Two-dimensional extended warranty cost analysis for multi-component repairable products

Kamran Shahanaghi, Majeed Heydari

***Abstract*—** Providing extended warranty (EW) contracts for increasingly complex products with multiple failure processes and almost high repair costs are challenging and risky task. In such circumstances, modeling products as multi-component systems and estimating their failures and servicing costs more accurately are of great interest to EW providers. In this paper, a repairable product sold with two-dimensional warranty, like an automobile, is considered as a multi-component system composed of a series of heterogeneous components with conditionally independent time to failures. For this product, the effect of usage rate on the failure processes of its components and in turn on the product's time to failure during the EW period is modeled. Then, considering cost of a repair as function of failed component and its age and usage, the expected servicing cost of the EW contract is estimated. A failure history of a commercial vehicle is also used to illustrate the proposed model. In the provided case study, vehicle's components are categorized into five main subsystems and the time to the first failures of each subsystem and in turn vehicle's time to failure is derived. Through this example the effects of EW's length and usage limits on its servicing cost are also studied and iso-warranty cost curves are obtained.

***Keywords*—** Two-dimensional Extended warranty contract, Multi-component product, Failure modeling, Warranty cost analysis..

I. Introduction

Extended warranty has become increasingly important part of after-sale market and there is increasing interest in the offering [1] and purchasing [2] of EW contracts for a wide range of products. The EW market study reveals that selling of EW contracts is among the biggest profit producers of all [3]. For example the profit margins of offering EW contracts for electrical devices estimated to be 40% to 77% [4] while for new vehicles this may reach upwards of 100 percent [5]. However, most of products sold in the market are complex products and rapidly changing and increasing technological complexity of them over time makes predicting future failures of products and upcoming servicing costs more challenging task. On the other hand, increasing competition in the EW market enforce the EW providers to estimate products' failures and servicing costs more accurately and offer EW contracts with competitive prices and quality of services.

Comparing raised practical problems in the offering of EW contracts for complex products with conducted researches on this area reveals huge gap between academic researches and practical needs. In the following, we show this gap by reviewing the reported researches in the EW literature since 2000. The researches into the EW contracts can be broadly categorized into market-related and mathematical modeling researches. The majority of these researches belong to the market-related researches (see for example [6-10]). However, mathematical modeling and optimization of EW contracts has become an active area of research especially over the last decade. Considerable growth in the conducted researches on this area shows that there is increasing interest on the mathematical modeling and optimization of EW contracts. In the following lines we briefly review these researches.

Considering only corrective maintenance Kumar and Chattopadhyay [11] determined minimum beneficial length of the EW contract from consumer's perspective under minimal repair at failure assumption. Jack and Murthy [12] assuming minimal repair and Lam and lam [13] in the presence of perfect repair at failures applied game models to derive the optimal policies of the manufacturer and the consumer during an EW period. Wu and Longhurst [14] also minimized expected life cycle cost of a product when minor and catastrophic failures were rectified by minimal repair and replacement respectively. In the above researches, only corrective maintenance is considered. However, assuming preventive maintenance with imperfect repair and minimal corrective repairs, Bouguerra et al. [15] determined the maximum acceptable price of an EW contract from the consumer's perspective as well as minimum acceptable price of the EW contract from the manufacturer's perspective. Chang and Lin [16] also maximized an EW provider's profit by deriving the optimal length of EW contract as well as controlled-limit preventive maintenance strategy during the base and EW periods. In this research, the corrective and preventive repairs were assumed to be minimal and imperfect respectively.

In all above mentioned researches, warranty policy assumed to be one-dimensional i.e. warranty coverage region solely defined by an interval (usually of time). However, for some

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products like automobiles time (age) and usage (mileage) are used simultaneously to determine the warranty coverage region in a two-dimensional plane. Considering this type of warranty policy which is called two-dimensional warranty, Majid et al. [17] optimized the EW's coverage region and its expected servicing cost from service provider's point of view. To do this, assuming minimal repair at failure, they used two-dimensional delay time model and determined optimal preventive maintenance strategy during the EW contract. In the presence of three different corrective repair options, Su and Shen [18] derived the expected cost and profit models of an EW contract when a manufacturer offers non-renewing one-dimensional and non-renewing two-dimensional warranties. For a product sold with a two-dimensional warranty, Shahanaghi et al. [19] also minimized the expected servicing cost of an EW contract from the EW provider's point of view. To do this, assuming minimal repairs at failures during the base and EW periods as well as assuming a preventive imperfect maintenance during the EW period, they derived the optimal number and degrees of preventive repairs. In another research, considering the possibility of an upgrade action at the beginning of the EW period as well as preventive maintenance with the fixed and pre-specified repair levels during the EW period, Heydari et al. [20] maximized the EW provider's profit. To do this, by incorporating the demand for EW contracts, they obtained the optimal upgrade improvement levels and optimal number of preventive repairs during the EW period.

In the all previously reviewed researches, the black-box approach is used to model the failure process of product and its corresponding costs during the EW period. In this approach, regardless of product's internal structure and its components characteristics, the product treated as a simple single-unit product. Since in this approach available information like system structure, the number and characteristics of its components are not taking into account, the provided results should be only considered as an approximation [21]. However, most of products sold in the market are complex systems composed of many heterogeneous components with different characteristics. In such situations, modeling a product as multi-components system and utilizing available information may considerably improves the warranty provider's insight on the failure process of the product. This may helps the service provider to estimate the product failures more accurately and derive more effective maintenance and inspection policies during the warranty period.

The research on the multi-component systems' failure modeling and maintenance optimization is vast (see for example [22-26] and references therein). However, warranty modeling and optimization by considering a product as a complex system is almost new topic [21] and there are limited reported researches in the warranty literature. Assuming one-dimensional free repair and pro-rata warranty policies, Bai and Pham [21] derived discounted warranty cost (DWC) models of a repairable series system. To do this, considering continues and discrete discount functions and minimal repair at failures they obtained expected and variance of DWCs. In another research, Bai and Pham [27] proposed a renewable full-service warranty (RFSW) policy for repairable series, parallel, series–parallel and parallel–series systems and obtained the first and second centered moments of base warranty servicing costs from manufacturer's point of view. In the proposed model, it is assumed that all failed components under warranty are replaced with new ones and at the same time system undergoes perfect repair all free of charge to consumers. Assuming imperfect repair at failures, Park and Pham [28] also applied quasi-renewal processes to obtain reliability and warranty cost models for several systems, including multi-component systems.

