3e^{-\frac{\lambda}{x}}\omega, \end{aligned} \tag{31}$$

and the ODE

$$\begin{aligned} &8k\omega^{\frac{8}{3}}(-\omega'''' + 4k\omega''' - 6k^2\omega'' + 4k^3\omega') - \\ &3e^{-\frac{8k}{3}z}\omega' - 8k^5\omega^{\frac{11}{3}} = 0 \end{aligned} \tag{32}$$

**11** Reduction with $\lambda^2\mathbf{v}_1 + \mathbf{v}_5$

$$z = t, \quad (x^2 + \lambda^2)^{\frac{3}{2}}\omega, \tag{33}$$

and the ODE

$$\omega' - 9\lambda^4\omega^{\frac{11}{3}} = 0 \tag{34}$$

### D. Reductions for $f = e^{-u}$

Besides the previous reductions for $f$ arbitrary we get

**12** Reduction with $\mathbf{v}_3 + \lambda \mathbf{v}_4$

$$z = xt^{-\frac{1}{\lambda_1}}, \qquad u = \frac{\lambda - 4}{\lambda}\ln|t| + \omega, \tag{35}$$

and the ODE

$$\lambda e^{-\omega}\omega'''' + z\omega' - \lambda + 4 = 0 \tag{36}$$

**13** Reduction with $\lambda \mathbf{v}_2 + \mathbf{v}_3$

$$z = e^{-\frac{t}{\lambda}}x, \qquad u = \omega - \frac{4t}{\lambda}, \tag{37}$$

and the ODE

$$\lambda e^{-\omega}\omega'''' + z\omega' + 4 = 0 \tag{38}$$

**14** Reduction with $\mathbf{v}_3$

$$z = t, \qquad u = -4\ln|x| + \omega, \tag{39}$$

and the ODE

$$\omega' - 24e^{-\omega} = 0 \tag{40}$$

**15** Reduction with $\mu \mathbf{v}_1 + \mathbf{v}_4$

$$z = x - \mu\ln|t|, \qquad u = \omega + \ln|t|, \tag{41}$$

and the ODE

$$e^{-\omega}\omega'''' + \mu\omega' - 1 = 0 \tag{42}$$

**16** Reduction with $\mu \mathbf{v}_1 + \mathbf{v}_4$

$$z = x - \mu t, \qquad u = \omega, \tag{43}$$

and the ODE

$$e^{-\omega}\omega'''' + \mu\omega' = 0 \tag{44}$$

**17** Reduction with $\mathbf{v}_1$

$$z = t, \qquad u = \omega, \tag{45}$$

and the ODE

$$\omega' = 0 \tag{46}$$

In [13] we have discussed some interpretation of the similarity variables in the reductions of the lubrication equation as well as in the reductions of the modified lubrication equation and we have provided some particular solutions.

- For $f(u) = u^a$, Eq. (18) is a first order equation that can be easily solved, in this way we have obtained a family of waiting-time solutions (if $a \neq 2$ or $4$) given by

$$u(x,t) = \begin{cases} x^{\frac{4}{a}}\left[A(t_0 - t)\right]^{-\frac{1}{a}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$A = 4(\frac{4}{a}+1)(\frac{4}{a}-1)(\frac{4}{a}-2)(\frac{4}{a}-3)$$

- For $f(u) = e^{-u}$ Eq. (40) is a first order equation, solving it we get that the corresponding similarity solution

$$u(x,t) = -\ln\frac{(x-x_0)^4}{24(t+t_0)}$$

describes a localized blow-up at $x = x_0$.

- For $f(u) = u^{\frac{8}{3}}$, using reduction (33) we get a new solution with blow-up at $t = t_0$, given by

$$u(x,t) = \left[\frac{3(x^2+\lambda^2)^4}{8\lambda^4(t_0-t)}\right].$$

## III. Adjoint and self-adjoint nonlinear equations

The following definitions of adjoint equations and self-adjoint equations are applicable to any system of linear and non-linear differential equations, where the number of equations is equal to the number of dependent variables (see [20]), and contain the usual definitions for linear equations as a particular case. Since we will deal in our paper with scalar equations, we will formulate these definitions in the case of one dependent variable only.

