Analysis of Group Performance Using Common Weights

A. Payan and B. Rahmani Parchikolaei

***Abstract*—**This article is aimed at analyzing group performance based on common weights. The recommended methods in this area are just capable of comparing the groups under evaluation and are not capable of measuring group performance. To evaluate group performance, in addition to group efficiency, some other indicators are required as efficiency of each unit in the related group and the efficiency spread. In this way, none of the methods have considered all these cases for group efficiency analysis. In this study, a method is presented in order to analyze group performance using common weights method to be able to estimate all the mentioned indices. Moreover, the recommended approach will be compared with the current methods by an illustrative example. At the end, the method will be applied for the evaluation of group efficiency performance in banks.

***Keywords*—**Performance analysis, Group efficiency, Efficiency spread, Common weights.

I. Introduction

SINCE the introduction of data envelopment analysis (DEA) by Charnes et al. [1] and its consequent development by Banker et al. [2], new branches have been developed in this particular scientific field most of which were abstract concepts from other sciences whose transition to measurable mathematical expression was made possible by data envelopment analysis. Ranking, return to scale, efficiency, cost efficiency, group or program performance, etc. are some examples of abstract concepts mentioned above. For further information refer to the article by Cook and Seiford [3]. Group or program performance consists of assessment of executed program within a system which includes homogenous decision making units (DMU). The university colleges provide the ideal groups to be assessed, in which the decision making units are their departments. It is also possible to consider bank branches in different regions of urban location as groups of decision making units.

Although wide spread studies have not been conducted on group performance, but some of the following may be pointed out as relevant. Comanho and Dyson [4] presented a method for pairwise comparison of groups that were subjects of evaluation based on Malmquist productivity index. Even though this method has considerable economic analysis benefits such as calculation of efficiency spread of groups, its incapability to make a relevant comparison of entire groups, which is one of the more positive sides of data envelopment analysis, is its most serious drawback. Also this method does not provide score for determining the overall efficiency of groups. Another method is the idea of common weights which Cook and Zhu [5] proposed to calculate the performance of each group and the units within each one, and applied the method to analyze the performance of power plants as groups of DMUs, in Canada. In this method, the efficiency of each unit within each group is identified, but a suitable outcome is not obtained for the overall efficiency of groups and the proposed index is solely for the relevant comparison of groups. In his article "Group performance evaluation, an application of data envelopment analysis" Bagharzadeh Valami [6] presented an index for comparing and ranking groups and used this index to compare the performance of various bank branches under different regions. However this method cannot be utilized to compute the overall efficiency of groups and the efficiency of units within them. For this purpose only geometric means of CCR efficiency of units up to specific efficiency frontiers are used to evaluate group performance. The advantage of this method is in the simplicity of performing its calculations and the ease by which they are comprehended. When comparing the two, it can readily be seen that the Comanho and Dyson method [4] provides a more comprehensive set of information as compared to group performance evaluation method presented by Bagharzadeh Valami [6]. The proposed method in this article is to evaluate group performance using the common weights concept which is able to compute the overall efficiency of each group and the units contained within them in comparison to other groups as well as the efficiency spread of groups. A difference between the method proposed by Cook and Zhu [5] and the method proposed by this paper is that the Cook and Zhu method [5] uses separate common weights to evaluate the performance of different groups and the efficiency of units contained within them while the proposed method of this article uses a single set of common weights to evaluate the performance of all groups and their relevant units.

In this article first an overall review on the conducted studies on group efficiency is done and then the preliminary idea for computing the overall efficiency of each group and the units within each one is presented. Afterwards, a method

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to calculate efficiency spread based on the amount of inter-group efficiency is proposed. In order to solve the presented model, a linear programming process is described that can produce an efficient solution for the multi-objective programming problem which was presented at the outset of this section. In Section V an example is solved using the method proposed by this article, the results for which will be compared to the results obtained using available methods in the literature and finally a practical example for the bank branches in Iran is presented.

II. Preliminaries

Consider the following multi-objective linear fractional programming problem as:

$$\begin{aligned} &\max\ z_k(x) = (c_k x + \alpha_k)/(d_k x + \beta_k),\ \ k = 1,\dots,K, \\ &s.t\ \ x \in S = \{x \mid Ax \le b, x \ge 0\} \end{aligned} \tag{1}$$

such that $\alpha_k, \beta_k\ (k = 1,\dots,K)$ are scalar and S is a nonempty set, and $b \in R^m, x, c_k, d_k \in R^n$, and A is an $m \times n$ matrix and z_k is the kth objective function.

Definition1. $\bar{x} \in S$ is called an efficient solution of model (1) if and only if there is no x, such that $x \in S, x \ne \bar{x}$ and $z_k(x) \ge z_k(\bar{x})(k = 1,\dots,K)$ and at least for one k, $z_k(x) > z_k(\bar{x})$.

Definition2. $x^* \in S$ is called a complete solution of model (1) if and only if for all x, such that $x \in S, x \ne x^*$ and $z_k(x^*) \ge z_k(x)(k = 1,\dots,K)$

III. Literature Review

A. Group Performance Based on Productivity Index

Using Malmquist productivity index for group performance was firstly suggested by Comanho and Dyson [4]. Assume that there are N groups available to be evaluated and group i has δ_i DMUs $(i = 1,\dots,N)$; All units within different groups are homogenous and by consuming m inputs yield s outputs.