In the reviewed researches on warranty analysis of a multi-component product, warranty policy was assumed to be one-dimensional. However, for some multi-component products, like automobiles, warranty coverage region is two-dimensional. To the best of authors' knowledge, warranty analysis of a multi-component product under a two-dimensional base warranty has become subject of the only two reported researches in the warranty literature. In the first research, Manna et al. [29] applied accelerated failure time (AFT) model at components level to study the effect of use-rate on the components' time to failures and in turn product's failure times. In the second research, for a k-out-of-n repairable system with a two-dimensional warranty, Park and Pham [30] minimized the long-run expected warranty cost when a periodic preventive maintenance as well as corrective maintenances are carried out on the system. In the proposed model, free repair/replacement and pro-rata warranty policies are studied and warranty period, repair time limit, and periodic preventive maintenance cycles are assumed to be manufacturer's decision variables.

As provided literature review reveals despite of the EW contracts' importance, there is no reported research on the modeling and analysis of EW contracts for complex multi-component products. In this paper, for the first time in EW literature, considering a product as a multi-component system composed of a series of heterogeneous components, we derive the failure process of the product and its servicing cost during the EW period. In the proposed model the base and EW policies assumed to be two-dimensional non-renewable free repair warranties. The provided model aims to help the EW provider to effectively utilize available information and more accurately estimate the product failures and its upcoming costs during the EW period. To achieve this goal, product's internal structure, heterogeneous characteristics of components as well as product's age and usage are investigated to build more accurate failure and in turn cost models. Since automobile is the most well-known example of such a multi-component system with widely offered two-dimensional EW contracts, we use the failure history of a commercial vehicle produced in a plant in Iran to illustrate the proposed model. Through this case study and using the proposed formulations, we analyze the effect of EW's coverage region on the product's servicing cost. We also derive iso-warranty cost curves which may seriously help the EW provider to offer a wide range of flexible two-dimensional EW contracts.

The remainder of this paper is organized as follows: in the next section, the coverage region of a two-dimensional EW contract is determined. In the first two subsections of section III, two-dimensional failure model at components and system levels are derived and the EW's servicing cost is modeled. Then, for a special case where random variables follow from Weibull distribution, the proposed formulations are derived and presented in the final subsection of section III. Through a case study, the proposed model is illustrated in section IV. Finally, conclusions are drawn and some future research directions are presented in section V.

II. Two-dimensional extended warranty coverage region

For a given product sold with a two-dimensional warranty, like an automobile, let the base warranty coverage region is a rectangle region in a two-dimensional plane represented by $\Omega_{\mathrm{BW}} = (0, W_0] \times (0, U_0]$ where W_0 and U_0 are warranty length (time/age) and usage (mileage). For this product, base warranty policy is non-renewable free repair warranty (NRFRW) with minimal repairs at failures and warranty expires when the age or usage of the product respectively exceeds from W_0 or U_0 whichever occurs first. Suppose that before the expiration of base warranty, the consumer purchases an EW contract with length W_1 and usage U_1 to cover product failures right after the expiration of base warranty period. The EW policy is also assumed to be the same as the base warranty i.e. it is NRFRW.

Despite one-dimensional warranty which its expiration time is known in advance, in two-dimensional case the base warranty expiration time and equivalently the EW beginning time are unknown and depends on the consumer's usage rate. For convenience, suppose that for a given consumer, his/her usage rate over product lifecycle is almost constant and different consumers have randomly different usage rates. Therefore, assuming $r_0 = \frac{U_0}{W_0}$ and $r_1 = \frac{U_1}{W_1}$ the base and extended warranties' coverage regions regarding r, the consumer's usage rate, will be different (see Fig 1).

As a result, regarding r_0 and r_1 the EW coverage region will be as follows:

$$\{\Omega_{EW} | r; r_0 \le r_1\} = \begin{cases} [W_0, W_0 + W_1] \times [rW_0, rW_0 + rW_1] & if\ r \le r_0 \\ \left[\frac{U_0}{r}, \frac{U_0}{r} + W_1\right] \times [U_0, U_0 + rW_1] & if\ r_0 < r \le r_1 \\ \left[\frac{U_0}{r}, \frac{U_0}{r} + \frac{U_1}{r}\right] \times [U_0, U_0 + U_1] & if\ r > r_1 \end{cases} \quad (1)$$

$$\{\Omega_{EW} | r; r_1 < r_0\} = \begin{cases} [W_0, W_0 + W_1] \times [rW_0, rW_0 + rW_1] & if\ r \le r_1 \\ \left[W_0, W_0 + \frac{U_1}{r}\right] \times [rW_0, rW_0 + U_1] & if\ r_1 < r \le r_0 \\ \left[\frac{U_0}{r}, \frac{U_0}{r} + \frac{U_1}{r}\right] \times [U_0, U_0 + U_1] & if\ r > r_0 \end{cases} \quad (2)$$

A seen in Fig 1, consumer's usage rate affects the expiration times of both base and extended warranties. Besides, as one can expect, it may considerably affects the product's degradation process, its reliability and in turn warranty servicing cost. In the next section, the effect of consumer's usage rate on the time to the first failure of product at components and system levels are modeled and the EW servicing cost will be obtained.

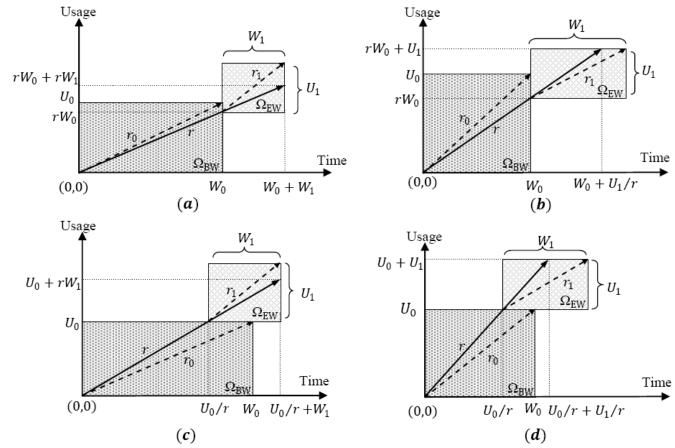

Fig 1: The base and extended warranties' coverage regions when a) $r \le \min\{r_0, r_1\}$, b): $r_1 < r \le r_0$ and c): $r_0 < r \le r_1$ and d) $r \ge \max\{r_0, r_1\}$.