Consider an sth-order partial differential equation

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0 \tag{47}$$

with independent variables $x = (x^1, \ldots, x^n)$ and a dependent variable $u$, where $u_{(1)} = \{u_i\}$, $u_{(2)} = \{u_{ij}\}, \ldots$ denote the sets of the partial derivatives of the first, second, etc. orders, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. The adjoint equation to (47) is

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{48}$$

with

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta(v\,F)}{\delta u}, \tag{49}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}} \tag{50}$$

denotes the variational derivatives (the Euler-Lagrange operator), and $v$ is a new dependent variable. Here

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots$$

are the total differentiations.

Eq. (47) is said to be *self-adjoint* if the equation obtained from the adjoint equation (48) by the substitution $v = u$ :

$$F^*(x, u, u, u_{(1)}, u_{(1)}, \ldots, u_{(s)}, u_{(s)}) = 0,$$

is identical with the original equation (47). In other words, if

$$\begin{array}{c} F^*(x,u,u_{(1)},u_{(1)},\ldots,u_{(s)},u_{(s)}) = \\ \phi(x,u,u_{(1)},\ldots)\,F(x,u,u_{(1)},\ldots,u_{(s)}). \end{array} \tag{51}$$

### A. General theorem on conservation laws

We use the following theorem on conservation laws proved in [21].

**Theorem** Any Lie point, Lie-Bäcklund or non-local symmetry

$$X = \xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i} + \eta(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u} \tag{52}$$

of Eqs.(47) provides a conservation law $D_i(C^i) = 0$ for the simultaneous system (47), (48). The conserved vector is given by

$$\begin{aligned} C^i = \;& \xi^i\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) - \cdots\right] \\ & + D_j(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) + \cdots\right] \\ & + D_jD_k(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}} - \cdots\right] + \cdots, \end{aligned} \tag{53}$$

where $W$ and $\mathcal{L}$ are defined as follows:

$$W = \eta - \xi^j u_j, \quad \mathcal{L} = v\,F\left(x,u,u_{(1)},\ldots,u_{(s)}\right). \tag{54}$$

### B. The class of self-adjoint equations

Let us single out quasi-self-adjoint equations from the equations of the form (4),

$$u_t = f(u)u_{xxxx}$$

The result was given in [8] by the following statement.

**Theorem**

Eq. (4) is self-adjoint if and only

$$f(u) = \frac{a}{u}.$$

**Proof.** Eq. (49) yields

$$\begin{aligned} F^* &= \frac{\delta}{\delta u}[v(u_t - fu_{xxxx})] \\ &= -D_tv - D_x^4(fv) - f'vu_{xxxx}, \end{aligned} \tag{55}$$

where

$$\begin{aligned} D_x^4\,(fv) = \;& fv_{xxxx} + 4f'u_xv_{xxx} \\ & +6f'u_{xx}v_{xx} + 6f''(u_x)^2v_{xx} \\ & +4f'u_{xxx}v_x + 12f''u_xu_{xx}v_x \\ & +4f'''(u_x)^3v_x + f'u_{xxxx}v \\ & +4f''u_xu_{xxx}v + 3f''(u_{xx})^2v \\ & +6f'''(u_x)^2u_{xx}v + f''''(u_x)^4v. \end{aligned} \tag{56}$$