Let us compare groups A and B which has δ_A and δ_B units respectively. The performance comparison of the two groups is as follows:

$$I_{adj}^{A,B} = IE^{A,B} \times IF_{adj}^{A,B} \tag{2}$$

where

$$IE^{A,B} = \left(\prod_{j=1}^{\delta_A} D^A\left(x_j^A, y_j^A\right)\right)^{\frac{1}{\delta_A}} \Bigg/ \left(\prod_{j=1}^{\delta_B} D^B\left(x_j^B, y_j^B\right)\right)^{\frac{1}{\delta_B}} \tag{3}$$

$$IF_{adj}^{A,B} = \left[\prod_{i=1}^{N}\left(\prod_{j=1}^{\delta_i} D^B\left(x_j^i, y_j^i\right) \Bigg/ \prod_{j=1}^{\delta_i} D^A\left(x_j^i, y_j^i\right)\right)^{\frac{1}{\delta_i}}\right]^{\frac{1}{N}} \tag{4}$$

in which $D^A(x^i, y^i)$ is the efficiency of unit (x, y) in group i with respect to the technology of group A and has been obtained by utilizing the following:

$$\begin{aligned} &D^A(x^i, y^i) = \min\ \theta, \\ &s.t\ \sum_{j=1}^{\delta_A} \lambda_j x_j^A \le \theta x^i, \\ &\sum_{j=1}^{\delta_A} \lambda_j y_j^A \ge y^i, \\ &\lambda_j \ge 0,\ \ j = 1,\dots,\delta_A \end{aligned} \tag{5}$$

Ratio (3) compares efficiency spread of groups A and B, and (4) is the productivity gap between the technologies of the two groups. Therefore it can be concluded that the best performance is greatly influenced by two factors: (a) less dispersion and (b) technology domination. If the calculated index in (2) is less than the unit, then A has a better performance than B.

The value $IE^{A,B}$ less than one indicates a low efficiency spread. In other words if $IE^{A,B} < 1$ then this means that the efficiency of units within group A is closer than the efficiency of those in group B so compatibility of overall efficiency in group A is at a higher level than group B. In this way units in group A have a more homogenous performance as compared to the ones in group B. Formula (4) makes the comparison of technologies of the two groups possible and the value $IF_{adj}^{A,B}$ less than one indicates a higher productivity at the technology of group A compared to the technology of group B.

B. Group Performance Based on Common Weights

Cook and Zhu [5] proposed utilizing the common set of weights method for evaluating group performance. Fractional CCR model for evaluation of unit p from group k is as follows:

$$\begin{aligned} &\max\ uy_p^k / vx_p^k, \\ &s.t\ uy_j^i / vx_j^i \le 1,\ \ j = 1,\dots,\delta_i,\ i = 1,\dots,N, \\ &u \ge 0, \\ &v \ge 0 \end{aligned} \tag{6}$$

Assume that θ_p^k stands for the efficiency of unit p from group k then the coefficients (v^k, u^k) used for the evaluation of group k must be determined in such manner that the yielded efficiency of these coefficients has the least amount of distance to the CCR efficiency of units in group k. Therefore they proposed the following multi-objective linear fractional programming problem for evaluating the performance of group k in which $\theta_1^k,\dots,\theta_{\delta_k}^k$ are considered as the goals of the objectives:

$$\begin{aligned}
&\max\ uy_j^k \big/ vx_j^k,\ \ j=1,\dots,\delta_k,\\
&s.t\ \ uy_j^i \big/ vx_j^i \le 1,\ \ j=1,\dots,\delta_i,\ \ i=1,\dots,N,\\
&u\ge 0,\\
&v\ge 0
\end{aligned} \tag{7}$$

The goal programming equivalent to model (8) using infinity norm is as follows:

$$\begin{aligned}
&\min\ d^k,\\
&s.t\ \ uy_j^k \big/ vx_j^k + d^k \ge \theta_j^k,\ \ j=1,\dots,\delta_k,\\
&uy_j^i \big/ vx_j^i \le 1,\ \ j=1,\dots,\delta_i,\ \ i=1,\dots,N,\\
&u\ge 0,\\
&v\ge 0,\\
&d^k\ge 0
\end{aligned} \tag{8}$$

Let (v^k,u^k) be the optimal solution for model (8). The optimal value of the objective function of model (8) is considered as a score to evaluate the performance of group k and the efficiency of units in group k are determined through multipliers (v^k,u^k).

C. Group Performance Based on Group Frontier

Bagharzadeh Valami [6] presented an index for group performance in 2009 based on the comparison of entire units in all groups with the efficient frontier of evaluating the group. Consider the production possibility set as follows:

$$T_c=\left\{(x,y)\mid x\ge\sum_{j=1}^{n}\lambda_j x_j,\ y\le\sum_{j=1}^{n}\lambda_j y_j,\ \lambda_j\ge 0,\ \ j=1,\dots,n\right\}$$

Output-oriented CCR model based on the above production possibility set is as follows:

$$\begin{aligned}
&\max\ \varphi,\\
&s.t\ \ \sum_{j=1}^{n}\lambda_j\, x_{ij}\le x_{io},\ \ i=1,\dots,m,\\
&\sum_{j=1}^{n}\lambda_j\, y_{rj}\ge \varphi\, y_{ro},\ \ r=1,\dots,s,\\
&\lambda_j\ge 0,\ \ j=1,\dots,n
\end{aligned} \tag{9}$$

