Admissibility Conditions and Asymptotic Behavior of Strongly Regular Graphs

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***Abstract*—We consider a strongly regular graph, G, with adjacency matrix A, and associate a three dimensional Euclidean Jordan algebra to A. Then, by considering convergent series of Hadamard powers of the idempotents of the unique complete system of orthogonal idempotents of the Euclidean Jordan algebra associated to A, we establish new admissibility conditions for the existence of strongly regular graphs. Finally, we extract some asymptotic conclusions about the spectrum of G.**

***Keywords*-Combinatorial Mathematics, Graph theory, Linear algebra, Symmetric matrices, Linear matrix inequalities.**

I. INTRODUCTION

Strongly regular graphs are a relatively new class of graphs firstly introduced in a 1963 paper form R. C. Bose, entitled *Strongly regular graphs, partial geometries and partially balanced designs*, [1]. One of the main problems on the study of these graphs is to find suitable feasibility conditions over their parameters. The most used and not trivial feasibility conditions are the Krein conditions and the absolute bounds (see [2]). In this paper we explore the close and interesting relationship of a three dimensional Euclidean Jordan algebra to the adjacency matrix of a strongly regular graph, in order to obtain some new inequalities for the existence of strongly regular graphs

Euclidean Jordan algebras were born back in 1934, when Pascual Jordan, John von Neumann and Eugene Wigner published their entitled paper *On an algebraic generalization of the quantum mechanical formalism* [3]. In this paper, the authors tried to deduce the Hermitian matrices properties, in a quantum mechanics context. It is remarkable that since then, Euclidean Jordan algebras have had such a wide range of applications. For instance there are applications to the theory of statistics (see [4]), to interior point methods (see [5] or [6]) and to combinatorics (see [7]). Detailed literature on Euclidean Jordan algebras can be found in Koecher's lecture notes, [8], and in the monograph by Faraut and Korányi, [9].

This paper is organized as follows. In Section II and III we present some basic concepts concerning Euclidean Jordan algebras and strongly regular graphs, respectively. In Section IV, we associate a three dimensional Euclidean Jordan algebra to the the adjacency matrix of a strongly regular graph. Then, in Section V we deduce new admissibility conditions over the parameters of a strongly regular graph. We finish this paper with some experimental results and some asymptotic conclusions, in Section VI.

II. A BRIEF INTRODUCTION ON EUCLIDEAN JORDAN ALGEBRAS

In this section we present relevant concepts for our work which can be seen, for instance, in [9].

Let $\mathcal{V}$ be a real vector space with finite dimension and a bilinear mapping $(u, v) \mapsto u \bullet v$. If $\mathcal{V}$ contains an element, e, such that for all u in $\mathcal{V}$ we have $e \bullet u = u \bullet e = u$, then e is called the *unit* element of $\mathcal{V}$. We define the powers of all elements u of $\mathcal{V}$ recursively as follows:

$$u^0 = e;\ u^n := u \bullet u^{n-1}, \forall n \in \mathbb{N}.$$

If for all u in $\mathcal{V}$ and any nonnegative integers p, q we have

$$u^p \bullet u^q = u^{p+q},$$

then $\mathcal{V}$ is a *power associative* algebra.

For each u in V let k be the minimal positive integer such that the set

$$\{e, u, u^2, \ldots, u^k\}$$

is linear dependent. Then k is called the *rank* of u and we write rank$(u) = k$. Since $\mathcal{V}$ is a finite dimensional algebra, we define the rank of $\mathcal{V}$ as the number r such that

$$r = \text{rank}(\mathcal{V}) = \max\{\text{rank}(u) : u \in \mathcal{V}\}.$$

If for all u and v in $\mathcal{V}$ we have

(J_1) $u \bullet v = v \bullet u$ and
(J_2) $u \bullet (u^2 \bullet v) = u^2 \bullet (u \bullet v)$,

with $u^2 = u \bullet u$, then $\mathcal{V}$ is called a *Jordan algebra*.

In general a Jordan algebra with unit element is non associative but is always a power associative algebra. On another hand, given an associative algebra, one can define a Jordan algebra like in Example 1.

Example 1: Let $\mathcal{V}$ be an associative algebra and define the new product

$$u \bullet v = \frac{1}{2}(uv + vu), \tag{1}$$

for all u, v in $\mathcal{V}$. Then, $(\mathcal{V}, \bullet)$ is a Jordan algebra. The new product (1) is called the *Jordan product*.