By substituting (56) into (55) it follows that the class of adjoint equations to class of equations (4) is

$$\begin{aligned} & -v_t - fv_{xxxx} - 4f'u_xv_{xxx} - 6f'u_{xx}v_{xx} \\ & -6f''(u_x)^2v_{xx} - 4f'u_{xxx}v_x \\ & -12f''u_xu_{xx}v_x - 4f'''(u_x)^3v_x - f'u_{xxxx}v \\ & -4f''u_xu_{xxx}v - 3f''(u_{xx})^2v \\ & -6f'''(u_x)^2u_{xx}v - f''''(u_x)^4v - f'vu_{xxxx} = 0. \end{aligned} \tag{57}$$

After setting $v = u$ in (57) we obtain that $F^* = -(u_t - fu_{xxxx})$ if and only if $f(u)$ satisfies

$$f + uf' = 0,$$

whose solution is

$$f = \frac{a}{u}.$$

Many equations having remarkable symmetry properties and physical significance are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from conservation laws of these equations by setting $v = u$. In [21] the concept of self-adjoint equation has been generalized by introducing the definition of quasi-self-adjoint equations.

Equation (47) is said to be quasi-self-adjoint if the the adjoint equation (48) is equivalent to the original equation (47) upon the substitution $v = h(u)$ with a certain function $h(u)$ such that $h'(u) \neq 0$. We consider again (4) and we substitute

$$\begin{aligned} & v = h(u) \\ & v_t = h'u_t \\ & v_x = h'u_x \\ & v_{xx} = h'u_{xx} + h''u_x^2 \\ & v_{xxx} = h'u_{xxx} + 3\,h''\,u_x\,u_{xx} + h'''\,(u_x)^3 \\ & v_{xxxx} = h'u_{xxxx} + 4h''u_xu_{xxx} + 3h_{uu}\,(u_{xx})^2 \\ & \qquad +6\,h_{uuu}\,(u_x)^2\,u_{xx} + h''''\,(u_x)^4 \end{aligned}$$

in the adjoint equation (57) and we get

$$\begin{aligned} & -f\,h'\,u_{xxxx} - 2\,f'\,h\,u_{xxxx} - 4\,f\,h''\,u_x\,u_{xxx} \\ & -8\,f''\,h'\,u_x\,u_{xxx} - 4\,f''\,h\,u_x\,u_{xxx} \\ & -3\,f\,h''\,(u_{xx})^2 - 6\,f'\,h'\,(u_{xx})^2 \\ & -3\,f''\,h\,(u_{xx})^2 - 6\,f\,h'''\,(u_x)^2\,u_{xx} \\ & -18\,f'\,h''\,(u_x)^2\,u_{xx} - 18\,f''\,h'\,(u_x)^2\,u_{xx} \\ & -6\,f'''\,h\,(u_x)^2\,u_{xx} - f\,h''''\,(u_x)^4 \\ & -4\,f'\,h'''\,(u_x)^4 - 6\,f''\,h''\,(u_x)^4 \\ & -4\,f'''\,h'\,(u_x)^4 - f''''\,h\,(u_x)^4 - h'\,u_t = 0. \end{aligned}$$

Hence the condition of quasi-self-adjointness is written as follows

$$\begin{aligned} & -f\,h'\,u_{xxxx} - 2\,f'\,h\,u_{xxxx} - 4\,f\,h''\,u_x\,u_{xxx} \\ & -8\,f''\,h'\,u_x\,u_{xxx} - 4\,f''\,h\,u_x\,u_{xxx} \\ & -3\,f\,h''\,(u_{xx})^2 - 6\,f'\,h'\,(u_{xx})^2 \\ & -3\,f''\,h\,(u_{xx})^2 - 6\,f\,h'''\,(u_x)^2\,u_{xx} \\ & -18\,f'\,h''\,(u_x)^2\,u_{xx} - 18\,f''\,h'\,(u_x)^2\,u_{xx} \\ & -6\,f'''\,h\,(u_x)^2\,u_{xx} - f\,h''''\,(u_x)^4 \\ & -4\,f'\,h'''\,(u_x)^4 - 6\,f''\,h''\,(u_x)^4 \\ & -4\,f'''\,h'\,(u_x)^4 - f''''\,h\,(u_x)^4 - h'\,u_t = 0 \\ & -\lambda[u_t - f(u)u_{xxxx}] = 0 \end{aligned}$$

where $\lambda$ is an undetermined coefficient.