III. Modeling the product as a multi-component series system

In essence most of products are multi-component systems composed of a collection of interrelated heterogeneous components where any failure of these components may lead to system failure. In this paper, we consider a product as a multi-component system with a series of K heterogeneous components. All components are repairable and in the case of any failure, only failed component is minimally repaired. Repair time and the probability of occurring two or more failures at the same time are negligible. The warranty policy over both base and extended warranty periods are non-renewable free repair warranties (NRFRW). We also assume that failure processes of components are conditionally independent i.e. conditioning on the usage rate, for a given consumer with usage rate r the time to first failures of components are independent. It is worth to note that when the usage rate varies between consumers the times to the first failures of components will also vary and this makes failure processes of components unconditionally dependent. For more information on the made assumption and raised issues see [29].

Let X_S and U are random variables denoting the time to the first failure and total usage of the product respectively. R is also random variable representing consumers' usage rate and as previously stated in this paper we assume that for a given consumer his/her usage rate over product lifecycle is almost constant. Therefore, under given usage rate r the total usage of the product will be $U(r) = rX_S(r)$, where $X_S(r) = [X_S | R = r]$. Considering these assumptions, in the next subsection, the failure process of the product at components and system levels are presented.

A. Two-dimensional failure model of a product during an EW period

In this paper, we use marginal approach to model the failure process of the product in terms of its age and usage at components level and in turn at the system level. To do this, we apply accelerated failure time (AFT) model at components level to investigate the effect of usage rate on the time to the first failures of components. Then, based on the system structure we model the time to the first failure of product in terms of its components failures times. For more information on two-dimensional failure modeling approaches and related researches please see [31].

Let the time to the first failure of ith component under nominal usage rate r_n is $X_i(r_n) = [X_i|R = r_n],\ i = 1, \dots, K$, and $F_{X_i(r_n)}\left(x; \alpha_i^{(n)}, \beta_i^{(n)}\right)$ is its distribution function with scale parameter $\alpha_i^{(n)}$ and shape parameter $\beta_i^{(n)}$. Then, based on the AFT model, the distribution of $X_i(r) = [X_i|R = r]$, the time to the first of this component under usage rate r, will be as follows:

$$F_{X_i(r)}(x; \alpha_i(r), \beta_i(r)) = F_{X_i(r_n)}\left(x; \alpha_i^{(n)}\left(\frac{r_n}{r}\right)^{\gamma_i}, \beta_i^{(n)}\right), \quad i = 1, \dots, K \tag{3}$$

where $\alpha_i(r) = \alpha_i^{(n)}\left(\frac{r_n}{r}\right)^{\gamma_i}$ and γ_i is the AFT parameter of ith component and $\beta_i(r) = \beta_i^{(n)}$. As one can see, in the AFT model it is assumed that changing the usage rate only affects the scale parameter and has no effect on the shape parameter of the time to the first failures. For convenience, in the following we use notations $F_{X_i(r)}(x; \alpha_i(r), \beta_i)$ or $F_{X_i(r)}(x)$ interchangeably to represent the conditional distribution function of the time to the first failure of ith component. Considering components' time to the first failures distribution functions and assuming conditional independency of components' failure processes, the conditional distribution function of the time to the first of the product under usage rate r will be as follows:

$$X_S(r) = \min(X_1(r), \dots, X_K(r))$$
$$F_{X_S(r)}(x; \mathbf{A}, \mathbf{B}) = 1 - \prod_{i=1}^{K}\left(1 - F_{X_i(r)}(x; \alpha_i(r), \beta_i)\right) \tag{4}$$

where $\mathbf{A} = (\alpha_1(r), \dots, \alpha_K(r))$ and $\mathbf{B} = (\beta_1, \dots, \beta_K)$. Consequently, conditional probability density and hazard functions of the product will be as follows:

$$f_{X_S(r)}(x) = \frac{dF_{X_S(r)}(x)}{dx}$$
$$h_{X_S(r)}(x) = \frac{f_{X_S(r)}(x)}{1 - F_{X_S(r)}(x)} \tag{5}$$

As a result, the conditional expected number of system failures up to time x will constitute a non-homogenous Poisson process (NHPP) where its conditional intensity function equals to conditional hazard rate of the time to first failure of product i.e. $\lambda(x|r) = h_{X_S(r)}(x)$. Recall that all failures are assumed to be minimally repaired at negligible times. Therefore, considering presented EW coverage region scenarios in equations (1) and (2) and their corresponding probabilities of occurrences and conditioning on $R = r$ the conditional expected number of failures during the EW period will be as follows:

$$\begin{aligned} E[N_S(\Omega_{EW}|r; r_0 \le r_1)] &= F_R(r_0)\int_{W_0}^{W_0+W_1} h_{X_S(r)}(x)dx \\ &+ \left(F_R(r_1) - F_R(r_0)\right) \\ &\times \int_{\frac{U_0}{r}}^{\frac{U_0}{r}+W_1} h_{X_S(r)}(x)dt \\ &+ R_R(r_1)\int_{\frac{U_0}{r}}^{\frac{U_0}{r}+\frac{U_1}{r}} h_{X_S(r)}(x)dt \end{aligned} \tag{6}$$

$$\begin{aligned} E[N_S(\Omega_{EW}|r; r_1 < r_0)] &= F_R(r_1)\int_{W_0}^{W_0+W_1} h_{X_S(r)}(x)dx \\ &+ \left(F_R(r_0) - F_R(r_1)\right) \\ &\times \int_{W_0}^{W_0+\frac{U_1}{r}} h_{X_S(r)}(x)dt \\ &+ R_R(r_0)\int_{\frac{U_0}{r}}^{\frac{U_0}{r}+\frac{U_1}{r}} h_{X_S(r)}(x)dt \end{aligned} \tag{7}$$

where $F_R(r)$and $R_R(r) = 1 - F_R(r)$ respectively are cumulative distribution function and reliability function associated with usage rate random variable R. Finally, by removing the conditioning the overall expected number of product failures during the EW period can be obtained.

