Construction of the α_3-automorphism

S. Abdelalim, M. Zeriouh and M. Ziane

***Abstract*—**Let λ a monomorphism from A to A' where $A, A' \in \Gamma[6]$, we consider B' a basic subgroup of A'

$$B' = \bigoplus_{i \geq k} B_k' \text{ with } \begin{cases} B_k' & = & \bigoplus_{i\in I_k} < x_{k,i} > \\ o(x_{k,i}) & = & p^k \quad ; \ \forall i \in I_k \end{cases}, \text{ we}$$

suppose there exists $n_0 \in \mathrm{IN}^*$ such that the restriction of λ to $p^{n_0}A$ is an isomorphism from $p^{n_0}A$ to $p^{n_0}A'$ and we pose: $\lambda(A) = A_1$ [6] and $A_2 = A_1 + B_1'$ [2].

We show that if $\alpha \in Aut(A)$ is written in the form: $\alpha = \pi id_A + \rho$, where π is an invertible p-adic number and $\rho \in Hom(A, A^1)$ with A^1 is the first Ulm subgroup of A then, there exists an automorphism α_3 of $A_3 = A_2 + B_2'$ such that for all $a_3 \in A_3$: $\alpha_3(a_3) = \pi a_3 + p^{n_0^4 - 1} a_1^0$ where $a_1^0 \in A_1$ and $\alpha_3 \lambda = \lambda \alpha$.

***Keywords*—**About Abelian goups, p-group, order, direct sums of cyclic groups, basic subgroups, monomorphism group, automorphism group.

I. Introduction

In 1987, P. Schupp showed, in [4], that the extension property in the category of groups, characterizes the inner automorphisms. M. R. Pettet gives, in [5], a simpler proof of Schupp's result and shows that the inner automorphisms of a group are also characterized by the lifting property in the category of groups. The automorphisms of abelian p-groups having the extension property in the category of abelian p-groups are characterized in [1].

This work was supported in part by the Department of Mathematical and Computer Sciences , Faculty of sciences , University of Mohamed first, BP.717 60000, Oujda, Morocco.

S. Abdelalim: Laboratory of Mathematics, Computing and Application, Department of Mathematical and computer, Faculty of sciences University of Mohamed V Agdal, BP.1014 . Rabat, Morocco. seddikabd@hotmail.com.

M. Zeriouh: Department of Mathematical and Computer Sciences , Faculty of sciences , University of Mohamed first, BP.717 60000, Oujda, Morocco. zeriouhmostafa@gmail.com

M. Ziane: Department of Mathematical and Computer Sciences , Faculty of sciences , University of Mohamed first, BP.717 60000, Oujda, Morocco.

II. Main result

Proposition 2.1

(i) $A_2 \cap B_2'$ is the direct sum of cyclic groups.

(ii) If $A_2 \cap B_2' = B_{2,1}^* \oplus B_{2,2}^\times$

$$\begin{cases} B_{2,1}^* & = & \bigoplus_{i\in I_{2,1}} < x_{2,1,i} > ; \ o(x_{2,1,i}) = p \ , \forall i \in I_{2,1} \\ B_{2,2}^\times & = & \bigoplus_{i\in I_{2,2}} < x_{2,2,i} > ; \ o(x_{2,2,i}) = p^2, \forall i \in I_{2,1} \end{cases}$$

then there exists a subgroup $B_{2,1}^\times$ of B_2' such that

$$B_{2,1}^\times \supset B_{2,1}^*$$

(iii) $B_{2,1}^\times \oplus B_{2,2}^\times$ is a direct factor of B_2'.

Proof

(i) Since we have $B_2' = \bigoplus_{i\in I_2} < x_{2,i} >$ and $o(x_{2,i}) = p^2$, $\forall i \in I_2$

while $p^2 B_2' = 0$ i.e. B_2' is a bounded group therefore $A_2 \cap B_2'$ is also a bounded group

then by theorem 17.2 [3]: $A_2 \cap B_2'$ is direct sum of cyclic groups.