Hence the following conditions must be satisfied

$$\lambda + h' = 0$$
$$f\,\lambda - f\,h' - 2\,f'\,h = 0$$
$$f\,h'' + 2\,f'\,h' + f''\,h = 0$$
$$f\,h''' + 3\,f'\,h'' + 3\,f''\,h' + f'''\,h = 0$$
$$3\,f\,h'' + 6\,f'\,h' + 3\,f''\,h = 0$$
$$f\,h'''' + 4\,f'\,h''' + 6\,f''\,h'' + 4\,f'''\,h' + f''''\,h = 0$$

Hence $\lambda = -h'$ and $h = \frac{k}{f(u)}$, namely the adjoint equation becomes equivalent to the original equation upon the substitution $v = \frac{k}{f(u)}$.

### *C. Conservation laws for a subclass of quasi-self-adjoint lubrication equations*

**1** Let us apply the Theorem in conservation-laws to the quasi-self-adjoint equation (4) with $f(u)$ arbitrary: in this case we have

$$\mathcal{L} = \Big(u_t - f(u)u_{xxxx})\Big)v. \tag{58}$$

We will write generators of point transformation group admitted by Eq. (4) in the form

$$X = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}$$

by setting $t = x^1,\ x = x^2$. The conservation law will be written

$$D_t(C^1) + D_x(C^2) = 0. \tag{59}$$

Since we will deal with fourth-order equations, we will use Eqs. (53) in the following form:

$$\begin{aligned} C^i = \ & \xi^i\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right)\right. \\ & \left. + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) - D_jD_kD_l\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right)\right] \\ & + D_j(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right)\right. \\ & \left. + D_kD_l\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right)\right] \\ & + D_jD_k(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}} - D_l\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right)\right] \\ & + D_jD_kD_l(W)\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right). \end{aligned} \tag{60}$$

Let us find the conservation law provided by the following obvious scaling symmetry of Eq. (4):

$$X = 4t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{61}$$

In this case we have $W = -4tu_t - xu_x$ and Eqs. (53) yield the conservation law (59) with

$$C^1 = -\frac{k\,u_x\,x}{f} - D_x\left(4\,k\,u_{xxxx}\right),$$

$$C^2 = \frac{k\,u_t\,x}{f} - k\,u_{xxx} - D_t\left(4\,k\,u_{xxxx}\right).$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = -\frac{k\,u_x\,x}{f},$$

$$C^2 = \frac{k\,u_t\,x}{f} - k\,u_{xxx}.$$

**2** Let us find the conservation law for $f(u) = u^a$ provided by the following obvious symmetry of Eq. (4):

$$X = -at\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \tag{62}$$

In this case we have $W = atu_t + u$ and Eqs. (53) yield the conservation law (59) with

$$C^1 = -ku^{1-a} + D_x(aktu_{xxx}),$$

$$C^2 = k(1-a)u_{xxx} - D_t(aktu_{xxx}).$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = -ku^{1-a},$$

$$C^2 = k(1-a)u_{xxx}.$$

**3** Let us find the conservation law for $f(u) = u^a$ provided by the following symmetry of Eq. (4):

$$X = x\partial_x + (4 - \frac{3}{2}a)t\partial_t + \frac{3}{2}u\partial_u. \tag{63}$$