Given a Jordan algebra $\mathcal{V}$ with unit element, if there is an inner product $< \cdot, \cdot >$ that verifies the equality

$$< u \bullet v, w >=< v, u \bullet w >,$$

for any u, v, w in $\mathcal{V}$, then $\mathcal{V}$ is called an *Euclidean Jordan algebra*.

Example 2: Let $\mathcal{V} = Sym(n, \mathbb{R})$ be the space of real symmetric matrices of order n equipped with the Jordan product. By Example 1, since $\mathcal{V}$ is associative, $\mathcal{V}$ is a Jordan algebra.

Now we consider the inner product

$$< B, C >= \mathrm{tr}(BC)$$

for all B, C in $\mathcal{V}$, where BC is the usual matrix product. Then, since

$$\begin{aligned}
< B \bullet C, D > &= \mathrm{tr}((B \bullet C)D) \\
&= \mathrm{tr}\left(\frac{BC + CB}{2} D\right) \\
&= \mathrm{tr}\left(\frac{BCD + CBD}{2}\right) \\
&= \mathrm{tr}\left(\frac{BCD}{2}\right) + \mathrm{tr}\left(\frac{CBD}{2}\right) \\
&= \mathrm{tr}\left(\frac{CBD}{2}\right) + \mathrm{tr}\left(\frac{CBD}{2}\right) \\
&= \mathrm{tr}\left(\frac{CBD + CBD}{2}\right) \\
&= \mathrm{tr}\left(C\frac{BD + BD}{2}\right) \\
&= \mathrm{tr}(C(B \bullet D)) \\
&= < C, B \bullet D >,
\end{aligned}$$

for all matrices B, C and D in $\mathcal{V}$, we conclude, that $\mathcal{V}$ is an Euclidean Jordan algebra.

From now on $\mathcal{V}$ is an Euclidean Jordan algebra with unit element e. If there is c in $\mathcal{V}$ such that $c^2 = c$, then c is called an *idempotent*. Two idempotents c and d are *orthogonal* if $c \bullet d = 0$. If two idempotents c and d of $\mathcal{V}$ are orthogonal, then they are orthogonal with respect to the inner product. In fact, we have

$$< c, d >=< c^2, d >=< c, c \bullet d > .$$

Then, if $c \bullet d = 0$ we conclude that $< c, d >= 0$.

A set $\{c_1, c_2, \dots, c_k\}$ is called a *complete system of orthogonal idempotents* if

1) $c_i^2 = c_i, \forall i \in \{1, \dots, k\}$,
2) $c_i \bullet c_j = 0, \forall i \neq j$,
3) $c_1 + c_2 + \cdots + c_k = e$.

An idempotent is called *primitive* if it is nonzero and cannot be written as the sum of two orthogonal nonzero idempotents.

Example 3: The matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is an idempotent of $Sym(3, \mathbb{R})$ that is not primitive, because it can be written as the following sum of primitive idempotents of $Sym(3, \mathbb{R})$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A *Jordan frame* is a complete system of orthogonal idempotents $\{c_1, c_2, \dots, c_k\}$, such that each c_i, for all i in $\{1, \dots, k\}$, is primitive. We observe that if S is a complete system of orthogonal idempotents of an Euclidean Jordan algebra $\mathcal{V}$, that is a basis of $\mathcal{V}$, then S is also a Jordan frame.

Now we present an example of a complete system of orthogonal idempotents that is also a Jordan frame.

Example 4: Let $\mathcal{V} = Sym(2, \mathbb{R})$. Now consider

$$c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathcal{V}$. The matrices c and d are idempotents, since $c^2 = c$ and $d^2 = d$, and are orthogonal, because $c \bullet d = 0$. Now, since

$$c + d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

then $\{c, d\}$ is a complete system of orthogonal idempotents. Finally, c and d are primitive and therefore $\{c, d\}$ is a Jordan frame.

For all u in $\mathcal{V}$, there exist unique distinct real numbers $\lambda_1, \lambda_2, \dots, \lambda_k$, and a unique complete system of orthogonal idempotents $\{c_1, c_2, \dots, c_k\}$ such that

$$u = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_k c_k, \tag{2}$$

with c_j, $j = 1, \dots, k$, real numbers (see [9], Theorem III.1.1). The λ_j's are the eigenvalues of u and (2) is the *spectral decomposition* of u. If $u = \sum_{i=1}^{k} \lambda_i c_i$ is the spectral decomposition of u, then the minimal polynomial of u is given by

$$p(U, u) = \prod_{i=1}^{k} (U - \lambda_i). \tag{3}$$

III. General Concepts on Strongly Regular Graphs

Along this paper we only consider non empty, simple graphs (graphs with no loops nor parallel edges) and not complete graphs (graphs that have some non-adjacent pair of vertices), herein called graphs.