However, since the components of the product are assumed to be heterogeneous, regarding the type of failed component and age and usage of the component at the time of failure the repair cost might be different. In the next subsection, considering this situation we estimate the expected cost of product's failures.

B. Expected servicing cost of a product during a two-dimensional EW contract

In this paper we assume that when the product fails during the base and EW periods, only failed component minimally repaired and no interventions are made on the other working components. The repair cost also depends on the type of failed component and its age and usage at the time of failure. Therefore, for a product under usage rate $R = r$, the cost of performing a minimal repair on ith, $i = 1, \dots, K$, component at the time x with total usage $u = rx$ can be given by:

$$C_i(x|r) = c_{i0} + c_i x^{\omega_i}(rx)^{\xi_i},\ i = 1, \dots, K \tag{8}$$

where, c_{i0} is the fixed cost of performing a minimal repair on ith component and c_i, ω_i and ξ_i are real positive numbers which their values can be estimated using the failure history of ith component. It is worth to mention that the primary version of this cost function is introduced by Chattopadhyay and Murthy [32] and later extended and used by Shahanaghi et al. [19] and Heydari et al. [20] for two-dimensional case.

Consequently, the conditional expected cost of failure rectifications of ith component during the time interval $[a, b]$, under usage rate $R = r$, will be as follows:

$$E[C_i([a,b]|r)] = \int_a^b C_i(x|r) h_{X_i(r)}(x)dx \text{ , } i = 1,..,K \tag{9}$$

As a result, the conditional expected servicing cost of the product with K heterogeneous components during the time interval $[a,b]$ will be:

$$\begin{aligned} E[C_S([a,b]|r)] &= \sum_{i=1}^{K} \int_a^b C_i(x|r) h_{X_i(r)}(x)dx \\ &= \int_a^b \sum_{i=1}^{K} \left[C_i(x|r) h_{X_i(r)}(x)\right] dx \end{aligned} \tag{10}$$

Finally, considering the usage rate scenarios in equations (1) and (2) and its corresponding probabilities of occurrences and by removing the conditioning on $R = r$, the overall expected servicing cost of tjhe product during the EW period will be as follows:

$$\begin{aligned} &E[C(\Omega_{EW}|r_0 \le r_1)] \\ &= F_R(r_0) \int_0^{r_0} \int_{W_0}^{W_0+W_1} \sum_{i=1}^{K} \left[C_i(x|r) h_{X_i(r)}(x)\right] g(r)dxdr \\ &+ \left(F_R(r_1) - F_R(r_0)\right) \\ &\times \int_{r_0}^{r_1} \int_{\frac{U_0}{r}}^{\frac{U_0}{r}+W_1} \sum_{i=1}^{K} \left[C_i(x|r) h_{X_i(r)}(x)\right] g(r)dxdr \\ &+ \ R_R(r_1) \int_{r_1}^{\infty} \int_{\frac{U_0}{r}}^{\frac{U_0}{r}+\frac{U_1}{r}} \sum_{i=1}^{K} \left[C_i(x|r) h_{X_i(r)}(x)\right] g(r)dxdr \end{aligned} \tag{11}$$

$$\begin{aligned} &E[C(\Omega_{EW}|r_1 < r_0)] \\ &= F_R(r_1) \int_0^{r_1} \int_{W_0}^{W_0+W_1} \sum_{i=1}^{K} \left[C_i(x|r) h_{X_i(r)}(x)\right] g(r)dxdr \\ &+ \left(F_R(r_0) - F_R(r_1)\right) \\ &\times \int_{r_1}^{r_0} \int_{\frac{U_0}{r}}^{W_0+\frac{U_1}{r}} \sum_{i=1}^{K} \left[C_i(x|r) h_{X_i(r)}(x)\right] g(r)dxdr \\ &+ R_R(r_0) \int_{r_0}^{\infty} \int_{\frac{U_0}{r}}^{\frac{U_0}{r}+\frac{U_1}{r}} \sum_{i=1}^{K} \left[C_i(x|r) h_{X_i(r)}(x)\right] g(r)dxdr \end{aligned} \tag{12}$$

It is clear that in the provided equations in (11) and (12), if $C_i(x|r)$, $i = 1, \dots, K$ is replaced with 1, the overall expected number of product's failures during the EW period will be obtained. As seen, in the provided EW cost model, the type of failed component, its age and usage at the time of failure as well as the usage rate of consumer are incorporated. To the best of authors' knowledge, for a first time in EW literature this is done in this paper.

In the next subsection, for a special case where the distributions of consumers' usage rate and the time to the first failures of components are Weibull, the presented formulations are evaluated.

C. *Proposed Mathematical model for a special case*

Let R, consumers usage rate random variable, follows from a Dagum distribution with scale parameter α_R and shape parameters β_R and k_R. The time to the first failure of ith component, $i = 1, \dots, K$ under nominal usage rate r_n i.e. $X_i(r_n)$ also assumed to be follows from a Weibull distribution with shape parameter $\alpha_i^{(n)}$ and scale parameter $\beta_i^{(n)}$. In such a situation, $F_R(r)$ and $f_R(r)$, CDF and PDF of usage rate, and $F_{X_i(r)}$, conditional time to the first failure of ith component, will be as follows:

$$F_R(r) = \left(1 + \left(\frac{r}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R} \text{, } r, \alpha_R, \beta_R, k_R > 0 \tag{13}$$

$$f_R(r) = \frac{\beta_R k_R \left(\frac{r}{\alpha_R}\right)^{\beta_R k_R - 1}}{\alpha_R \left(1 + \left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R+1}} \text{ , } \quad r, \alpha_R, \beta_R, k_R > 0 \tag{14}$$

$$F_{X_i(r)}(x; \alpha_i(r), \beta_i) = 1 - e^{-\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i^{(n)}}} \qquad i = 1, \dots, K \tag{15}$$

where $\alpha_i(r) = \alpha_i^{(n)} \left(\frac{r_n}{r}\right)^{\gamma_i}$ and $\beta_i^{(n)} = \beta_i$. The conditional CDF, PDF and hazard function of the product also are given by

$$F_{X_S(r)}(x) = 1 - e^{-\sum_{i=1}^{K}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i}} \tag{16}$$

$$f_{X_S(r)}(x) = \sum_{i=1}^{K} \left[\frac{\beta_i}{\alpha_i(r)} \left(\frac{x}{\alpha_i(r)}\right)^{\beta_i - 1}\right] e^{-\sum_{i=1}^{K}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i}} \tag{17}$$

$$h_{X_S(r)}(x) = \sum_{i=1}^{K} \left[\frac{\beta_i}{\alpha_i(r)} \left(\frac{x}{\alpha_i(r)}\right)^{\beta_i - 1}\right] \tag{18}$$