(ii) We have $A_2 \cap B_2' = B_{2,1}^* \oplus B_{2,2}^\times$ (5) with:

$$\begin{cases} B_{2,1}^* & = & \bigoplus_{i\in I_{2,1}} < x_{2,1,i} > ; \ o(x_{2,1,i}) = p \ , \forall i \in I_{2,1} \\ B_{2,2}^\times & = & \bigoplus_{i\in I_{2,2}} < x_{2,2,i} > ; \ o(x_{2,2,i}) = p^2, \forall i \in I_{2,1} \end{cases} \text{ And}$$

we have:

$$x_{2,1,i} \in B_{2,1}^* \subset B_2' = \bigoplus_{i\in I_2} < x_{2,i} > \ ; \ o(x_{2,i}) = p^2, \forall i \in I_2$$

then $x_{2,1,i} = \sum_{j=1}^{r} m_j x_{2,i_j}$ where $m_j \in \mathbb{Z}$

therefore $px_{2,1,i} = 0 = \sum_{j=1}^{r} pm_j x_{2,i_j}$

hence $\forall j = 1,\ldots,r : \ pm_j x_{2,i_j} = 0$

then $\forall j = 1,\ldots,r : \ p \mid m_j$

hence $\forall j = 1,\ldots,r \ ; \exists m_j' \in \mathbb{Z}: \ m_j = pm_j'$

then $x_{2,1,i}=\sum_{j=1}^{r} pm_j^{'}x_{2,i_j}=p\sum_{j=1}^{r} m_j^{'}x_{2,i_j}=py_{2,1,i}$ (6)

where $y_{2,1,i}=\sum_{j=1}^{r} m_j^{'}x_{2,i_j}\in B_2^{'};o(y_{2,1,i})=p^2,\forall i\in I_{2,1}$ then

$B_{2,1}^{*}\subset B_{2,1}^{\times}=\bigoplus_{i\in I_{2,1}}<y_{2,1,i}>;o(y_{2,1,i})=p^2,\forall i\in I_{2,1}$

(iii) Since $B_{2,1}^{\times}\oplus B_{2,2}^{\times}$ is a subgroup of $B_2^{'}$ a $B_{2,1}^{\times}\oplus B_{2,2}^{\times}$ is the direct sum of cyclic groups of the same order p^2 and

$(B_{2,1}^{\times}\oplus B_{2,2}^{\times})\cap p^2B_2^{'}=0$

then by proposition 27.1 [3]:

$B_{2,1}^{\times}\oplus B_{2,2}^{\times}$ is a direct summand of $B_2^{'}$.

We pose: $B_2^{'}=(B_{2,1}^{\times}\oplus B_{2,2}^{\times})\oplus B_2$ (7) where B_2 is a subgroup of $B_2^{'}$.

Definition 2.2

We define the homomorphism $\overline{\alpha_3}$ from $B_2^{'}$ to $A^{'}$ as follows:

$$\begin{cases}\overline{\alpha_3}\,|_{\overline{B_2}} &= \pi id_{\overline{B_2}}\\ \overline{\alpha_3}|_{B_{2,2}^{\times}} &= \alpha_2.\end{cases}$$

Under the conditions of Theorem 1.4 see [2] p, 251, we will enunciate and prove the following lemmas.

Lemma 2.3

$$\overline{\alpha_3}\,|_{B_{2,1}^{\times}}=\alpha_2$$

Proof

By (6) and (5): $py_{2,1,i_j}=x_{2,1,i_j}\in B_{2,1}^{*}\subset A_2$

and by theorem 1.4 [2], we have:

$\alpha_2(x_{2,1,i_j})=\pi x_{2,1,i_j}+p^{n_0^4}a_1$ (8) where $a_1\in A_1$

We pose: $\overline{\alpha_3}(y_{2,1,i_j})=\pi y_{2,1,i_j}+p^{n_0^4-1}a_1$ (9)

while

$$\begin{aligned}\overline{\alpha_3}(x_{2,1,i_j}) &= \overline{\alpha_3}(py_{2,1,i_j})\\ &= p\overline{\alpha_3}(y_{2,1,i_j})\\ &= p(\pi y_{2,1,i_j}+p^{n_0^4-1}a_1);a_1\in A_1\\ &= p\pi y_{2,1,i_j}+p^{n_0^4}a_1\\ &= \pi x_{2,1,i_j}+p^{n_0^4}a_1\\ &= \alpha_2(x_{2,1,i_j})\end{aligned}$$

