An Algebraic Approach to Two-Degree-of-Freedom Controller Design for Systems with Parametric Uncertainty

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Abstract—The main aim of this paper is to present a possible application of two-degree-of-freedom (2DOF) control design to systems with real parametric uncertainty. The control synthesis is based on algebraic tools and the robust stability analysis utilizes mainly the value set concept in combination with the zero exclusion condition. The set of illustrative simulation examples includes the control of first and third order controlled plants with interval parameters.

Keywords—2DOF Control Structure, Algebraic Approach, Parametric Uncertainty, Value Set Concept, Zero Exclusion Condition.

I. INTRODUCTION

The control loops which have separated feedback and feedforward parts of the controller are usually called as systems with two degrees of freedom (2DOF) and they have considerable advantages against traditional configuration [1] – [3]. Their possible applications include also the systems affected by uncertainties (some robustness problems are solved e.g. in [4] – [9]).

The main aim of the paper is to present the possibilities of application of 2DOF controllers to systems with parametric uncertainty which are supposed to have known structure but their parameters can lie within given bounds. The control design approach is based on algebraic ideas invented in [10] and [11] and subsequently studied e.g. in [12] – [16]. This synthesis technique is combined with robust stability analysis based on the value set concept and the zero exclusion condition [17]. The theoretical foundations are followed by computational and simulation examples with first and third order controlled interval plants. This work is a follow-up of the papers [18], [19] with extension of 2DOF control to plants with parametric uncertainty. The paper is the extended version of the conference contribution [20].

II. SYSTEMS WITH PARAMETRIC UNCERTAINTY

Systems with parametric uncertainty represent effective and popular way of considering the uncertainty in the mathematical model of a real plant [17], [21]. Their usage supposes known structure (order) but imprecise knowledge of real physical parameters, which are usually bounded by intervals with minimal and maximal possible values. They can be described by a transfer function:

$$G(s,q) = \frac{b(s,q)}{a(s,q)}$$

(1)

where \(b(s,q)\) and \(a(s,q)\) denote polynomials in \(s\) with coefficients depending on \(q\), which is a vector of real uncertain parameters. Typically, this vector is confined by some uncertainty bounding set which is generally a ball in some appropriate norm. The most common case supposes \(L_\infty\) norm which implies that a ball in this norm is a box. The combination of the uncertain system (e.g. transfer function (1)) with an uncertainty bounding set gives so-called family of systems [17].

A special and frequently used case of system with parametric uncertainty is the interval plant. Its parameters can vary independently on each other within given bounds, i.e.:

$$G(s,b,a) = \frac{\sum_{i=1}^{n} [b_i^-; b_i^+] s^i}{\sum_{i=0}^{n} [a_i^-; a_i^+] s^i}$$

(2)

where \(b_i^-, b_i^+, a_i^-, a_i^+\) represent lower and upper limits for parameters of numerator and denominator, respectively.

The paper is organized as follows. In section 2, the systems with parametric uncertainty and their description are briefly presented. The section 3 then provides the fundamentals on robust stability analysis with special accent to the value set concept in combination with the zero exclusion condition. An algebraic approach to 2DOF control design is summarized in section 4. Further, several simulation examples for first and third order controlled plants are presented in the extensive section 5. And finally, section 6 offers some conclusion remarks.
III. ROBUST STABILITY ANALYSIS

Stability is the crucial requirement in all control applications. Subsequently, one speaks about robust stability if not only one fixed closed-loop system but also whole family of closed-loop control systems is ensured to be stable.

Since the stability of linear systems can be investigated by means of stability of its characteristic polynomials, the primary object of interest from the robust stability viewpoint is the uncertain continuous-time closed-loop characteristic polynomial:

\[ p(s, q) = \sum_{i=0}^{n} \rho_i(q)s^i \]

where \( \rho_i(q) \) are coefficient functions. Then, the family of closed-loop characteristic polynomials can be denoted as:

\[ P = \{ p(. , q) : q \in Q \} \]

Generally, the family of polynomials (4) is robustly stable if and only if \( p(s, q) \) is stable for all \( q \in Q \), i.e. all roots of \( p(s, q) \) must be located in the strict left half of the complex plane for all \( q \in Q \). Thus, the sufficient instrument for robust stability analysis seems to be just the computation of all roots of \( p(s, q) \) for \( q \in Q \). This is right, but the direct calculation of roots can be impractical due to potentially enormously long computation times. The idea is applicable only for the most primitive cases and so the more efficient techniques had to be studied [22], [23].