Finally, the expected overall cost of EW servicing for each product are obtained as follows:

$$
\begin{aligned}
&E[C(\Omega_{EW}|r_0 \le r_1)] \\
&= \left(1+\left(\frac{r_0}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R} \\
&\times \int_0^{r_0}\int_{W_0}^{W_0+W_1}\sum_{i=1}^{K}\Bigg[\Big(c_{i0} \\
&+ c_i x^{\omega_i}(rx)^{\xi_i}\Big)\frac{\beta_i}{\alpha_i(r)}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i-1}\Bigg]\frac{\beta_R k_R\left(\frac{r}{\alpha_R}\right)^{\beta_R k_R-1}}{\alpha_R\left(1+\left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R+1}}dx \\
&+ \left(\left(1+\left(\frac{r_1}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R} - \left(1+\left(\frac{r_0}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R}\right) \\
&\times \int_{r_0}^{r_1}\int_{\frac{U_0}{r}}^{\frac{U_0}{r}+W_1}\sum_{i=1}^{K}\left[\left(c_{i0}+c_i x^{\omega_i}(rx)^{\xi_i}\right)\frac{\beta_i}{\alpha_i(r)}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i-1}\right] \\
&\times \frac{\beta_R k_R\left(\frac{r}{\alpha_R}\right)^{\beta_R k_R-1}}{\alpha_R\left(1+\left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R+1}}dxdr \\
&+ \left(1-\left(1+\left(\frac{r_0}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R}\right) \\
&\times \int_{r_1}^{\infty}\int_{\frac{U_0}{r}}^{\frac{U_0}{r}+\frac{U_1}{r}}\sum_{i=1}^{K}\left[\left(c_{i0}+c_i x^{\omega_i}(rx)^{\xi_i}\right)\frac{\beta_i}{\alpha_i(r)}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i-1}\right] \\
&\times \frac{\beta_R k_R\left(\frac{r}{\alpha_R}\right)^{\beta_R k_R-1}}{\alpha_R\left(1+\left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R+1}}dxdr
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&E[C(\Omega_{EW}|r_1 < r_0)] \\
&= \left(1+\left(\frac{r_1}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R} \\
&\times \int_0^{r_1}\int_{W_0}^{W_0+W_1}\sum_{i=1}^{K}\Bigg[\Big(c_{i0} \\
&+ c_i x^{\omega_i}(rx)^{\xi_i}\Big)\frac{\beta_i}{\alpha_i(r)}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i-1}\Bigg]\frac{\beta_R k_R\left(\frac{r}{\alpha_R}\right)^{\beta_R k_R-1}}{\alpha_R\left(1+\left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R+1}}dx \\
&+ \left(\left(1+\left(\frac{r_0}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R} - \left(1+\left(\frac{r_1}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R}\right) \\
&\times \int_{r_1}^{r_0}\int_{\frac{U_0}{r}}^{W_0+\frac{U_1}{r}}\sum_{i=1}^{K}\left[\left(c_{i0}+c_i x^{\omega_i}(rx)^{\xi_i}\right)\frac{\beta_i}{\alpha_i(r)}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i-1}\right] \\
&\times \frac{\beta_R k_R\left(\frac{r}{\alpha_R}\right)^{\beta_R k_R-1}}{\alpha_R\left(1+\left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R+1}}dxdr \\
&+ \left(1-\left(1+\left(\frac{r_0}{\alpha_R}\right)^{-\beta_R}\right)^{-k_R}\right) \\
&\times \int_{r_0}^{\infty}\int_{\frac{U_0}{r}}^{\frac{U_0}{r}+\frac{U_1}{r}}\sum_{i=1}^{K}\left[\left(c_{i0}+c_i x^{\omega_i}(rx)^{\xi_i}\right)\frac{\beta_i}{\alpha_i(r)}\left(\frac{x}{\alpha_i(r)}\right)^{\beta_i-1}\right] \\
&\times \frac{\beta_R k_R\left(\frac{r}{\alpha_R}\right)^{\beta_R k_R-1}}{\alpha_R\left(1+\left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R+1}}dxdr
\end{aligned} \tag{20}
$$

IV. NUMERICAL EXAMPLE ON AUTOMOBILE WARRANTY DATA

Automobile is the most well-known example of a multi-component repairable product with widely offered two-dimensional EW contracts. Therefore, in order to illustrate the proposed model, we use the failure history of a commercial vehicle produced in a plant in Iran to reproduce a numerical example. For studied vehicle, the manufacturer provides two-dimensional base warranty with length $W_0 = 3$ years and total usage $U_0 = 20\ (\times 10^4\ Km)$ and warranty expires when the age or usage of vehicle exceeds three years or 200,000 kilometer respectively, whichever occurs first. During the first two years of the base warranty period i.e. our observation period, from 11200 vehicles under coverage 682 vehicles were failed. For failed vehicles information on VIN, date of sale, date of failure, km at failure, failed component, cause of failure, failed components supplier and etc were recorded. However, no information on service agent, servicing costs and usage environments were available.

Based on the available failure data the following steps are conducted to illustrate the model.

- Preliminary analyze the data and provide some graphical representations to extract needed information in building intended models.
- Estimate the consumers' usage rate distribution function and its parameters
- Categorize the vehicle failures into five main subsystems failures
- Categorize the consumers into three groups based on their usage rate
- Estimate the conditional distributions of the time to the first failures of subsystems
- Model and estimate the studied vehicle's expected servicing cost under different EW contracts
- Analyze the effect of EW's length and usage on the servicing cost and drive iso-warranty cost curves

In the first step, the primary evaluation of failure data showed that the number of vehicles with more than one claim is negligible so they are ignored in subsequent analysis. Then, to visualize data and help in model building, a scatter plot of failure times and corresponding total usages as well as marginal histograms are plotted (see Fig 2).