Considering a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. An edge of G with endpoints x

and y is denoted by xy. In this case the vertices are called *adjacent* or *neighbors*. The number of vertices of G, $|V(G)|$, is called the *order* of G. If all vertices of G have k neighbors, then G is a k-*regular* graph. The *complement* of a graph G is a graph, denoted by $\overline{G}$, such that $V(\overline{G}) = V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Let G be a graph of order n. Then G is a (n,k,a,c)-*strongly regular graph* if it is k-regular, any pair of adjacent vertices have a common neighbors and any pair of non-adjacent vertices have c common neighbors.

If G is a strongly regular graph, then $\overline{G}$ is also strongly regular with parameters $(n,\overline{k},\overline{a},\overline{c})$, such that

$$\begin{aligned}\overline{k} &= n-k-1,\\ \overline{a} &= n-2-2k+c,\\ \overline{c} &= n-2k+a.\end{aligned}$$

The parameters of a (n,k,a,c)-strongly regular graph are not independent and are related by the equality

$$k(k-a-1) = (n-k-1)c. \tag{4}$$

We associate to G a n by n matrix $A = [a_{ij}]$, where each $a_{ij} = 1$, if $v_iv_j \in E(G)$, otherwise $a_{ij} = 0$, called the *adjacency matrix* of G. The eigenvalues of A are simply called the eigenvalues of G.

It is well known (see, for instance, [2]) that the eigenvalues of the (n,k,a,c)-strongly regular graph G are k, θ and τ, where θ and τ are given by

$$\theta = \frac{a-c+\sqrt{(a-c)^2+4(k-c)}}{2} \tag{5}$$

and

$$\tau = \frac{a-c-\sqrt{(a-c)^2+4(k-c)}}{2}. \tag{6}$$

Equation (4) is an example of a feasibility condition that must be satisfied by the parameters of any strongly regular graph. The Krein conditions, obtained in 1973 by Scott, Jr., [10], are among the most important feasibility conditions for strongly regular graphs. However, there are still many parameter sets for which we do not know if they correspond to a strongly regular graph. In this work we obtain some new conditions to establish the unfeasibility of certain parameter sets of strongly regular graphs. We deduce them by associating an Euclidean Jordan algebra to the adjacency matrix of a strongly regular graph.

IV. Associating an Euclidean Jordan algebra to a strongly regular graph

From now on, we consider the Euclidean Jordan algebra $\mathcal{V} = Sym(n,\mathbb{R})$ equipped with the Jordan product and with the inner product

$$< B, C > = \text{tr}(BC)$$

for all B and C in $\mathcal{V}$.

Let G be a (n,k,a,c)-strongly regular graph such that $0 < c < k < n-1$ and let A be the adjacency matrix of G with three distinct eigenvalues, namely k, θ, and τ, given by the formulae (5) and (6) in Section III. Let k and θ be the positive eigenvalues and τ be the negative eigenvalue of A.

Now we consider the Euclidean Jordan subalgebra of $\mathcal{V}$, $\mathcal{V}'$, spanned by I_n, and the natural powers of A. Since A has three distinct eigenvalues, then $\mathcal{V}'$ is a three dimensional Euclidean Jordan algebra with rank$(\mathcal{V}') = 3$.

Let $S = \{E_1, E_2, E_3\}$ be the unique complete system of orthogonal idempotents of $\mathcal{V}$ associated to A, with

$$\begin{aligned}E_1 &= \frac{1}{(k-\theta)(k-\tau)}A^2 - \frac{\theta+\tau}{(k-\theta)(k-\tau)}A + \\ &+ \frac{\theta\tau}{(k-\theta)(k-\tau)}I_n,\\ E_2 &= \frac{1}{(\theta-\tau)(\theta-k)}A^2 - \frac{k+\tau}{(\theta-\tau)(\theta-k)}A + \\ &+ \frac{k\tau}{(\theta-\tau)(\theta-k)}I_n,\\ E_3 &= \frac{1}{(\tau-\theta)(\tau-k)}A^2 - \frac{k+\theta}{(\tau-\theta)(\tau-k)}A + \\ &+ \frac{k\theta}{(\tau-\theta)(\tau-k)}I_n,\end{aligned}$$

where J_n is the matrix whose entries are all equal to 1. We rewrite the idempotents under the basis $\{I_n, A, J_n - A - I_n\}$ of $\mathcal{V}'$ obtaining