Lemma 2.4

$$\overline{\alpha_3}\,|_{B_{2,1}^{*}\oplus B_{2,2}^{\times}}=\alpha_2.$$

Proof

Let $b_{2,1}^{*}\in B_{2,1}^{*}$ we have:

$$\begin{aligned}\overline{\alpha_3}(b_{2,1}^{*}) &= \overline{\alpha_3}(\sum_{j=1}^{r} m_jx_{2,1,i_j})\\ &= \sum_{j=1}^{r} m_j\overline{\alpha_3}(x_{2,1,i_j})\\ &= \sum_{j=1}^{r} m_j\alpha_2(x_{2,1,i_j})\\ &= \alpha_2(\sum_{j=1}^{r} m_jx_{2,1,i_j})\\ &= \alpha_2(b_{2,1}^{*})\end{aligned}$$

Let $b=b_{2,1}^{*}+b_{2,2}^{\times}\in B_{2,1}^{*}\oplus B_{2,2}^{\times}$

we have:

$$\begin{aligned}\overline{\alpha_3}(b) &= \overline{\alpha_3}(b_{2,1}^{*}+b_{2,2}^{\times})\\ &= \overline{\alpha_3}(b_{2,1}^{*})+\overline{\alpha_3}(b_{2,2}^{\times})\\ &= \alpha_2(b_{2,1}^{*})+\alpha_2(b_{2,2}^{\times})\\ &= \alpha_2(b)\end{aligned}$$

Proposition 2.5

(i) $\forall b_{2,1}^{\times}\in B_{2,1}^{\times}:\overline{\alpha_3}(b_{2,1}^{\times})=\pi b_{2,1}^{\times}+p^{n_0^4-1}a_1^{'};a_1^{'}\in A_1$

(ii) $\forall b_{2,2}^{\times}\in B_{2,2}^{\times}:\alpha_2(b_{2,2}^{\times})=\pi b_{2,2}^{\times}+p^{n_0^4}a_1^{''};\ a_1^{''}\in A_1$

Proof

(i) Since we have:

$b_{2,1}^{\times}\in B_{2,1}^{\times}=\bigoplus_{i\in I_{2,1}}<y_{2,1,i}>$ then $b_{2,1}^{\times}=\sum_{t=1}^{r} m_ty_{2,1,i_t}$

therefore:

$$\begin{aligned}\overline{\alpha_3}(b_{2,1}^{\times}) &= \overline{\alpha_3}(\sum_{t=1}^{r} m_t\,y_{2,1,i_t})\\ &= \sum_{t=1}^{r} m_t\overline{\alpha_3}(y_{2,1,i_t})\\ &= \sum_{t=1}^{r} m_t(\pi y_{2,1,i_t}+p^{n_0^4-1}a_1);\ a_1\in A_1 \text{ (by (9))}\\ &= \pi b_{2,1}^{\times}+p^{n_0^4-1}\sum_{t=1}^{r} m_ta_1\\ &= \pi b_{2,1}^{\times}+p^{n_0^4-1}a_1^{'};a_1^{'}=\sum_{t=1}^{r} m_ta_1\in A_1\end{aligned}$$

(ii) Since we have:

$b_{2,2}^{\times}\in B_{2,2}^{\times}=\bigoplus_{i\in I_{2,2}}<x_{2,2,i}>\subset A_2$ by (5)

then $b_{2,2}^{\times}=\sum_{t=1}^{r} m_tx_{2,2,i_t}$ where $x_{2,2,i_t}\in A_2$

therefore :

$$\alpha_2(b_{2,2}^{\times}) = \alpha_2(\sum_{t=1}^{r} m_t x_{2,2,i_t})$$
$$= \sum_{t=1}^{r} m_t \alpha_2(x_{2,2,i_t})$$
$$= \sum_{t=1}^{r} m_t(\pi x_{2,2,i_t} + p^{n_0^4} a_1^{'}); a_1^{'} \in A_1 \ [2]$$
$$= \pi b_{2,2}^{\times} + p^{n_0^4} \sum_{t=1}^{r} m_t a_1^{'}$$
$$= \pi b_{2,2}^{\times} + p^{n_0^4} a_1^{''}; a_1^{''} = \sum_{t=1}^{r} m_t a_1^{'} \in A_1$$

Definition 2.6

We define the endomorphism α_3 of $A_3 = A_2 + B_2^{'}$ as follows:
$$\begin{cases} \alpha_{3|_{A_2}} = \alpha_2 \\ \alpha_{3|_{B_2^{'}}} = \overline{\alpha_3} \end{cases}$$

Remark 2.7

α_3 is well defined

because: if $a_2 + b_2^{'}$ and $x_2 + y_2^{'}$ are two elements of A_3 such that $a_2 + b_2^{'} = x_2 + y_2^{'}$

then $a_2 - x_2 = -b_2^{'} + y_2^{'}$

therefore $\overline{\alpha_3}(a_2 - x_2) = \overline{\alpha_3}(-b_2^{'} + y_2^{'})$

i.e. $\alpha_2(a_2 - x_2) = \overline{\alpha_3}(-b_2^{'} + y_2^{'})$

i.e. $\alpha_2(a_2) + \overline{\alpha_3}(b_2^{'}) = \alpha_2(x_2) + \overline{\alpha_3}(y_2^{'})$

i.e. $\alpha_3(a_2 + b_2^{'}) = \alpha_3(x_2 + y_2^{'})$

Proposition 2.8

For all $a_3 \in A_3$ there exists $n_0 \in \mathrm{IN}^*$ and $a_1^0 \in A_1$ such that $\alpha_3(a_3) = \pi a_3 + p^{n_0^4 - 1} a_1^0$

Proof

We have: $a_3 \in A_3 = A_2 + B_2^{'}$

then $\exists (a_2, b_2^{'}) \in (A_2 \times B_2^{'})$ such that $a_3 = a_2 + b_2^{'}$ hence
$\alpha_3(a_3) = \alpha_3(a_2) + \alpha_3(b_2^{'})$

and by definition 2.6, we have :
$$\alpha_3(a_3) = \alpha_2(a_2) + \overline{\alpha_3}(b_2^{'})$$

and by theorem 1.4, [2] there exists $n_0 \in \mathrm{IN}^*$ and $a_1 \in A_1$ such that: $\alpha_3(a_3) = \pi a_2 + p^{n_0^4} a_1 + \overline{\alpha_3}(b_2^{'})$

and by (7) we have $b_2^{'} = b_{2,1}^{\times} + b_{2,2}^{\times} + \overline{b_2}$

then by definition 2.2, we have:
$$\overline{\alpha_3}(b_2^{'}) = \overline{\alpha_3}(b_{2,1}^{\times} + b_{2,2}^{\times} + \overline{b_2})$$

i.e. $\overline{\alpha_3}(b_2^{'}) = \overline{\alpha_3}(b_{2,1}^{\times}) + \alpha_2(b_{2,2}^{\times}) + \pi \overline{b_2}$

The proposition 2.5 and definition 2.2 show that:
$$\begin{aligned} \overline{\alpha_3}(b_2^{'}) &= \pi b_{2,1}^{\times} + p^{n_0^4 - 1} a_1^{'} + \pi b_{2,2}^{\times} + p^{n_0^4} a_1^{''} + \pi \overline{b_2}; a_1^{'}, a_1^{''} \in A_1 \\ &= \pi(b_{2,1}^{\times} + b_{2,2}^{\times} + \overline{b_2}) + p^{n_0^4 - 1}(a_1^{'} + p a_1^{''}) \\ &= \pi b_2^{'} + p^{n_0^4 - 1} a_1^{'''}; a_1^{'''} = a_1^{'} + p a_1^{''} \in A_1 \end{aligned}$$