As one can see, the provided scatter plot shows that age and usage are positively correlated and their marginal histograms are positively skewed. In order to build two-dimensional failure model, the distribution function of consumers' usage rate is estimated. To do this, for failed vehicles usage rates i.e. $r_i = \frac{u_i}{x_i}$ (10^4 Km / year) , $i = 1, \dots, 682$ are calculated where x and u are age and usage of vehicles at the time of failures. Then, fifteen different parametric distributions are fitted on the usage rate data and their adjusted Anderson–Darling (AD) statistics are calculated. These distributions and their ranks in the well fitting of consumers' usage rate based on the AD statistics are presented in Table 1.

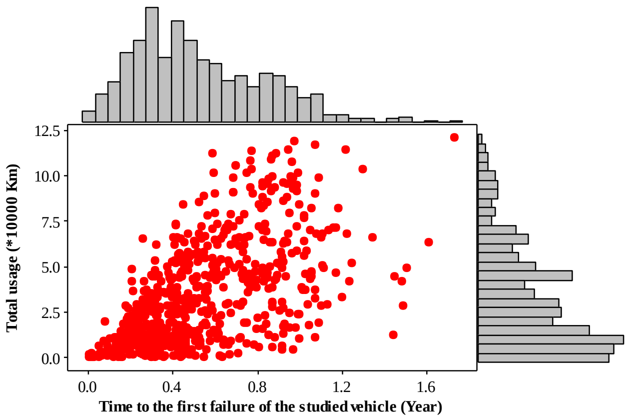

Fig 2: plot and marginal histograms of failure times and usages of the studied vehicle

Table 1: Fitted distributions on consumers' usage rate and their Anderson–Darling statistics

Rank	Distribution	AD (adj)
1	Dagum	0.463
2	Nakagami	2.096
3	3-Parameter Weibull	3.074
4	Generalized extreme value	3.221
5	Weibull	3.565
6	3-Parameter Lognormal	4.085
7	Gamma	4.329
8	3-Parameter Loglogistic	5.393
9	Logistic	7.313
10	Normal	7.326
11	Loglogistic	12.227
12	2-Parameter Exponential	18.943
13	Exponential	19.246
14	Lognormal	20.222
15	Smallest Extreme Value	24.499

As seen in Table 1, Dagum distribution function with scale parameter $\alpha_R = 12.455$ and shape parameters $\beta_R = 6.6652$ and $k_R = 0.13956$ is the best fitted distribution on the consumers' usage rate random variable. In order to validate whether the consumers' usage rate follows from Dagum distribution function or not, Anderson-Darling, Kolmogorov-Smirnov and Chi-Square goodness of fit tests are performed. The results show that, all tests do not reject the null hypothesis, i.e. H_0: consumers' usage rate follows from Dagum distribution, at 0.01 up to 0.2 significant levels. The probability plot of Dagum distribution function with mentioned parameters is depicted in Fig 3.

It is worth to note that in researches reported in the literature, distributions like Uniform, Weibull, lognormal, Gamma and Normal were considered as usage rate distribution functions (see for example [33-36]). However, in the provided case study none of these distributions passed the goodness of fit tests.

As previously stated, a vehicle is a multi-component product composed of thousands parts and modeling the failure process of vehicle at parts level and in terms of their failure

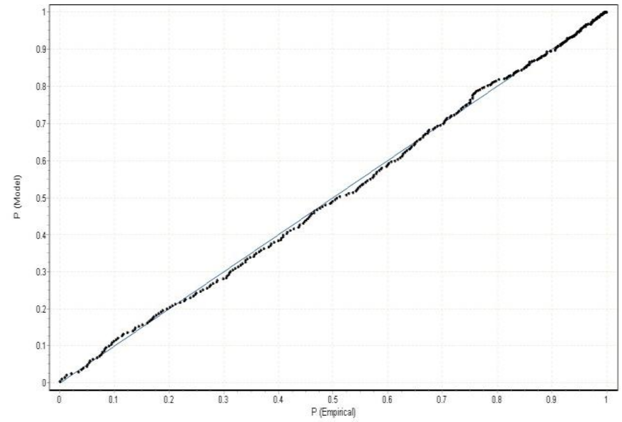

Fig 3: Probability plot of Dagum distribution function

processes are nearly impossible. Therefore, we model the failure process of the vehicle at subsystems level and treat each subsystem as a component. To do this, we consider the vehicle as a series system composed of following five main subsystems: (1) engine, (2) transmission and drive axle, (3) steering, suspension, wheels and brakes, (4) body (including chassis frame and body) and (5) electrical subsystem. Then, regarding the failed component, we categorize vehicle failures into five mentioned subsystems' categories. The scatter plot of the time to the first failures and corresponding usages for these subsystems are depicted in Fig 4.

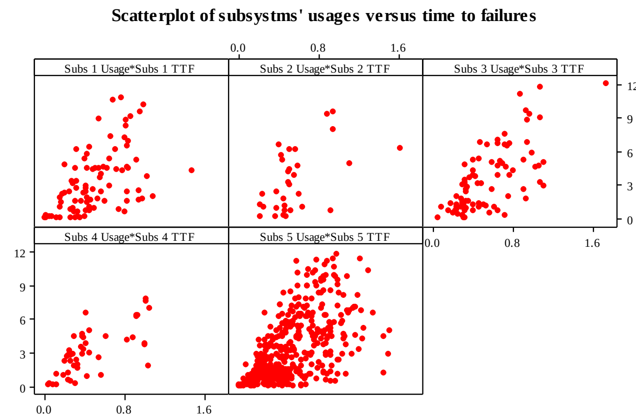

Fig 4: Scatter plots of five main subsystems (horizontal and vertical axes are TTF and usage respectively)

As seen in Fige 4, subsystems' failures are indexed by age and usage and their failure processes should be derived using the two-dimensional failure modeling approaches. To do this, we use marginal approach through a three-step procedure proposed by Baik and Murthy [37]. At the first step, regarding consumers' usage rate they are categorized into the following three groups; light users (LU), normal users (NU) and heavy users (HU). For these groups usage rate intervals are defined and average usage rates and the probabilities that a given consumer falls into one of these groups are calculated (see Table 2).