$$\begin{aligned}E_1 &= \frac{1}{n}I_n + \frac{1}{n}A + \frac{1}{n}\left(J_n - A - I_n\right),\\ E_2 &= \frac{-\tau n+\tau-k}{n(\theta-\tau)}I_n + \frac{n+\tau-k}{n(\theta-\tau)}A\\ &+ \frac{\tau-k}{n(\theta-\tau)}\left(J_n - A - I_n\right),\\ E_3 &= \frac{\theta n+k-\theta}{n(\theta-\tau)}I_n + \frac{-n+k-\theta}{n(\theta-\tau)}A\\ &+ \frac{k-\theta}{n(\theta-\tau)}\left(J_n - A - I_n\right).\end{aligned}$$

Let p be a natural number such that $p \geq 1$ and denote by $M_n(\mathbb{R})$ the set of square matrices of order n with real entries. For B in $M_n(\mathbb{R})$, we denote by $B^{\circ p}$ and $B^{\otimes p}$ the Hadamard power of order p of B and the Kronecker power of order p of B, respectively, with $B^{\circ 1} = B$ and $B^{\otimes 1} = B$.

V. New Admissibility Conditions for Strongly Regular Graphs

Consider the following spectral decomposition of A,

$$A = kE_1 + \theta E_2 + \tau E_3.$$

For $l \in \mathbb{N}$, let:

$$\begin{aligned} S_{2l-1}^{\otimes} &= E_3 \otimes J_n^{\otimes(2l-2)} + E_3^{\otimes 3} \otimes J_n^{\otimes(2l-4)} + \\ &+ \cdots + E_3^{\otimes(2l-1)}, \end{aligned} \quad (7)$$

where each summand is a Kronecker product with $2l-1$ factors. The sum $S_{2l-1}^{\otimes}$ has a principal submatrix given by:

$$\begin{aligned} S_{2l-1}^{\circ} &= E_3 \circ J_n^{\circ(2l-2)} + E_3^{\circ 3} \circ J_n^{\circ(2l-4)} + \\ &+ \cdots + E_3^{\circ(2l-1)}. \end{aligned} \quad (8)$$

Observe that, since J_n is the identity for the Hadamard product of matrices which is associative, it follows that $S_{2l-1}^{\circ} = \sum_{i=1}^{l} E_3^{\circ(2i-1)}$. Let q_{2l-1}^1, q_{2l-1}^2 and q_{2l-1}^3 be the real numbers such that $S_{2l-1}^{\circ} = \sum_{i=1}^{3} q_{2l-1}^i E_i$. Since the set $\mathcal{B} = \{E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_{2l-1}} : i_1, i_2, \cdots, i_{2l-1} \in \{1,2,3\}\}$ is a complete system of orthogonal idempotents that is a basis of the real Euclidean Jordan algebra $(\mathcal{V}')^{\otimes(2l-1)}$ spanned by $I^{\otimes(2l-1)}$ and the natural powers of $A^{\otimes(2l-1)}$, then the minimal polynomial of $S_{2l-1}^{\otimes}$ is

$$p(\lambda, S_{2l-1}^{\otimes}) = (\lambda - 0) \prod_{i=1}^{l} (\lambda - n^{2(l-i)}). \quad (9)$$

Note that, to obtain (9) we use formula (3), applying the system of orthogonal idempotents in each summand of (7), see [9, p. 44].

The interlacing Theorem (see [11], Theorem 4.3.15), states that given a real symmetric matrix, B, of order n, whose eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and a principal submatrix of B, C of order m, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, then, the eigenvalues of C relate with those of B in the following manner:

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i},$$

for $i = 1, 2, \ldots, m$.

The matrix (8) is a principal submatrix of $S_{2l-1}^{\otimes}$ and p is the minimal polynomial of $S_{2l-1}^{\otimes}$. By the interlacing Theorem, the eigenvalues of S_{2l-1}° are all nonnegative. Regarding that

$$\begin{aligned} S_{2l-1}^{\circ} &= \sum_{j=1}^{l} \left(\frac{\theta n + k - \theta}{n(\theta - \tau)} \right)^{2j-1} I_n + \\ &+ \sum_{j=1}^{l} \left(\frac{-n + k - \theta}{n(\theta - \tau)} \right)^{2j-1} A + \\ &+ \sum_{j=1}^{l} \left(\frac{k - \theta}{n(\theta - \tau)} \right)^{2j-1} (J_n - A - I_n) \end{aligned} \quad (10)$$

and since $|\tau| > 1$, then

$$\left| \frac{\theta n + k - \theta}{n(\theta - \tau)} \right| < 1,$$
$$\left| \frac{-n + k - \theta}{n(\theta - \tau)} \right| < 1,$$
$$\left| \frac{k - \theta}{n(\theta - \tau)} \right| < 1,$$

and therefore the series $\sum_{i=1}^{+\infty} E_3^{\circ(2i-1)}$ is convergent with sum s. Consider the real numbers q_∞^1, q_∞^2, q_∞^3 such that

$$s = \lim_{l \to +\infty} S_{2l-1}^{\circ} = q_\infty^1 E_1 + q_\infty^2 E_2 + q_\infty^3 E_3. \quad (11)$$