The choice of technique for investigation of robust stability depends primarily on the structure of the uncertainty, in other words on the way how the uncertain parameters \( q \) enter into the coefficients of the uncertain polynomial (3). According to this, it can be distinguished among several basic structures of uncertainty with increasing generality, i.e. independent (interval) uncertainty structure, affine linear uncertainty structure, multilinear uncertainty structure, and nonlinear uncertainty structure (polynomial, general). The higher level of relation among coefficients means the more complex robust stability analysis. The classical feedback connection of interval plant (2) and a fixed controller leads to the closed-loop characteristic polynomial with affine linear uncertainty structure. An interested reader can find further information e.g. in [17], [21], [23].

As it has been already mentioned, the structure of uncertainty is a key factor for determination of the suitable choice of tool for its robust stability analysis. However, there is a very universal graphical approach applicable for all, even the most complicated, cases. It is known as the value set concept in combination with the zero exclusion condition [17].

Suppose a family of polynomials (4). The value set at frequency \( \omega \in \mathbb{R} \) is given by [17]:

\[ p(j\omega, Q) = \{ p(j\omega, q) : q \in Q \} \]

In other words, \( p(j\omega, Q) \) is the image of \( Q \) under \( p(j\omega, .) \). Practical construction of the value sets then means to substitute \( s \) for \( j\omega \), fix \( \omega \) and let the vector of uncertain parameters \( q \) range over the set \( Q \).

The zero exclusion condition for Hurwitz stability of family of continuous-time polynomials (4) says [17]: Assume invariant degree of polynomials in the family, pathwise connected uncertainty bounding set \( Q \), continuous coefficient functions \( \rho_i(q) \) for \( i = 0, 1, 2, \ldots, n \) and at least one stable member \( p(s, q^0) \). Then the family \( P \) is robustly stable if and only if the complex plane origin is excluded from the value set \( p(j\omega, Q) \) at all frequencies \( \omega \geq 0 \), that is \( P \) is robustly stable if and only if:

\[ 0 \notin p(j\omega, Q) \quad \forall \omega \geq 0 \quad (6) \]

More details can be found especially in [17] or e.g. in [21]. A gallery of value sets for various uncertainty structures can be found e.g. in [23].

IV. 2DOF CONTROL DESIGN VIA ALGEBRAIC TOOLS

The 2DOF closed-loop control system with separated feedback and feedforward parts of the controller is depicted in fig. 1. The transfer functions \( G(s), C_f(s), \text{ and } C_b(s) \) represent controlled plant, feedback part of the controller, and feedforward part of the controller, respectively and the signals \( w(s), n(s), \text{ and } v(s) \) are reference, load disturbance, and disturbance signal.

![Fig. 1 two-degree-of-freedom control loop](image-url)

The control synthesis itself is based on the algebraic ideas of Vidyasagar [10] and Kučera [11]. Subsequently, the specific tuning rules has been developed and analyzed e.g. in [12]– [16].

Besides, the controller tuning rules for the case of first order controlled plant under assumption of either purely reference tracking problem or reference tracking and load disturbance rejection together have been already elaborated e.g. in [18], [19] and so this paper will only summarize the most important results and then it will apply them to the systems with parametric uncertainty.

Primarily, the control design technique supposes the description of linear systems in fig. 1 by means of the ring of proper and stable rational functions (Rps). The conversion
from the ring of polynomials to $\mathbb{R}_{PS}$ can be performed very simply according to:

$$G(s) = \frac{b(s)}{a(s)} = \frac{b(s)}{(s + m)^{\max\{\deg(a), \deg(b)\}}} = B(s) \frac{1}{A(s)}$$

(7)

$m > 0$

The parameter $m > 0$ will be later used as a controller tuning knob.

The algebraic analysis [18], [19] leads to the first and most important Diophantine equation:

$$A(s)P(s) + B(s)Q(s) = 1$$

with a general solution $P(s) = P_0(s) + B(s)T(s)$, $Q(s) = Q_0(s) - A(s)T(s)$, where $T(s)$ is an arbitrary member of $\mathbb{R}_{PS}$ and the pair $P_0(s)$, $Q_0(s)$ represents particular solution of (8). This principle is known as Youla – Kučera parameterization of all stabilizing controllers. Thus, all possible solutions of the Diophantine equation give all stabilizing feedback controllers. Since the feedback part of the controller is responsible not only for stabilization but also for disturbance rejection, the convenient controller from the set of all stabilizing ones can be chosen on the basis of divisibility conditions.