In this case we have

$$W = \frac{3}{2}u - xu_x - (4 - \frac{3}{2}a)tu_t$$

and Eqs. (53) yield the conservation law (59) with

$$C^1 = -\left(\frac{kxu_x}{u^a} - \frac{3ku^{1-a}}{2}\right) + D_x\left(kt(4 - \frac{3a}{2})u_{xxx}\right),$$

$$C^2 = \frac{kxu_t}{u^a} + \frac{k}{2}(5-3a)u_{xxx} - D_t\left(kt(4 - \frac{3a}{2})u_{xxx}\right).$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = -\left(\frac{kxu_x}{u^a} - \frac{3ku^{1-a}}{2}\right),$$

$$C^2 = \frac{kxu_t}{u^a} + \frac{k}{2}(5-3a)u_{xxx}.$$

**4** Let us find the conservation law for $f(u) = u^{\frac{8}{3}}$ provided by the following symmetry of Eq. (4):

$$X = x\partial_x + (4 - \frac{3}{2}a)t\partial_t + \frac{3}{2}u\partial_u. \tag{64}$$

In this case we have

$$W = 3xu - x^2 u_x$$

and Eqs. (53) yield the conservation law (59) with

$$\begin{aligned} C^1 &= \frac{3\,k\,u_x\,x^2}{2\,u^{\frac{8}{3}}} + D_x\left(\frac{3\,k\,x^2}{2\,u^{\frac{5}{3}}}\right), \\ C^2 &= -\frac{3\,k\,u_t\,x^2}{2\,u^{\frac{8}{3}}} - 3\,k\,u_{x\,x\,x}\,x + 3\,k\,u_{x\,x} \\ &\quad -D_t\left(\frac{3\,k\,x^2}{2\,u^{\frac{5}{3}}}\right). \end{aligned}$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$\begin{aligned} C^1 &= \frac{3\,k\,u_x\,x^2}{2\,u^{\frac{8}{3}}}, \\ C^2 &= -\frac{3\,k\,u_t\,x^2}{2\,u^{\frac{8}{3}}} - 3\,k\,u_{x\,x\,x}\,x + 3\,k\,u_{x\,x}. \end{aligned}$$

**5** Let us find the conservation law for $f(u) = e^{-u}$ provided by the following symmetry of Eq. (4):

$$X = x\partial_x - 4\partial_u. \tag{65}$$

In this case we have

$$W = -4 - xu_x$$

and Eqs. (53) yield the conservation law (59) with

$$\begin{aligned} C^1 &= 3\,k\,e^u\,u_x\,x + D_x\left(-4\,k\,e^u\,x\right), \\ C^2 &= -3\,k\,e^u\,u_t\,x - 3\,k\,u_{x\,x\,x} \\ &\quad -D_t\left(-4\,k\,e^u\,x\right). \end{aligned}$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$\begin{aligned} C^1 &= 3\,k\,e^u\,u_x\,x, \\ C^2 &= -3\,k\,e^u\,u_t\,x - 3\,k\,u_{x\,x\,x}. \end{aligned}$$

## IV. Conclusions

In this work we have considered the class of modified nonlinear diffusion equations. By using free software Maxima, we have derived the Lie classical symmetries. If $f(u) = u^a$ or $f(u) = e^{-u}$ the equation admits additional classical symmetries. We have determined the subclasses of this equations which are self-adjoint and quasi-self adjoint. By using a general theorem on conservation laws proved by Nail Ibragimov we found conservation laws for some of these partial differential equations without classical Lagrangians