Assume that A is a subset of DMUs which has been considered as a group of DMUs. Then the production possibility set based on A will be:

$$T_c^A=\left\{(x,y)\mid x\ge\sum_{j\in A}\lambda_j x_j,\ y\le\sum_{j\in A}\lambda_j y_j,\ \lambda_j\ge 0,\ \ j=1,\dots,n\right\}$$

The distance of unit (x,y) to T_c^A is obtained as follows,

$$\begin{aligned}
&D^A(x,y)=\max\varphi,\\
&s.t\ \ \sum_{j\in A}\lambda_j\, x_{ij}\le x_i,\ \ i=1,\dots,m,\\
&\sum_{j\in A}\lambda_j\, y_{rj}\ge\varphi\, y_r,\ \ r=1,\dots,s,\\
&\lambda_j\ge 0,\ \ j\in A
\end{aligned} \tag{10}$$

and group performance of group A is calculated by the following formula:

$$P(A)=\left(\prod_{j=1}^{n} D^A(x_j,y_j)\right)^{\frac{1}{n}} \tag{11}$$

Accordingly the performance of group A is measured based on geometric mean of efficiency of all units with respect to production possibility set of group A. It can therefore be concluded that group performance of A is better than group that of B if $P(A)>P(B)$.

IV. Common Weights Model for Analyzing Group Performance

Assume that $v=(v_1,\dots,v_m)$ and $u=(u_1,\dots,u_s)$ are respectively input and output weight vectors. Then the efficiency of unit j in group h is obtained as outlined below:

$$E_j^h=uy_j^h\big/vx_j^h=\sum_{r=1}^{s}u_r\,y_{rj}^h\Big/\sum_{i=1}^{m}v_i\ x_{ij}^h,\ j=1,\dots,\delta_h,\ h=1,\dots,N \tag{12}$$

The overall efficiency of group h can be defined as convex combination of efficiency of whole units within that group and so we have,

$$E_{G_h}=\sum_{j=1}^{\delta_h}w_j^h E_j^h,\ \ h=1,\dots,N \tag{13}$$

In other words, the performance of group h is the product of the efficiency of all the units within it. From a statistical viewpoint, E_{G_h} is the weighted mean of the efficiency of units in group h and from an economic viewpoint, w_j^h is the share of unit j in group h to calculate the overall efficiency of group h.

Then following multi-objective programming problem is used as a common weights model to calculate the overall efficiency of groups as:

$$\begin{aligned}&\max\ E_{G_h},\ h=1,...,N,\\&s.t\ E_j^h \le 1,\ j=1,...,\delta_h,\ h=1,...,N,\\&u \ge 0,\\&v \ge 0\end{aligned} \tag{14}$$

In order to solve the above-mentioned problem, more detailed arguments shall be put later on. Consider v^* and u^* are the optimal weights yielded from the above model, the overall efficiency of groups and their relevant inner units are obtained as follows:

$$\begin{aligned}E_j^{h^*} &= u^* y_j^h \big/ v^* x_j^h\\&= \sum_{r=1}^{s} u_r^{\ *} y_{rj}^h \bigg/ \sum_{i=1}^{m} v_i^{\ *} x_{ij}^h,\ j=1,...,\delta_h, h=1,...,N\end{aligned} \tag{15}$$

$$E_{G_h}^{\ *} = \sum_{j=1}^{\delta_h} w_j^h E_j^{h^*},\ \ h=1,...,N \tag{16}$$

A. Performance Analysis

With due attention to management issues, an effective management is a well-balanced one which has high performance. According to model (14), performance for each group can be calculated by estimating the score of its efficiency, but in order to evaluate a well-balanced management it should be examined whether different parts of a group (DMUs of groups) have similar performance or not. If the answer is affirmative, then it can be claimed that the performance of the group is balanced, otherwise if some units of group have a higher level of performance while other units show low performance, overall group performance is not balanced. The existence of such situation results in consequential damages which in many instances are not recoverable. Compensation for weak performance of an inefficient unit with low performance requires immense time and expenditure, but if done in an appropriate manner, it can lower the costs involved.

According to the points outlined above dispersion value of efficiency of units within a group indicate a well-balanced or unbalanced management system. The lower the dispersion level, the more balanced are the levels of management and implementation. Based on the dispersion level which was called within-group efficiency spread by Comanho and Dyson [4] the following index to determine the level of management balance in groups is presented.

Assume that $E_1^{h^*},...,E_{\delta_h}^{h^*}$ signifies the efficiency of units in group h then $e_{pj}^h = \left|E_p^{h^*} - E_j^{h^*}\right|$ indicates the performance gap of unit p and unit j. In such case we will have:

$$R_h = \delta_h - 1 + \delta_h - 2 + ... + 2 + 1 = \sum_{t=1}^{\delta_h - 1} t = ((\delta_h)(\delta_h - 1))/2$$

as the number of such gaps and efficiency spread of group h can be explained that

$$e^h = \frac{1}{R_h} \sum_{p=1}^{\delta_h} \sum_{j=p+1}^{\delta_h} e_{pj}^h \tag{17}$$

In this relation e^h is the mean of efficiency gaps of units in group h. It is obvious that if e^h has the tendency to move towards zero then the efficiencies of DMUs in group h is extremely close. The larger e^h becomes, the bigger the efficiency gaps for units in group h turn, in fact the smaller the efficiency spread is, the higher is the uniformity and balance of performance. Formula (17) is comparable to IE index of formula (4) in Comanho and Dyson's method [4].