Table 2: Characteristics of defined light, normal and heavy users

Consumers' group	Usage rate interval ($10^4 Km/Year$)	Average usage rate ($\bar{r}$) ($10^4 Km/Year$)	Probability (p) ($10^4 Km/Year$)
Light users (LU)	$0 < r_{LU} \leq 3$	$\bar{r}_{LU} = 1.4458$	$p_{LU} = 0.2660$
Normal users (NU)	$3 < r_{NU} \leq 9$	$\bar{r}_{NU} = 5.9159$	$p_{NU} = 0.4620$
Heavy users (HU)	$r_{HU} > 9$	$\bar{r}_{HU} = 12.1079$	$p_{HU} = 0.2719$

Then, in the second step considering only failed vehicles during the first two years of the base warranty period, the conditional distributions of the time to the first failures of subsystems regarding usage rate scenarios are estimated. By doing this on each subsystem, 2-parameter Weibull distributions with common shape parameters found to be the best fitted distributions on the time to the first failures of considered subsystem under light, normal and heavy usage rates. For example, the Weibull probability plots of the time to the first failures of electrical subsystem under light, normal and heavy usage rates and their estimated scale parameters as well as common shape parameter are presented in Fig 5.

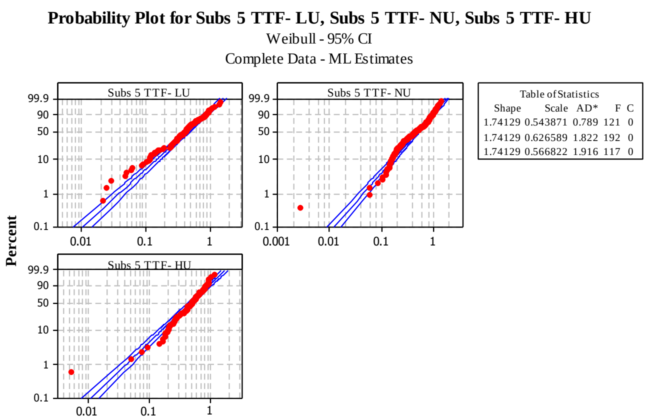

Fig 5: Weibull probability plots of the time to the first failures of electrical subsystem under light, normal and heavy usage rates

However, in the estimation of the time to the first failures distribution functions we have not take into account considerable number of vehicles (i.e. $10518 = 11200 - 682$) that were not failed during our observation period. It is clear that ignoring this censored information may seriously underestimate the true reliability of subsystems and lead to bias and unrealistic results. Therefore, in the third step considering right censored information we update the scale parameters of the estimated time to failures distribution functions. To do this, for ith subsystem $i = 1, \ldots, 5$ we solve equation (21) to obtain an updated estimation of the scale parameter $\hat{\alpha}_i^{(n)}$.

$$\hat{\varphi}_i = \int_0^{\infty} e^{-\left(\frac{\min(\frac{20}{r}, 2)}{\alpha_i^{(n)} \left(\frac{6.66}{r}\right)^{\hat{\gamma}_i}}\right)^{\hat{\beta}_i}} \times \frac{\beta_R k_R \left(\frac{r}{\alpha_R}\right)^{\beta_R k_R - 1}}{\alpha_R \left(1 + \left(\frac{r}{\alpha_R}\right)^{\beta_R}\right)^{k_R + 1}} dr \qquad (21)$$

In equation (21) φ_i is the unconditional reliability of ith subsystem at censored time 2 (years) and its estimated value is $\hat{\varphi}_i = \frac{10518}{11200} \sim 0.94$. α_R and β_R, k_r are respectively scale and shape parameters of consumers usage rate distribution function and as previously presented their values are $\alpha_R = 12.455$, $\beta_R = 6.6652$ and $k_R = 0.13956$. Note that in equation (21), $\alpha_i^{(n)}$ is the only unknown parameter and solving this equation yields a new estimation which takes into account censored information as well. By doing this procedure for five mentioned subsystems, new estimates of scale parameters can be obtained. All estimated parameters of subsystems are presented in Table 3.

Table 3: Estimated parameters of the time to first failures distribution functions and corresponding AFT parameters for five mentioned subsystems of the studied vehicle

Vehicle subsystem	Distribution of the time to the first failure under nominal usage rate ($r_n = 6.66$)	Estimated AFT parameter
Engine	$W(\hat{\alpha}_1^{(n)} = 32.29, \hat{\beta}_1 = 1.78)$	$\gamma_1 = 0.04$
Transmission and drive axle	$W(\hat{\alpha}_2^{(n)} = 23.13, \hat{\beta}_2 = 2.25)$	$\gamma_2 = 0.16$
Steering, Suspension, Wheels and Brakes	$W(\hat{\alpha}_3^{(n)} = 34.52, \hat{\beta}_3 = 1.93)$	$\gamma_3 = 0.10$
Body (chassis frame and body)	$W(\hat{\alpha}_4^{(n)} = 13.19, \hat{\beta}_4 = 1.90)$	$\gamma_4 = 0.31$
Electrical subsystem	$W(\hat{\alpha}_5^{(n)} = 11.95, \hat{\beta}_5 = 1.74)$	$\gamma_5 = 0.07$
Consumers usage rate	$D(\alpha_R = 12.455, \beta_R = 6.6652, k_R = 0.13956)$	

By implementing the steps mentioned above, we have estimated the distribution function of consumers' usage rate and conditional distribution functions of the time to the first failures of five subsystems. In the following, considering assumptions presented in Table 4, we used the proposed cost models in (19) and (20) to estimate expected servicing cost of EW contracts under different combinations of EW's length and total usage limits. These results are presented in Table 5.

Table 4: Base warranty coverage region and parameters of cost functions' regarding failed subsystems

Base warranty characteristics	$W_0 = 3\ (years)\ U_0 = 20\ (10^4 km)$
Base and extended warranties policies	Free repair warranty
Cost functions parameters of subsystems as	$c_0 = (90, 70, 50, 30, 20)$ $c = (140, 12, 100, 75, 25)$ $\omega = (0.75, 0.7, 0.6, 0.5, 0.2)$ $\psi = (0.5, 0.5, 0.35, 0.2, 0.15)$

As seen in Table 5, the expected servicing costs of EW contracts considerably increase when the length and total usage of EW contracts increases. In order to provide a clear insight about the effect of EW's length and total usage on the expected servicing cost of EW contracts, we plot the EW servicing cost surface versus covered length and total usage of EW contracts (see Fig 6).