As

$$S_{2l-1}^{\circ} = q_{2l-1}^1 E_1 + q_{2l-1}^2 E_2 + q_{2l-1}^3 E_3, \quad (12)$$

applying limits to (12) and comparing expressions (11) and (12) we obtain

$$\begin{aligned} q_\infty^1 &= \lim_{l \to \infty} q_{2l-1}^1, \\ q_\infty^2 &= \lim_{l \to \infty} q_{2l-1}^2 \\ q_\infty^3 &= \lim_{l \to \infty} q_{2l-1}^3. \end{aligned}$$

As the eigenvalues of S_{2l-1}° are nonnegative, it follows that $q_\infty^1 \geq 0$, $q_\infty^2 \geq 0$ and $q_\infty^3 \geq 0$. Then from identity (10) and doing some algebraic manipulations we obtain:

$$\begin{aligned} q_\infty^1 &= \frac{n(\theta - \tau)(n\theta - \theta + k)}{n^2(\theta - \tau)^2 - (n\theta - \theta + k)^2} + \\ &+ \frac{n(\theta - \tau)(-n + k - \theta)}{n^2(\theta - \tau)^2 - (-n + k - \theta)^2} k + \\ &+ \frac{n(\theta - \tau)(k - \theta)}{n^2(\theta - \tau)^2 - (k - \theta)^2} (n - k - 1); \\ q_\infty^2 &= \frac{n(\theta - \tau)(n\theta - \theta + k)}{n^2(\theta - \tau)^2 - (n\theta - \theta + k)^2} + \\ &+ \frac{n(\theta - \tau)(-n + k - \theta)}{n^2(\theta - \tau)^2 - (-n + k - \theta)^2} \theta + \\ &+ \frac{n(\theta - \tau)(k - \theta)}{n^2(\theta - \tau)^2 - (k - \theta)^2} (-\theta - 1); \\ q_\infty^3 &= \frac{n(\theta - \tau)(n\theta - \theta + k)}{n^2(\theta - \tau)^2 - (n\theta - \theta + k)^2} + \\ &+ \frac{n(\theta - \tau)(-n + k - \theta)}{n^2(\theta - \tau)^2 - (-n + k - \theta)^2} \tau + \\ &+ \frac{n(\theta - \tau)(k - \theta)}{n^2(\theta - \tau)^2 - (k - \theta)^2} (-\tau - 1). \end{aligned}$$

Now, since

$$\lambda_{min}(A \circ B) \geq \lambda_{min}(A) \lambda_{min}(B), \quad (13)$$

for any matrices A, B in $\mathcal{M}_n(\mathbb{R})$, (see [12], p. 312), and the parameters q_∞^1, q_∞^2 and q_∞^3 are nonnegative then the eigenvalues of $s^{\circ x}$ are also nonnegative, for $x \in \mathbb{N}$. Let q_{xs}^i, for $i \in \{1,2,3\}$, be the real numbers such that

$s^{\circ x} = \sum_{i=1}^{3} q_{xs}^i E_i$. Analyzing the parameters q_{xs}^1 and q_{xs}^3 and after some algebraic manipulation we establish the following theorem that contains new feasibility conditions for the existence of strongly regular graphs.

Theorem 1: Let G be a (n,k,a,c)-strongly regular graph, such that $0 < c < k < n-1$, whose adjacency matrix has the eigenvalues k, θ and τ. Let $x \in \mathbb{N}$, then

$$\begin{aligned} 0 &\leq \left(\frac{n(\theta-\tau)(n\theta-\theta+k)}{n^2(\theta-\tau)^2-(n\theta-\theta+k)^2}\right)^{2x-1} + \\ &+ \left(\frac{n(\theta-\tau)(-n+k-\theta)}{n^2(\theta-\tau)^2-(-n+k-\theta)^2}\right)^{2x-1} k + \\ &+ \left(\frac{n(\theta-\tau)(k-\theta)}{n^2(\theta-\tau)^2-(k-\theta)^2}\right)^{2x-1} (n-k-1), \quad (14) \\ 0 &\leq \left(\frac{n(\theta-\tau)(n\theta-\theta+k)}{n^2(\theta-\tau)^2-(n\theta-\theta+k)^2}\right)^{2x} + \\ &+ \left(\frac{n(\theta-\tau)(-n+k-\theta)}{n^2(\theta-\tau)^2-(-n+k-\theta)^2}\right)^{2x} \tau + \\ &+ \left(\frac{n(\theta-\tau)(k-\theta)}{n^2(\theta-\tau)^2-(k-\theta)^2}\right)^{2x} (-\tau-1). \quad (15) \end{aligned}$$