B. A Method for Solving the Proposed Model

Consider model (15), it can be outlined as shown below:

$$\begin{aligned}&\max\ \sum_{j=1}^{\delta_h} w_j^h \left(u y_j^h \big/ v x_j^h\right),\ h=1,...,N,\\&s.t\ u y_j^h \big/ v x_j^h \le 1,\ \ j=1,...,\delta_h,\ h=1,...,N,\\&u \ge 0,\\&v \ge 0\end{aligned} \tag{18}$$

Assume that,

$$w_j^h = v x_j^h \bigg/ v \sum_{p=1}^{\delta_h} x_p^h,\ \ j=1,...,\delta_h,\ h=1,...,N \tag{19}$$

then we have,

$$\begin{aligned}E_{G_h} &= \sum_{j=1}^{\delta_h} w_j^h E_j^h = \sum_{j=1}^{\delta_h} w_j^h \left(u y_j^h / v x_j^h\right) = \sum_{j=1}^{\delta_h} \left(v x_j^h \bigg/ v \sum_{p=1}^{\delta_h} x_p^h \right)\left(u y_j^h / v x_j^h\right)\\&= u \sum_{j=1}^{\delta_h} y_j^h \bigg/ v \sum_{j=1}^{\delta_h} x_j^h,\ \ h=1,...,N\end{aligned} \tag{20}$$

Three separate analyses of the above discussion can be conducted:

1. E_{G_h} is the weighted-sum of outputs to that of inputs which is the same as efficiency concept of data envelopment analysis. In other words, group h is considered as a DMU

whose outputs and inputs are respectively the collective outputs and inputs of all units in group h.

2. E_{G_h} is a linear fractional function. We know that the properties of problems with linear fractional objective functions are very close to linear programming problems. According to implemented changes, problem (18) is transformed to a multi-objective linear fractional programming problem. Several reliable techniques with proper calculation timing to solve and find solution are available for such problems.

3. w_j^h which is the contribution of unit j $\left(j=1,\ldots,\delta_h\right)$ of group $h\left(h=1,\ldots,N\right)$ in calculating the overall efficiency of the group is obtained based on the optimal solution of the problem. Moreover the fractional denomination in w_j^h for various units of one group is the same. The higher is this value for each unit, the more is the role of the unit in calculating the overall efficiency of a group. As it is known the weighted-sum of inputs within the economic concepts is cost of unit. So the more cost in a group, the more share in the overall efficiency of the group and of course from the economic outlook this is greatly meaningful.

Therefore based on the explanation presented, non-linear multi-objective programming (18) can be transformed into a multi-objective linear fractional programming problem as follows:

$$
\begin{aligned}
&\max\ u\sum_{j=1}^{\delta_h} y_j^h \Big/ v\sum_{j=1}^{\delta_h} x_j^h,\ h=1,\ldots,N,\\
&s.t\ \ uy_j^h \big/ vx_j^h \le 1,\ \ j=1,\ldots,\delta_h,\ h=1,\ldots,N,\\
&u\ge 0,\\
&v\ge 0
\end{aligned}
\tag{21}
$$

Here a method to calculate and obtain an efficient solution for the above problem is presented. Based on the solution obtained, the overall efficiency of each group and the efficiency of units in that group as well as the weight for each unit and efficiency spread can be calculated using formulae (16), (15), (19) and (17), respectively.

From the constraints of problem (18) we have,

$$
\begin{aligned}
&E_j^h \le 1,\ \ j=1,\ldots,\delta_h,\ h=1,\ldots,N \Rightarrow w_j^h\,E_j^h \le w_j^h\\
&\Rightarrow \sum_{j=1}^{\delta_h} w_j^h\,E_j^h \le \sum_{j=1}^{\delta_h} w_j^h \Rightarrow E_{G_h} \le 1,\ h=1,\ldots,N
\end{aligned}
$$

So the number one could be considered as the cause for target. Accordingly we have

$$
E_{G_h} = u\sum_{j=1}^{\delta_h} y_j^h \Big/ v\sum_{j=1}^{\delta_h} x_j^h \le 1 \Rightarrow u\sum_{j=1}^{\delta_h} y_j^h \le v\sum_{j=1}^{\delta_h} x_j^h,\ \ h=1,\ldots,N,
$$

and so there is $\theta_h \ge 0$ which

$$
u\sum_{j=1}^{\delta_h} y_j^h \le \theta_h,\ \theta_h \le v\sum_{j=1}^{\delta_h} x_j^h,\ \ h=1,\ldots,N
$$

So $d_h^+, d_h^- \ge 0$ are that

$$
u\sum_{j=1}^{\delta_h} y_j^h + d_h^- = \theta_h,\ \ v\sum_{j=1}^{\delta_h} x_j^h - d_h^+ = \theta_h,\ \ h=1,\ldots,N
$$