Subsequently, the requirement of the reference tracking is “hidden” in the second Diophantine equation:

$$F_o(s)Z(s) + B(s)R(s) = 1$$

(9)

In papers [18], [19], it was derived that the assumption of first order plant

$$G(s) = \frac{b_0}{s + a_0}$$

(10)

step-wise reference with $F_o(s) = s/(s + m)$ and no disturbances leads to the controllers:

$$C_o(s) = \frac{Q(s)}{P(s)} = \frac{q_o}{p_o}; \quad C_f(s) = \frac{R(s)}{P(s)} = \frac{r_o}{p_o}$$

(11)

where

$$p_o = 1; \quad q_o = \frac{m - a_0}{h_0}; \quad r_o = \frac{m}{h_0}$$

(12)

Moreover, if the step-wise load disturbance signal is considered, the controllers change to:

$$C_o(s) = \frac{Q(s)}{P(s)} = \frac{\hat{q}_o s + \hat{q}_i}{s}; \quad C_f(s) = \frac{R(s)}{P(s)} = \frac{\hat{r}_o s + \hat{r}_i}{s}$$

(13)

where

$$\hat{q}_i = \frac{2m - a_0}{h_0}; \quad \hat{q}_o = \frac{m^2}{h_0}; \quad \hat{r}_i = \frac{m}{h_0}; \quad \hat{r}_o = \frac{m^2}{h_0}$$

(14)

Note that the controller parameters depend (generally in a nonlinear way) on the tuning parameter $m > 0$. Its choice can influence robust stability as well as performance of the control loop. A possible method of parameter selection for 1DOF configuration based on the requested size of first overshoot of control output is presented e.g. in [14] – [16]. Nevertheless, this contribution will not use any exact technique for the choice of $m$ in simulations.

The tuning rules for the higher order controlled systems would be more complicated and they are not explicitly shown here. Certainly, they can be derived according to general principle outlined here and provided in more detail e.g. in [18], [19].

V. SIMULATION EXAMPLES

The algebraic theory of 2DOF control design and foundations of robust stability analysis described in the previous sections are going to be verified by means of simulation examples within this part. The aim is to compute 2DOF controller for step-wise reference tracking and potentially also step-wise load disturbance rejection which is able to robustly stabilize given system with parametric uncertainty.

A. First Order Plant

In the first instance, the simple first order plant with uncertain gain and time constant is considered as a controlled object:

$$G(s, K, T) = \frac{K}{Ts + 1} = \frac{[5; 15]}{[1; 3]}s + 1$$

(15)

The fixed nominal plant used for the controller design itself can be easily determined by means of the average values of the uncertain parameters:

$$G_n(s) = \frac{10}{2s + 1} = \frac{5}{s + 0.5} = \frac{h_0}{s + a_0}$$

(16)

First, two P controllers (11) with parameters (12) were calculated for 2DOF configuration using tuning parameter $m = 1$:

$$C_o(s) = 0.1; \quad C_f(s) = 0.2$$

(17)
It means that the controllers were designed only for (nominal) stabilization and reference tracking but not for disturbance rejection. Moreover, the robust stability can be easily verified via the family of closed-loop characteristic polynomials:

\[ p_{cl}(s, K, T) = (Ts + 1) p_0 + K q_0 = \]
\[ = [1; 3] s + 1 + [5; 15] 0.1 = \]
\[ = [1; 3] s + [1.5; 2.5] \]

Robust stability of the first order polynomial (18) is evident thanks to the positivity of both interval coefficients.

The simulations in fig. 2 show the output signals of the 225 “representative” systems (RS) from the interval family (15). Both interval parameters were divided into 14 subintervals of the same size and thus the obtained 15 values and 2 parameters lead to 215 = 225 systems for simulation. In addition, the red curve represents the output signal for the nominal system (16). Furthermore, it was assumed the step reference signal changing from 1 to 2 in one third of simulation time and the step load disturbance -0.05 which influences the input to the controlled plant during the last third of simulation.

\[ \text{Fig. 2 control of RS from interval system (15) by 2DOF controller (17) – output signals} \]

As can be seen, the family of systems (15) is really robustly stabilized by the 2DOF controller (17), but the permanent control error is observed due to the perturbations in controlled plant gain and also when the load disturbance is injected. So, the practical application of this controller is very limited as it is suitable only for the ideal nominal case without perturbations and without disturbances.