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# APPENDIX A
## APPENDIX

$(eta1[u1])*f, (eta1[u1u1])*f, (eta1[u1u1u1])*f,$
$(eta1[u1u1u1u1])*f, (eta2[u1])*f^2, (eta2[x1])*f^2,$
$(eta2[x1x1])*f^2, (eta2[x1x1x1])*f^2,$
$f*(2*(eta2[u1])*(f[u1])+(eta2[u1u1])*f),$
$f*((eta2[x1])*(f[u1])+(eta2[u1x1])*f),$
$f*((eta2[x1x1])*(f[u1])+(eta2[u1x1x1])*f),$
$f*((eta2[x1x1x1])*(f[u1])+(eta2[u1x1x1x1])*f-$
$eta1[u1]), f*(3*(eta2[u1])*(f[u1u1])+$
$3*(eta2[u1u1])*(f[u1])+$
$(eta2[u1u1u1])*f),$
$f*((eta2[x1])*(f[u1u1])+2*(eta2[u1x1])*(f[u1])+$
$(eta2[u1u1x1])*f), f*((eta2[x1x1])*(f[u1u1])+$
$2*(eta2[u1x1x1])*(f[u1])+(eta2[u1u1x1x1])*f),$
$f*(4*(eta2[u1])*(f[u1u1u1])+6*(eta2[u1u1])*(f[u1u1])+$
$4*(eta2[u1u1u1])*(f[u1])+(eta2[u1u1u1u1])*f),$
$f*((eta2[x1])*(f[u1u1u1])+3*(eta2[u1x1])*(f[u1u1])+$
$3*(eta2[u1u1x1])*(f[u1])+(eta2[u1u1u1x1])*f),$
$(f[u1])*phi1+(eta2[x1x1x1x1])*f^2+(eta2[x2])*f-$
$4*(eta1[x1])*f, f*(2*(phi1[u1x1])-3*(eta1[x1x1])),$
$f*(3*(phi1[u1x1x1])-2*(eta1[x1x1x1])),$
$4*f*(phi1[u1x1x1x1])-(eta1[x1x1x1x1])*f-eta1[x2],$
$f*(phi1[u1u1]-4*(eta1[u1x1])),$
$f*(2*(phi1[u1u1x1])-$
$3*(eta1[u1x1x1])), f*(3*(phi1[u1u1x1x1])-$
$2*(eta1[u1x1x1x1])), f*(phi1[u1u1u1]-4*(eta1[u1u1x1])),$
$f*(2*(phi1[u1u1u1x1])-3*(eta1[u1u1x1x1])),$
$f*(phi1[u1u1u1u1]-4*(eta1[u1u1u1x1])),$
$phi1[x2]+f*(phi1[x1x1x1x1])$

TABLE I
COMMUTATOR TABLE FOR THE LIE ALGEBRA $\mathbf{v_i}$.

| | $\mathbf{v_1}$ | $\mathbf{v_2}$ | $\mathbf{v_3}$ | $\mathbf{v_4}$ |
|---|---|---|---|---|
| $\mathbf{v_1}$ | 0 | 0 | 0 | $\mathbf{v_1}$ |
| $\mathbf{v_2}$ | 0 | 0 | $-a\mathbf{v_2}$ | 0 |
| $\mathbf{v_3}$ | 0 | $a\mathbf{v_2}$ | 0 | 0 |
| $\mathbf{v_4}$ | $-\mathbf{v_1}$ | $(\frac{3a}{2}-4)\mathbf{v_2}$ | 0 | 0 |

TABLE II
ADJOINT TABLE FOR THE LIE ALGEBRA $\mathbf{v_i}$.

| | $\mathbf{v_1}$ | $\mathbf{v_2}$ | $\mathbf{v_3}$ | $\mathbf{v_4}$ |
|---|---|---|---|---|
| $\mathbf{v_1}$ | $\mathbf{v_1}$ | $\mathbf{v_2}$ | $\mathbf{v_3}$ | $\mathbf{v_4}-\epsilon\mathbf{v_1}$ |
| $\mathbf{v_2}$ | $\mathbf{v_1}$ | $\mathbf{v_2}$ | $\mathbf{v_3}+\mathbf{a}\epsilon\mathbf{v_2}$ | $\mathbf{v_4}$ |
| $\mathbf{v_3}$ | 0 | $e^{a\epsilon}\mathbf{v_2}$ | 0 | 0 |
| $\mathbf{v_4}$ | $e^{\epsilon}\mathbf{v_1}$ | $e^{(\frac{3a}{2}-4)\epsilon}\mathbf{v_2}$ | $e^{\epsilon}\mathbf{v_3}$ | $e^{\epsilon}\mathbf{v_4}$ |