Proof: The proof follows from the statements made above. We use induction over x to prove that $q_{xs}^i \geq 0$, for $i \in \{1,2,3\}$. By definition $s^{\circ 1} = q_{1s}^1 E_1 + q_{1s}^2 E_2 + q_{1s}^3 E_3$, with $q_{1s}^i \geq 0$. Suppose that the eingenvalues of $s^{\circ x}$ are all nonnegative. Then $s^{\circ(x+1)} = s^{\circ x} \circ s^{\circ 1}$. Since the eingenvalues of $s^{\circ x}$ and $s^{\circ 1}$ are nonnegative, then by property (13), we conclude that the eingenvalues of $s^{\circ(x+1)}$ are all nonnegative. Therefore, for all $x \in \mathbb{N}$, $q_{xs}^i \geq 0$, for $i \in \{1,2,3\}$. In particular, inequalities (14) and (15) correspond precisely to $q_{xs}^1 \geq 0$ and $q_{xs}^3 \geq 0$, respectively. ■

Regard that we have used the idempotent E_3 and proceeding in a similar manner with the other idempotents of S we would obtain other necessary conditions for the existence of a (n,k,a,c)-strongly regular graph. The reason for this choice is that this idempotent makes algebraic manipulation simpler and we obtain better results in our computation experiments.

The following results are obtained from the inequalities of Theorem 1.

Theorem 2: Let G be a (n,k,a,c)-strongly regular graph, such that $0 < c < k < n-1$, whose adjacency matrix has the eigenvalues k, θ and τ. Then

$$(\theta-1)n + 2(k-\theta) \geq 0. \quad (16)$$

Proof: Considering $x = 1$, we can rewrite inequality (14) as:

$$\begin{aligned} &\frac{\theta n+k-\theta}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2} + \\ + \; &\frac{-n+k-\theta}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2} k + \\ + \; &\frac{k-\theta}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2} (n-k-1) \geq 0. \end{aligned}$$

Then, since $\theta n + k - \theta \geq k - \theta$, we can conclude that

$$\begin{aligned} &\frac{\theta n+k-\theta}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2} + \\ + \; &\frac{-n+k-\theta}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2} k + \\ + \; &\frac{k-\theta}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2} (n-k-1) \geq 0. \end{aligned}$$

Associating the first and third summands of the left hand side of the above inequality we obtain:

$$\begin{aligned} &\frac{(n-k+\theta)k}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2} - \\ - \; &\frac{(n-k+\theta)k}{n(\theta-\tau)} \cdot \frac{1}{1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2} \geq 0. \quad (17) \end{aligned}$$

After some straightforward calculations and simplifications, we conclude that (17) is equivalent to:

$$\begin{aligned} &\frac{k(n-k+\theta)}{n(\theta-\tau)} \times \\ \times \; &\frac{\frac{(\theta-1)n+2(k-\theta)}{n(\theta-\tau)} \cdot \frac{\theta}{\theta-\tau}}{\left[1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right]\left[1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right]} \geq 0. \end{aligned}$$

Analyzing the above inequality we conclude that

$$(\theta-1)n + 2(k-\theta) \geq 0.$$

■

This result is only relevant if $\theta < 1$, otherwise it represents a trivial inequality. Note that, if θ is near 0, then k has to be larger than $n/2$. Next, we present an inequality over the parameters of a strongly regular graph that satisfy $k < n/2$.