Moreover, fig. 3 shows the set of control signals (outputs of the controller) which corresponds to the set of output signals from fig. 2.

\[ \text{Fig. 3 control of RS from interval system (15) by 2DOF controller (17) – control signals} \]

Now, it have been calculated two PI controllers (13) with parameters (14) for tuning parameter \( m = 0.05 \):

\[ C_i(s) = \frac{-0.08 s + 0.0005}{s}; \quad C_f(s) = \frac{0.01 s + 0.0005}{s} \]

which lead to the family of closed-loop characteristic polynomials:

\[ p_{cl}(s, K, T) = (Ts + 1) s + K (\tilde{q}_s s + \tilde{q}_d) = \]
\[ = [1; 3] s^2 + [-0.2; 0.6] s + [0.0025; 0.0075] \]

This family is robustly unstable due to the potentially negative first power coefficient, which is demonstrated also on the set of output signals for the RS and corresponding control signals (see fig. 4 and fig. 5) obtained according to the previous simulation.

\[ \text{Fig. 4 control of RS from interval system (15) by 2DOF controller (19) – output signals} \]
Finally, the whole experiment with PI controllers is repeated, but now for tuning parameter $m=1$ with the following results:

$$C_s(s) = \frac{0.3s + 0.2}{s}; \quad C_f(s) = \frac{0.2s + 0.2}{s}$$

(21)

$$p_{cl}(s,K,T) = [1;3]s^2 + [2.5;5.5]s + [1;3]$$

(22)

The interval polynomial (22) and thus also the whole 2DOF control loop is robustly stable, control error tends to zero and load disturbance is rejected which is confirmed by fig. 6 with set of output signals for RS and fig. 7 with set of control signals.

B. Third Order Plant

In the second example, the third order interval plant is assumed to be controlled:

$$G(s,b_1,a_1) = \frac{[0.8;1.2]s^2 + [0.8;1.2]s + [0.8;1.2]}{[0.8;1.2]s^2 + [0.8;1.2]s^2 + [0.8;1.2]s + [0.8;1.2]}$$

(23)

The fixed nominal system is:

$$G_N(s) = \frac{s^2 + s + 1}{s^3 + s^2 + s + 1}$$

(24)

First, the tuning parameter $m=0.6$ leads to 2DOF controller (for step-wise load disturbance rejection):

$$C_s(s) = \frac{1.4227s^3 + 0.3001s^2 - 1.08s + 0.0467}{s^3 + 1.1773s^2 + 1.4999s}$$

$$C_f(s) = \frac{0.216s^3 + 0.3888s^2 + 0.2333s + 0.0467}{s^3 + 1.1773s^2 + 1.4999s}$$

(25)

The family of closed-loop characteristic polynomials has the sixth order and affine linear structure of uncertainty. So, the analysis of its robust stability is not as trivial as in the previous section anymore. The value sets plotted for the range of frequencies $\omega = 0:0.01:3$ according to the principle described in section 3 and with the practical assistance of the Polynomial Toolbox for Matlab [24] are shown in fig. 8 and their zoomed version for better view in the neighborhood of the origin of the complex plane is provided in fig. 9. The family is robustly unstable because the zero is included in the value sets. This fact is demonstrated also on the set of output signals given in fig. 10 (now the size of load disturbance is - 0.5 and the amount of RS for simulations is $2^7 = 128$). The set of control signals is presented by fig. 11.
Finally, choice of the tuning parameter $m = 0.9$ results in 2DOF controller:

$$C_s(s) = \frac{4.2699s^5 + 6.1685s^3 + 2.43s + 0.5314}{s^3 + 0.1301s^2 + 0.5815s}$$

(26)

$$C_f(s) = \frac{0.729s^3 + 1.9683s^2 + 1.7715s + 0.5314}{s^2 + 0.1301s^2 + 0.5815s}$$

The value sets of corresponding family of closed-loop characteristic polynomials are visualized for $\omega = 0 : 0.01 : 5$ in fig. 12. Its zoomed version is then in fig. 13. Since the family contains a stable member and the complex plane origin is excluded from the value sets, it is robustly stable. The sets of simulated output signals and control signals are depicted in figs. 14 and 15, respectively.
VI. CONCLUSION

The paper has been focused on application of continuous-time 2DOF control algorithms designed in RPS to systems with parametric uncertainty. The synthesis method itself is accompanied by the graphical approach to robust stability analysis based on the value set concept and the zero exclusion condition. Furthermore, the paper has presented several computational and simulation examples.

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