It is apparent that if the value $d_h^+ + d_h^-$ is zero then E_{G_h} equals unity which means the overall efficiency of group h has reached its peak (100% efficiency). But if at least one of d_h^- or d_h^+ is larger than zero or equivalently if $d_h^+ + d_h^-$ is positive then E_{G_h} will not reach its maximum peak. In a situation like this group h is inefficient. To calculate the maximum overall efficiency of group h or to reduce its gap to the one, $d_h^+ + d_h^-$ can be minimized. Model (21) therefore will be transformed to the multi-objective linear programming problem outlined herein below:

$$
\begin{aligned}
&\min\ \ d_h^+ + d_h^-,\ \ h=1,\ldots,N,\\
&s.t\ \ u\sum_{j=1}^{\delta_h} y_j^h + d_h^- = \theta_h,\ \ h=1,\ldots,N,\\
&v\sum_{j=1}^{\delta_h} x_j^h - d_h^+ = \theta_h,\ \ h=1,\ldots,N,\\
&uy_j^h - vx_j^h \le 0,\ \ j=1,\ldots,\delta_h,\ h=1,\ldots,N,\\
&u\ge 0,\\
&v\ge 0,\\
&d_h^+, d_h^- \ge 0,\ \ h=1,\ldots,N
\end{aligned}
\tag{22}
$$

which in turn can be transformed into the following linear programming problem:

$$
\begin{aligned}
&\min\ \ \sum_{h=1}^{N} d_h^+ + d_h^-,\\
&s.t\ \ u\sum_{j=1}^{\delta_h} y_j^h + d_h^- = \theta_h,\ \ h=1,\ldots,N,\\
&v\sum_{j=1}^{\delta_h} x_j^h - d_h^+ = \theta_h,\ \ h=1,\ldots,N,\\
&uy_j^h - vx_j^h \le 0,\ \ j=1,\ldots,\delta_h,\ h=1,\ldots,N,\\
&v\sum_{h=1}^{N}\sum_{j=1}^{\delta_h} x_j^h = N,\\
&u\ge 0,
\end{aligned}
$$

$$v \geq 0,$$
$$d_h^+, d_h^- \geq 0, \quad h = 1,\ldots,N \tag{23}$$

Constraint $v\sum_{h=1}^{N}\sum_{j=1}^{\delta_h} x_j^h = N$, has been added to the model as a normalized constraint to prevent zero weights.

Suppose that $(v^*, u^*) = (v_1^*,\ldots,v_m^*,u_1^*,\ldots,u_s^*)$ is the optimal weights of the above model with the optimal value z^*.

Theorem1. If z^* equals zero then (v^*, u^*) is the complete solution to the multi-objective linear fractional programming problem (21).

Proof: Since the objective function of model (23) is a summation of non-negative variables, if the optimum value of the problem equals zero then all the variables appearing in objective function will be zero. Therefore we have $d_h^+ = 0, d_h^- = 0\,(h = 1,\ldots,N)$. So,

$$u^*\sum_{j=1}^{\delta_h} y_j^h = \theta_h^*, \; v^*\sum_{j=1}^{\delta_h} x_j^h = \theta_h^*, \quad h = 1,\ldots,N$$
$$\Rightarrow u^*\sum_{j=1}^{\delta_h} y_j^h \Big/ v^*\sum_{j=1}^{\delta_h} x_j^h = 1, \; h = 1,\ldots,N$$

On the other hand the number one is the supremum of the objective functions of model (21) so the obtained solution (v^*, u^*) is the complete solution for model (21).

In an event that the optimal value of the above problem is larger than zero the model below which was first used by Hosseinzadeh Lotfi et al. [7] to test the efficiency in multi-objective linear fractional programming may be utilized to test the efficiency of the optimal solution of model (23):

$$\max \; \sum_{h=1}^{N} d_h^+ + d_h^-,$$
$$s.t \; u\sum_{j=1}^{\delta_h} y_j^h - d_h^- = \theta_h m_h, \; h = 1,\ldots,N,$$
$$v\sum_{j=1}^{\delta_h} x_j^h + d_h^+ = \theta_h n_h, \quad h = 1,\ldots,N,$$
$$u y_j^h - v x_j^h \leq 0, \; j = 1,\ldots,\delta_h, \; h = 1,\ldots,N,$$
$$v\sum_{h=1}^{N}\sum_{j=1}^{\delta_h} x_j^h = N,$$
$$u \geq 0,$$
$$v \geq 0,$$
$$d_h^+, d_h^- \geq 0, \quad h = 1,\ldots,N \tag{24}$$

in which

$$m_h = u^*\sum_{j=1}^{\delta_h} y_j^h, \; n_h = v^*\sum_{j=1}^{\delta_h} x_j^h, \; h = 1,\ldots,N$$

If the optimal value of the above problem equals zero, (v^*, u^*) is efficient solution of (21) [7], otherwise suppose that $(\bar{v}, \bar{u})$ is the optimal solution of the above problem with the objective function value is non-zero then the above model can be used to test the efficiency of $(\bar{v}, \bar{u})$. Let that $(v^1, u^1), (v^2, u^2), \ldots, (v^l, u^l)$ is the continuum of the solutions obtained from the model (24) which their optimum objective function values does not equal zero. Suppose that for (v^t, u^t) and inserting $m_h^t = u^t\sum_{j=1}^{\delta_h} y_j^h, \; n_h^t = v^t\sum_{j=1}^{\delta_h} x_j^h \; (h = 1,\ldots,N)$ in model (24) the optimal solution (v^{t+1}, u^{t+1}) with non-zero optimal value of objective function is obtained. So we have