Theorem 3: Let G be a (n,k,a,c)-strongly regular graph, such that $0<c<k<n-1$, whose adjacency matrix has the eigenvalues k, θ and τ. If $k<n/2$, then

$$(-2\tau-1)(4\theta-2\tau+1)<\frac{8n\theta(2\theta+1)(\theta+1)(\theta-\tau)}{n-2(k-\theta)}. \quad (18)$$

Proof: Considering $x=1$, we rewrite inequality (14) as:

$$\begin{aligned}&\frac{\theta n+k-\theta}{n(\theta-\tau)}\cdot\frac{1}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2}+\\ +\;&\frac{-n+k-\theta}{n(\theta-\tau)}\cdot\frac{1}{1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2}k+\\ +\;&\frac{k-\theta}{n(\theta-\tau)}\cdot\frac{1}{1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2}(n-k-1)\geq 0.\end{aligned}$$

Since

$$(k-\theta)(n-k-1)=-(\theta n+k-\theta)+(n-k+\theta)k$$

and after some straightforward calculations we obtain:

$$\begin{aligned}&\frac{\theta n+k-\theta}{n(\theta-\tau)}\times\\ \times\;&\left[\frac{1}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2}-\frac{1}{1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2}\right]-\\ -\;&k\frac{n-k+\theta}{n(\theta-\tau)}\times\\ \times\;&\left[\frac{1}{1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2}-\frac{1}{1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2}\right]\geq 0.\end{aligned}$$

Developing both expressions between brackets in the left hand side of the above expression and multiplying by

$$1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2,$$

we deduce:

$$\begin{aligned}&\frac{\theta n+k-\theta}{n(\theta-\tau)}\left[\frac{\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2}\right]-\\ -\;&k\frac{n-k+\theta}{n(\theta-\tau)}\left[\frac{\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2}{1-\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2}\right]\geq 0.\end{aligned}$$

Noting that

$$\frac{1}{1-\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2}>1,$$

we conclude that

$$\begin{aligned}&\frac{\theta n+k-\theta}{n(\theta-\tau)}\left[\frac{\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2}\right]-\\ -\;&k\frac{n-k+\theta}{n(\theta-\tau)}\left[\left(\frac{n-k+\theta}{n(\theta-\tau)}\right)^2-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right]>0,\end{aligned}$$

and therefore

$$\begin{aligned}&\frac{\theta n+k-\theta}{n(\theta-\tau)}\cdot\frac{\frac{(\theta n+2(k-\theta))\theta}{n(\theta-\tau)^2}}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2}-\\ -\;&k\frac{n-k+\theta}{n(\theta-\tau)}\cdot\frac{n-2(k-\theta)}{n(\theta-\tau)^2}>0.\end{aligned}$$

Because $k<n/2$ we obtain

$$\begin{aligned}&\frac{2\theta+1}{2(\theta-\tau)}\left(\frac{\frac{(\theta+1)\theta}{(\theta-\tau)^2}}{1-\left(\frac{2\theta+1}{2(\theta-\tau)}\right)^2}\right)-\\ -\;&k\frac{1}{2(\theta-\tau)}\left(\frac{n-2(k-\theta)}{n(\theta-\tau)^2}\right)>0. \quad (19)\end{aligned}$$

Now since $\theta-\tau<2k$ and after some algebraic manipulation of inequality (19), we conclude that

$$(-2\tau-1)(4\theta-2\tau+1)<\frac{8n\theta(2\theta+1)(\theta+1)(\theta-\tau)}{n-2(k-\theta)}.$$

■

Analyzing inequality (18) we observe that considering n, k and θ fixed, the left hand side is a polynomial in τ of degree 2, with positive coefficients, and the right hand side is a polynomial in τ of degree 1, with positive coefficients. Therefore we may conclude that if θ is smaller than $|\tau|$, then $|\tau|$ cannot be too large relatively to θ.

VI. CONCLUSION

In this section we present a few examples of parameter sets (n,k,a,c) that don't verify inequalities (14), (15) and (18). In Table I, II and III we consider the parameter sets:

1) $P_1=(201,100,2,97)$,
2) $P_2=(1585,784,33,735)$,
3) $P_3=(23989,11988,987,10989)$,
4) $P_4=(19999001,9999000,8999,9989001)$,
5) $P_5=(1024,385,36,210)$,
6) $P_6=(1225,456,39,247)$ and
7) $P_7=(1296,481,40,260)$.

For each set we present the respective eigenvalues θ and τ, and the parameters $q^1_{(2x+1)s}$ and $q^3_{(2x)s}$ for $x = 1$. We also consider

$$q_{\theta\tau kn} = \frac{8n\theta(2\theta+1)(\theta+1)(\theta-\tau)}{n-2(k-\theta)} + (2\tau+1)(4\theta-2\tau+1),$$

from the inequality of Theorem 3.