therefore
$$\begin{aligned} \alpha_3(a_3) &= \pi a_2 + p^{n_0^4} a_1 + \pi b_2^{'} + p^{n_0^4 - 1} a_1^{'''} \\ &= \pi(a_2 + b_2^{'}) + p^{n_0^4 - 1}(p a_1 + a_1^{'''}) \\ &= \pi a_3 + p^{n_0^4 - 1} a_1^0; a_1^0 = p a_1 + a_1^{'''} \in A_1 \quad (10) \end{aligned}$$

Proposition 2.9

α_3 is an automorphism of A_3

Proof

Let $a_3 \in ker\alpha_3$ then $\alpha_3(a_3) = 0$

and by (10) we have: $\pi a_3 + p^{n_0^4 - 1} a_1^0 = 0$

which is equivalent to $a_3 = -\pi^{-1} p^{n_0^4 - 1} a_1^0 \in A_1 \subset A_2$

and since $\alpha_{3|_{A_2}} = \alpha_2$ then $\alpha_3(a_3) = \alpha_2(a_3)$

i.e. $0 = \alpha_2(a_3)$ then $0 = a_3$

i.e. $ker\alpha_3 = 0$ then α_3 is a monomorphism.

On the other hand let $a_3 \in A_3$

then $\alpha_3(a_3) = \pi a_3 + p^{n_0^4 - 1} a_1^0$ where $a_1^0 \in A_1$

hence $a_3 = \pi^{-1} \alpha_3(a_3) - \pi^{-1} p^{n_0^4 - 1} a_1^0$

i.e. $a_3 = \alpha_3(\pi^{-1} a_3) - \pi^{-1} p^{n_0^4 - 1} a_1^0$

and since we have $-\pi^{-1} p^{n_0^4 - 1} a_1^0 \in A_1 \subset A_2$ and
$\alpha_2 \in Aut(A_2)$

then $\exists! a_2 \in A_2 \subset A_3$ such that:
$$-\pi^{-1} p^{n_0^4 - 1} a_1^0 = \alpha_2(a_2) = \alpha_3(a_2)$$

so $a_3 = \alpha_3(\pi^{-1} a_3) + \alpha_3(a_2) = \alpha_3(\pi^{-1} a_3 + a_2)$

which means that α_3 is an epimorphism and hence α_3 is an automorphism of A_3.

Proposition 2.10

The following diagram is commutative:
$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A_3 \\ \alpha \downarrow & & \downarrow \alpha_3 \\ A & \xrightarrow{\lambda} & A_3 \end{array}$$

Proof

We have $\forall a \in A, \quad \lambda(a) \in A_1 \subset A_2$

then $\alpha_3\lambda\left(a\right) = \alpha_2\lambda\left(a\right)$ because $\alpha_{3|_{A_2}} = \alpha_2$

$= \alpha_1\lambda\left(a\right)$ because $\alpha_{2|_{A_1}} = \alpha_1$

$= \lambda\alpha\lambda^{-1}\lambda\left(a\right)$ because $\alpha_1 = \lambda\alpha\lambda^{-1}$

$= \lambda\alpha\left(a\right)$

therefore $\alpha_3\lambda = \lambda\alpha$

ACKNOWLEDGMENT

We thank Professor Abdelhakim Chillali for his helpful comments and suggestions.We also thank the referee for his suggestions and comments in the revision of this paper.

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S. Abdelalim: Laboratory of Mathematics, Computing and Application, Department of Mathematical and computer, Faculty of sciences University of Mohamed V Agdal, BP.1014 . Rabat, Morocco. seddikabd@hotmail.com

M. Zeriouh: Department of Mathematical and Computer Sciences , Faculty of sciences , University of Mohamed first, BP.717 60000, Oujda, Morocco. zeriouhmostafa@gmail.com

M. Ziane: Department of Mathematical and Computer Sciences , Faculty of sciences , University of Mohamed first, BP.717 60000, Oujda, Morocco.