$$\left.\begin{aligned} u^{t+1}\sum_{j=1}^{\delta_h} y_j^h - d_h^{t+1^-} = \theta_h^{t+1} m_h^t, \; h = 1,\ldots,N \\ v^{t+1}\sum_{j=1}^{\delta_h} x_j^h + d_h^{t+1^+} = \theta_h^{t+1} n_h^t, \; h = 1,\ldots,N \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} u^{t+1}\sum_{j=1}^{\delta_h} y_j^h \geq \theta_h^{t+1} m_h^t, \; h = 1,\ldots,N \\ v^{t+1}\sum_{j=1}^{\delta_h} x_j^h \leq \theta_h^{t+1} n_h^t, \; h = 1,\ldots,N \end{aligned}\right\} \Rightarrow$$

$$u^{t+1}\sum_{j=1}^{\delta_h} y_j^h \Big/ v^{t+1}\sum_{j=1}^{\delta_h} x_j^h \geq m_h^t / n_h^t = u^t\sum_{j=1}^{\delta_h} y_j^h \Big/ v^t\sum_{j=1}^{\delta_h} x_j^h$$

Because the optimal value of objective function is positive then at least one of $d_h^{t+1^-}$ or $d_h^{t+1^+}\,(h = 1,\ldots,N)$ is positive yielding:

$$u^{t+1}\sum_{j=1}^{\delta_h} y_j^h \Big/ v^{t+1}\sum_{j=1}^{\delta_h} x_j^h \geq u^t\sum_{j=1}^{\delta_h} y_j^h \Big/ v^t\sum_{j=1}^{\delta_h} x_j^h, \; h = 1,\ldots,N,$$

and there is the strict inequality for at least one h, this point shows that (v^t, u^t) is not the efficient solution for model (21). On the other hand $u\sum_{j=1}^{\delta_h} y_j^h \Big/ v\sum_{j=1}^{\delta_h} x_j^h\,(h = 1,\ldots,N)$ to the number one is limited and for each repetition at least one of them is larger and other ratios do not deteriorate. Therefore in the worst case a solution like $(\tilde{v}, \tilde{u})$ is obtained which $\tilde{u}\sum_{j=1}^{\delta_h} y_j^h \Big/ \tilde{v}\sum_{j=1}^{\delta_h} x_j^h = 1\,(h = 1,\ldots,N)$ so $(\tilde{v}, \tilde{u})$ is the complete solution for multi-objective linear fractional programming problem (21). We could use the following algorithm to calculate an efficient solution to the multi-objective linear fractional programming problem (21).

Step1: Solve model (23) and name its optimal solution as (v^1, u^1) and its optimal value as z_1^* and move on to the next step.

Step2: If we have $z_1^* = 0$ then (v^1, u^1) is efficient solution, otherwise put $t = 1$ and move on to the next step.

Step3: Solve linear programming problem (24) by utilizing $m_h^t = u^t \sum_{j=1}^{\delta_h} y_j^h,\ n_h^t = v^t \sum_{j=1}^{\delta_h} x_j^h\ (h = 1,\ldots,N)$ and named its optimal solution (v^{t+1}, u^{t+1}) and its optimal value as z_{t+1}^* and move on to the next step.

Step 4: If we have $z_{t+1}^* = 0$ then $(v^{t+1}, u^{t+1})$is efficient, otherwise put $t = t+1$ and go back to step 3.

V. Example

Two examples are given to clarify this argument. The first example compares the new method with the existing group performance methods. For the other example which is a practical example, the capabilities of the recommended methods are used to evaluate bank branches as groups of DMUs.

A. *Illustrative Example*

Here an example to compare the aforementioned methods is presented. Data have been extracted from Bagahrzadeh Valami [6]. Consider one input and two outputs according to the following Table I.

Table I: Data for eight units

DMU	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6	DMU7	DMU8
I	1	1	1	1	1	1	1	1
O1	4	5	2	6	3	4.5	6	5
O2	0.5	5	3	3	6	2	1	4

Suppose there are three groups being assessed which are represented with *A*, *B* and *C* and are as:

$$A = \{DMU_5, DMU_6\},$$
$$B = \{DMU_6, DMU_7\},$$
$$C = \{DMU_7, DMU_8\}$$

Farrell frontier yielded from all units and also frontiers for three groups and the position of all DMUs related to these frontiers are provided in following Figures 1 to 5.

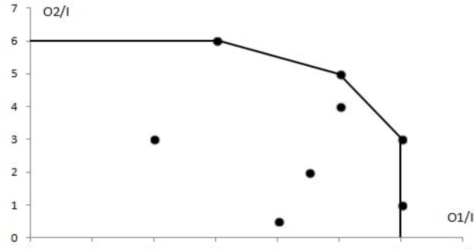

Fig. 1 Farrell frontier yielded from all units.

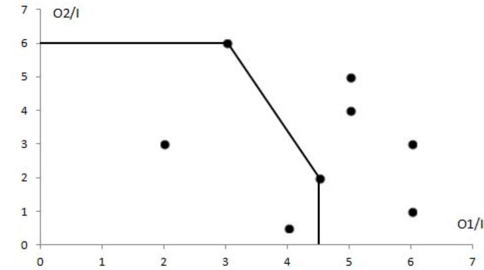

Fig. 2 Farrell frontier yielded from group *A*.