Param.	P_1	P_2
θ	0.032	0.07
τ	-95	-702
q^1_{3s}	-2.9×10^{-7}	-1.2×10^{-8}
q^3_{2s}	-5.2×10^{-5}	-1.5×10^{-5}
q^1_{5s}	-1.6×10^{-11}	-1.2×10^{-14}
q^3_{6s}	-1.2×10^{-13}	-1.2×10^{-17}
$q_{\theta\tau kn}$	-3.1×10^{4}	-1.9×10^{6}

Table I
NUMERICAL RESULTS FOR $P_1 = (201, 100, 2, 97)$ AND $P_2 = (1585, 784, 33, 735)$.

Param.	P_3	P_4
θ	$0,1$	0.001
τ	-10002	-1.0×10^{7}
q^1_{3s}	-3.2×10^{-12}	-2.5×10^{-19}
q^3_{2s}	-5.4×10^{-8}	-5.0×10^{-12}
q^1_{5s}	-1.6×10^{-20}	-1.3×10^{-33}
q^3_{6s}	-1.0×10^{-24}	-9.5×10^{-41}
$q_{\theta\tau kn}$	-3.8×10^{8}	-4.0×10^{14}

Table II
NUMERICAL RESULTS FOR $P_3 = (23989, 11988, 987, 10989)$ AND $P_4 = (19999001, 9999000, 8999, 9989001)$.

Observing the results obtained in Table I and II we note that all the parameter sets analyzed fail our inequalities because they satisfy the condition $k < n/2$ and they present sufficiently small values of θ, confirming our analysis of Theorem 2.

Param.	P_5	P_6	P_7
θ	1	1	1
τ	-175	-209	-221
q^1_{3s}	-1.1×10^{-5}	-7.7×10^{-6}	-7.0×10^{-6}
q^3_{2s}	-1.4×10^{-3}	-1.2×10^{-3}	-1.1×10^{-3}
q^1_{5s}	-1.6×10^{-10}	-8.4×10^{-11}	-6.9×10^{-11}
q^3_{4s}	-2.1×10^{-8}	-1.3×10^{-8}	-1.1×10^{-8}
$q_{\theta\tau kn}$	-9.0×10^{4}	-1.4×10^{5}	-1.6×10^{5}

Table III
NUMERICAL RESULTS FOR $P_5 = (1024, 385, 36, 210)$, $P_6 = (1225, 456, 39, 247)$ AND $P_7 = (1296, 481, 40, 260)$.

As for the results obtained in Table III we observe that all the parameters fail our inequalities because they satisfy $k < n/2$ and they present values for θ and τ such that $\theta << |\tau|$, which confirms our conclusions from Theorem 3.

Regarding our results we can show that our parameters $q^1_{(2x+1)s}$ and $q^3_{(2x)s}$, display an interesting asymptotic behavior when $k < n/2$ and $\theta \to 0$. In fact, with $x = 1$ we have:

$$\lim_{\theta\to0} q^1_{(2x+1)s} = -n\tau\left[\frac{k(n-k)}{n^2\tau^2-k^2} + \frac{(-n+k)k}{n^2\tau^2-(-n+k)^2}\right].$$

Then, in order for (14) to fail, in these circumstances, we must have

$$\begin{aligned} & \frac{k(n-k)}{n^2\tau^2-k^2} < \frac{(n-k)k}{n^2\tau^2-(-n+k)^2} \\ \Leftrightarrow\ & n^2\tau^2-(-n+k)^2 < n^2\tau^2-k^2 \\ \Leftrightarrow\ & -n^2+2nk < 0 \\ \Leftrightarrow\ & n(2k-n) < 0. \end{aligned}$$

Therefore we must have $2k - n < 0$, and so $k < n/2$.

As for the parameter $q^3_{(2x)s}$, for $x = 1$, we can proceed in a similar manner

$$\begin{aligned} \lim_{\theta\to0} q^3_{(2x)s} &= n^2\tau^2\left[\frac{-\tau k^2}{(n^2\tau^2-k^2)^2}\right. \\ &+ \left.\frac{\tau(-n+k)^2}{(n^2\tau^2-(-n+k)^2)^2}\right], \end{aligned}$$

to conclude that, in order for (15) to fail, we must have

$$\frac{-\tau k^2}{(n^2\tau^2-k^2)^2} < \frac{-\tau(-n+k)^2}{(n^2\tau^2-(-n+k)^2)^2}.$$

However, this inequality is trivially verified if $k < n/2$.

This proves an interesting asymptotic behavior: we can conclude that for a (n, k, a, c)-strongly regular graph such that $k < n/2$ and θ sufficiently small, inequalities (14) and (15) will fail. The parameter sets that we used in Table I, Table II and Table III match these requirements and the corresponding results are precisely as expected.

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