Table 5: Expected servicing cost of EW contracts with different coverage regions

EW's length (years)	EW's total usage limit (10,000 km)					
	10	20	30	40	50	60
0.5	6.30	8.22	11.31	15.87	22.23	30.70
1	13.19	15.11	18.20	22.77	29.12	37.59
1.5	21.96	23.88	26.97	31.54	37.89	46.36
2	32.85	34.78	37.87	42.43	48.79	57.26
2.5	46.12	48.04	51.13	55.69	62.05	70.52
3	62.00	63.93	67.02	71.58	77.94	86.41
3.5	80.77	82.69	85.79	90.35	96.71	105.17
4	102.69	104.61	107.70	112.26	118.62	127.09
4.5	128.01	129.93	133.03	137.59	143.94	152.41
5	157.02	158.94	162.03	166.60	172.95	181.42

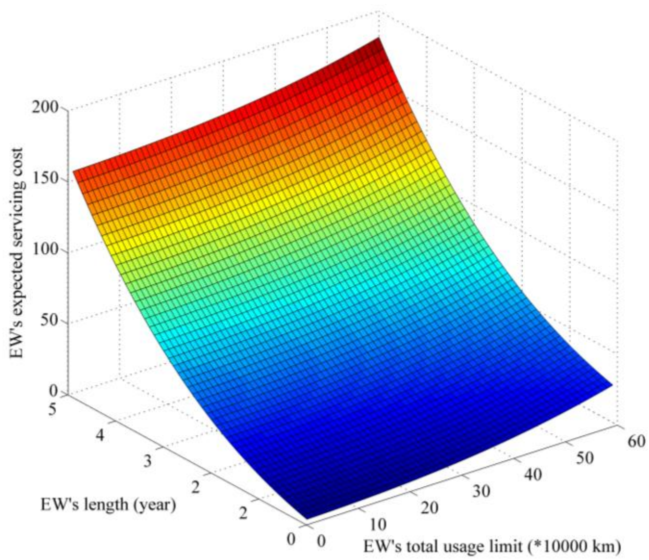

Fig 6: Expected EW's servicing cost surface versus covered length and total usage of EW contract for studied vehicle

Clearly, the provided cost surface will helps the EW provider to analyze the effect of offered EW coverage region on EW's expected servicing cost.

In the analysis of EW's servicing cost, obtaining iso-warranty cost curves is also of great interest to EW providers and consumers Typically, an iso-warranty cost curve includes all combinations of the EW's length and usage limitations which have the same pre-specified expected servicing costs [38]. Obviously, this information may help the EW provider to offer a wide range of two-dimensional EW contracts with the same servicing costs. In order to obtain such curves for a given expected servicing cost, we need to solve either equation (19) or equation (20) regarding r_0 and r_1 values and derive all possible combinations of EW's length and total usage. For the case at hand, due to complexity of these equations deriving closed form relation between EW's length and total usage are nearly impossible. Therefore, for limited expected servicing costs we obtained EW's iso-warranty cost curves using the numerical computations. For studied vehicle, iso-warranty cost curves are depicted in Fig 7.

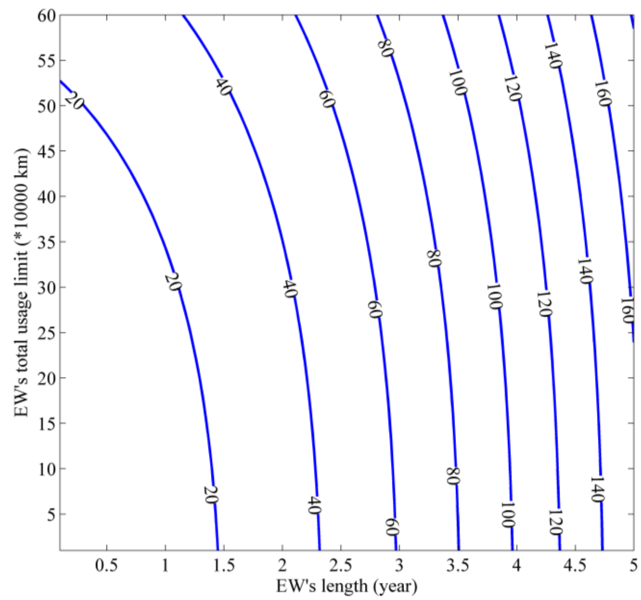

Fig 7: Iso-warranty cost curves for studied vehicle (expected servicing costs are shown on the arcs)

if you wish to use units of T, either refer to magnetic flux density B or magnetic field strength symbolized as $\mu_0 H$. Use the center dot to separate compound units, e.g., "A·m^2."

V. Conclusion and Future Research Directions

In this paper, for the first time in the EW literature, the failure process of a product sold with two-dimensional warranty, like an automobile, was modeled as a multi-component repairable system. The product assumed to be composed of a series of heterogeneous components with conditional independent failure processes. Then, the effect of consumers' usage rate on the failure processes of components and in turn on the product failures were studied. Besides, assuming repair cost as function of failed component as well as its age and usage at the time of failure, the expected servicing cost of the product during the EW period is derived.

The failure history of a commercial vehicle during the two first years of the base warranty period are also used to derive consumers usage rate distribution function as well as the components' time to the first failures distribution functions. The proposed cost model is applied on the provided case study to estimate the EW servicing cost for a vehicle under different EW contracts. The proposed cost model not only found to be useful in the estimation of the EW servicing cost, but it also successfully used to obtain iso-warranty cost curves. Cleary, deriving mathematical formulations and plots of iso-warranty cost curves for given expected servicing costs may considerably helps the EW provider to offer a wide range of EW contracts with the same servicing costs.

In the present paper, a two-dimensional failure model and expected servicing cost of a multi-component product during the EW period were proposed. However, there are many important un-answered questions where need to be investigated in the future researches. These research areas include, but are not limited to: considering stochastic, economic or structural dependency between product components, modeling and optimizing preventive maintenance strategy during the EW period as well as modeling and optimizing inspection policies, considering repair as an imperfect repair and etc. Currently, the authors work on the development of some models on this area and they will be reported elsewhere.

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