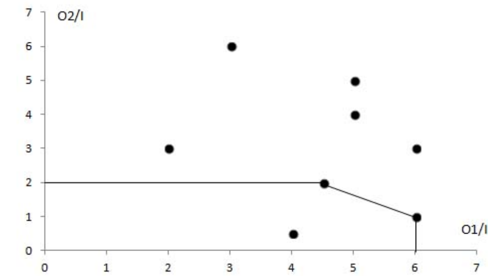

Fig. 3 Farrell frontier yielded from group *B*.

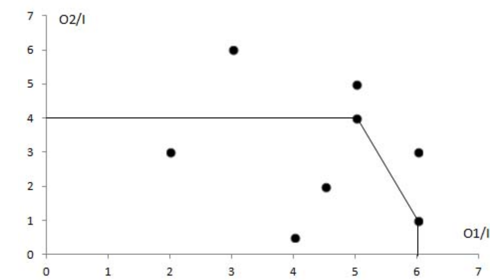

Fig. 4 Farrell frontier yielded from group *C*.

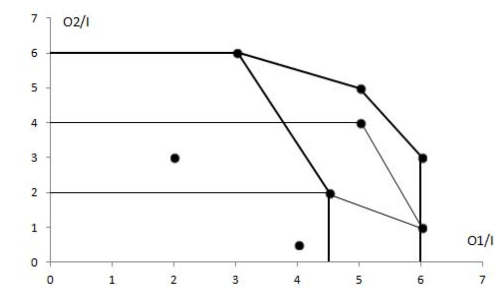

Fig. 5 Comparing all Farrell frontiers in Figure 1 to 4.

Through utilizing model (23) the weight vector $(v_1, u_1, u_2) = (0.75, 0.6, 0.3)$ is obtained. The optimal value of the model based on the solution is 0.8 so the obtained solution is not a complete solution. Because optimal value of model (24) is zero then based on the data represented in Table I the optimal solution of this problem which is $(v_1, u_1, u_2) = (0.75, 0.6, 0.3)$ will be an efficient solution for problem (21). Table II represents contribution of each unit within groups, the efficiency of those within their relevant group, the overall efficiency of groups and the efficiency spread for the three groups.

Table II: Analysis group performance by using the proposed method.

Group	DMU	Unit Contribution	Unit Efficiency	Group Efficiency	Efficiency Spread
A	5	0.5	0.8	0.767	0.0667
	6	0.5	0.733		
B	6	0.5	0.733	0.8	0.133
	7	0.5	0.867		
C	7	0.5	0.867	0.9	0.0667
	8	0.5	0.933		

As it can be seen in the second column of the above Table II, the weight of each unit in corresponding group which is used to calculate group efficiency is equal to 0.5. Since the number of units of each group is two, the efficiency of groups is precisely the arithmetic mean of the efficiency of units inside those groups. Additionally by using the data in the third column, the units inside each group can easily be compared and ranked which is one of the advantages of the method mentioned in this paper. Overall group efficiency can conveniently be calculated based on the data represented in the above mentioned columns as shown in column four of Table II. Therefore, group C is ranked first and groups B and A are ranked second and third respectively. The last column in Table II signifies the efficiency spread for groups which for A and C is the same, even though these two have different group efficiency. The aforementioned value is higher for group B which illustrates lack of management balance in that group as compared to the other two. Table III shows the ranking comparison obtained from this method versus methods described in section III.

Table III: Ranking Groups based on different methods.

Group	Method in A	Method in B	Method in C	New Method
A	2	3	2	3
B	1	1	3	2
C	3	2	1	1

As it can be seen group ranking based on the new method has similarities to Cook and Zhu [5] and Bagharzadeh Valami [6]. The difference in ranking between the new method and common weight method of Cook and Zhu [5] exists in high rankings; however the difference between the new method and Bagharzadeh Valami [6] method is in low rankings. And as far as Comaho and Dyson [4] method is concerned, the difference is quite noticeable and far apart.

B. Applicable Example

The objective of this section is to analyze the proposed method of the article in a wider scope and to illustrate the capability of the method to group performance in application case. Wider or larger scope here means the number of groups, the number of units within groups and additional evaluation indices of DMUs. To do so an example of Bagharzadeh Valami [6] including twenty four bank branches in Iran divided into five groups for evaluation has been considered. Each branch has three input indices as payable interest (I_1), personnel (I_2) and arrear claims (I_3) as well as five output indices including the total sum of four main deposits (O_1), other resources (O_2), facilities (O_3), receivable interest (O_4)

Table IV: Inputs for twenty four bank branches inside five groups [6]

Group	DMU	I1	I2	I3
	1	3105.05	35.47	121910
	2	1113.92	21.23	14310
G1	3	1476.28	20	10375
	4	1962.49	13.58	2109
	5	4521.03	26.08	41890
	6	442.42	21.58	18935
	7	4155.68	14.46	6513
	8	8727.27	18.47	48683
	9	309.69	28.51	32819
G2	10	858.96	18.43	264
	11	2497.29	27.03	9363
	12	5714.57	12.49	991
	13	1016.45	22.81	280852
	14	10176.79	15.54	12558
G3	15	2671.41	19.18	982
	16	3528.29	21.11	52888
	17	4833.09	21.81	20616
G4	18	2546.77	21.32	18918
	19	2134.71	36.97	57297
	20	3706.76	28.42	29362
	21	3078.37	25.76	41104
G5	22	492	13.15	19201
	23	3317.51	28.55	17575
	24	780.46	29.52	31492

Table V: Outputs for twenty four bank branches inside five groups [6]

Group	DMU	O1	O2	O3	O4	O5
	1	3109668	179168	1506247	112273.11	1087.79
	2	145446	11175	312514	7584.57	859.46
G1	3	335492	161235	322235	34195.15	167.82
	4	265645	27191	251604	1218.22	15.95
	5	802090	501363	1765008	7017.64	672.82
	6	1510181	254998	519720	10065.04	89
	7	391820	4791	180814	2315.37	134.13
	8	1172029	106362	640435	46433.47	1297.05
	9	270421	265635	389094	2563.16	465.45
G2	10	296173	65334	1068621	28712..99	454.82
	11	434360	651496	1119173	11520.54	2103.71
	12	594269	8740	2072894	57808.34	4.8
	13	457615	425490	2127581	110811.32	1948.88
	14	803008	9335	2733094	36242.9	7.53
G3	15	674783	53412	2927758	3288.39	511.1
	16	454623	33596	287705	4650.81	123.99
	17	550577	28837	447227	2083	97.47
G4	18	353411	28294	181693	1276.63	353.37
	19	338997	235339	602678	6487.29	116.56
	20	358475	100579	352019	1650.86	48.02
	21	777689	483724	1749329	1162.84	925
G5	22	338822	3767	27233	1559.35	97.12
	23	634772	776270	699417	89577.41	1567.95
	24	138304	14248	297669	2823.85	79.09

Table VI: Results obtained from the proposed methods

Group	DMU	Unit Contribution	Unit Efficiency	Group Efficiency	Efficiency Spread
G1	1	0.3099	1.0000	0.6677	4.5936
	2	0.1783	0.2031		
	3	0.1685	0.4003		
	4	0.1157	0.2744		
	5	0.2273	0.9775		
G2	6	0.1143	0.9365	0.7707	11.8567
	7	0.08069	0.2761		
	8	0.1091	0.8184		
	9	0.1510	0.3799		
	10	0.0971	0.4959		
	11	0.1444	1.0000		
	12	0.0719	0.9833		
	13	0.1375	1.0000		
	14	0.0938	1.0000		
G3	15	0.4676	1.0000	0.6066	0.7387
	16	0.5323	0.2612		
G4	17	0.2775	0.3090	0.2775	0.1701
	18	0.2652	0.2239		
	19	0.4572	0.2895		
G5	20	0.2284	0.2305	0.5329	5.1502
	21	0.2078	0.9942		
	22	0.10406	0.2310		
	23	0.2276	1.0000		
	24	0.2319	0.0943		

and service fee (O_5). As Tables IV and V illustrates the number of DMUs within groups is different, group two with nine units has the largest aggregation and group three with two units has the least, also the number of units for the first and last groups are equal and total five units.

The third column in Table VI shows weight of units used in calculation of overall group efficiency and the efficiency of units in each group is given in by the values shown in the fourth column. Based on the method proposed in this article the efficiency of each group is measured by the efficiency of units within that group which are represented by values in the fifth column. The last column indicates the efficiency spread.

By further reference to the fourth column of Table VI it can be seen that all groups except group four include one efficient unit. By utilizing the values of this column all units inside groups can be compared. The following group ranking can be performed based on overall group efficiency values:

$G_2 \succ G_1 \succ G_3 \succ G_5 \succ G_4$.

Group two with the most statistics for units has the highest efficiency score. It is obvious from the last column of Table VI that group four has the least score of efficiency spread therefore the efficiency for the units inside it is extremely close. However, since group four has the lowest efficiency, it can be concluded that efficiency of its units must be low. The efficiency values in column four namely $\theta_{19} = 0.2895$, $\theta_{18} = 0.2239, \theta_{17} = 0.3090$ are evidence for this fact. According to another study, group two has the highest efficiency spread which indicates that units with high and low efficiency must exist within that group. But since the group has the highest ranking then the number or score of units with higher efficiency must be greater than those with a low efficiency. This point is reaffirmed based on values in column four. Six units namely 6, 8, 11, 12, 13 and 14 have a higher efficiency than the corresponding group and three units 7, 9 and 10 have a lower efficiency than the overall efficiency of group two. In fact the number of units with high efficiency is twice the number of those with low efficiency which is effective in increasing the efficiency of group two.

VI. Conclusion

A method based on common weights has been proposed to analyze group performance and put forth in this article which is contrary to Comanho and Dyson's method [4] that evaluates group performance by utilizing paired comparison, and it performs relative comparison between the groups and therefore coincides with data envelopment analysis. In the method proposed in this article, common set of weights used is an efficient solution for the proposed multi-objective linear fractional programming problem which reinforces the theory put forth by the method. By referring to the solved example it is clear that the method from calculation point of view is more cost effective because common weights can only be yielded through utilizing linear models. Furthermore by using the obtained common weights overall group efficiency, unit efficiency within each group, the rate of the effect of units on overall group efficiency and the amount of efficiency spread are calculated. It is worth noting that none of the proposed methods are capable of calculating all these indices, so the capability in the method proposed by this article results in a more comprehensive evaluation as compared to the methods in the literature. The authors of the article are attempting to utilize the proposed method to evaluate group performance in which DMUs within groups are non-homogenous.

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