A mixed finite element method with new boundary condition

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Abstract—In this paper, we introduce the Stokes equations with a new boundary condition. In this context, we show the existence and uniqueness of the solution of the weak formulation associated with the proposed problem. To solve this problem, we use the discretization by mixed finite element method. In addition, two types of a posteriori error indicator are introduced and are shown to give global error estimates that are equivalent to the true discretization error. Computational results suggest that both error estimators seem to be able to correctly indicate the structure of the error.

Keywords—Stokes equations; $C_{a,b}$ Boundary condition; Mixed finite element method; Residual error estimator; Adina system.

I. INTRODUCTION

THIS paper describes a numerical solution of Stokes equations in an open bounded connected Ω to IR^2 and which we must add the boundary condition.

To date, most works have considered the standard boundary conditions: Dirichlet, Neumann [1, 2, 4, 5, 14, 21] and the mixed Direchlet-Neumman boundary condition [2, 4, 14, 15, 21]. In this paper, we propose a new boundary condition, noted Ca,b, which generalizes the standard ones. In addition, we prove that the weak formulation of the proposed modeling has a unique solution. To calculate this latter, we use the mixed finite element method.

In this modeling flow of porous media, it is essential to use a discretization method which satisfies the physics of the problem, i.e. conserve mass locally and preserve continuity of flux.

The Raviart-Thomas Mixed Finite Element (MFE) method of lowest order satisfies these properties.

Moreover, both the pressure and the velocity are approximated with the same order of convergence [4, 6]. The discretization of the velocity is based on the properties of Raviart-Thomas. Other works have been introduced by Brezzi, Fortin, Marini, Dougla and Robert [4, 5, 7]. This method was widely used for the prediction of the behavior of fluid in the hydrocarbons tank.

A posteriori error analysis in problems related to fluid dynamics is a subject that has received a lot of attention during the last decades. In the conforming case there are several ways to define error estimators by using the residual equation. In particular, for the Stokes problem, M. Ainsworth, J. Oden [9], R.E. Bank, B.D. Welfert [10], C. Carstensen, S.A. Funken [11], D.Kay, D. Silvester [12] and R. Verfurth [13], introduced several error estimators and provided that they are equivalent to the energy norm of the errors. Other works for the stationary Navier-Stokes problem had been introduced in [14, 17, 18, 20, 16]. For this paper two types of a posteriori error indicator are introduced and are shown to give global error estimates that are equivalent to the true discretisation error.

The plan of the paper is as follows. Section 2 presents the model problem used in this paper. The weak formulation is presented in section 3. In section 4, we show the existence and uniqueness of the solution. The discretization by mixed finite elements is described in section 5. Section 6 introduces two types of a posteriori error bounds of the computed solution. In section 7, numerical experiments within the framework of this publication were carried out.

II. GOVERNING EQUATIONS

We will consider the model of viscous incompressible flow in an idealized, bounded, connected domain in IR^2.

$$-\nabla^2\vec{u}+\nabla p=\vec{f} \quad in\ \Omega \tag{2.1}$$

$$\nabla.\vec{u}=0\ \ in\ \ \Omega \tag{2.2}$$

The boundary value problem which is posed on two dimensional domains Ω, is defined as:

$$C_{a,b}: a\vec{u}+b(\nabla\vec{u}-pI)\vec{n}=\vec{t}\ on\ \Gamma:=\partial\Omega\,. \tag{2.3}$$

We also assume that Ω has a polygonal boundary Γ, so $\vec{n}$ that is the usual outward pointing normal. The vector field $\vec{u}$, is the velocity of the flow the functional $\vec{f}$ in the space $\left[L^2(\Omega)\right]^2$ and the pressure p in the space $L^2(\Omega)$, ∇ is the gradient, $\nabla.$ is the divergence and ∇^2 is the Laplacien operator, a and b nonzero function defined on Γ verify:

There are two strictly positive constants α_1 and β_1 such that:

$$\forall x\in\Gamma:\alpha_1\le\frac{a(x)}{b(x)}\le\beta_2\quad. \tag{2.4}$$

Remark 2.1 If a and b are two strictly positive constants such that t $a\succ\succ b$ then $C_{a,b}$ is the Dirichlet boundary condition and if $a\prec\prec b$ then the $C_{a,b}$ is the Neumann boundary condition. For this, a is called the Dirichlet coefficient and b is the Neumann coefficient.

III. THE WEAK FORMULATION

We define the spaces:

$$h^1(\Omega)=\left\{ u:\Omega\to IR\ ;\ u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\in L^2(\Omega)\right\},$$

$$H^1(\Omega)=\left[h^1(\Omega)\right]^2$$

$$L_0^2(\Omega)=\left\{ q\in L^2(\Omega);\int_\Omega q d\Omega=0\ \right\}.$$

The standard weak formulation of the Stokes flow problem (2.1)-(2.2)-(2.3) is the following:

Find $\vec{u}\in H^1(\Omega)$ *and* $p\in L_0^2(\Omega)$ *such that*

$$\begin{cases}\int_\Omega \nabla\vec{u}:\nabla\vec{v}+\int_\Gamma \frac{a}{b}\vec{u}.\vec{v}-\int_\Omega p\nabla.\vec{v}=\int_\Omega \vec{f}.\vec{v}+\int_\Gamma \frac{1}{b}\vec{t}.\vec{v} \\ -\int_\Omega q\nabla.\vec{u}=0,\end{cases}\tag{3.4}$$

for all $\vec{v}\in H^1(\Omega)$ *and* $q\in L_0^2(\Omega)$.

Let the bilinear forms A and B

$$A(\vec{u},\vec{v})=\int_\Omega \nabla\vec{u}:\nabla\vec{v}+\int_\Gamma \frac{a}{b}\vec{u}.\vec{v}\,,\tag{3.5}$$

$$B(\vec{v},q)=-\int_\Omega q\nabla.\vec{v}\tag{3.6}$$

Given the functional L

$$L(\vec{v})=\int_\Omega \vec{f}.\vec{v}+\int_\Gamma \frac{1}{b}\vec{t}.\vec{v}\,,\tag{3.7}$$

The underlying weak formulation (3.4) may be restated as:

Find $\vec{u}\in H^1(\Omega)$ *and* $p\in L_0^2(\Omega)$ *such that*

$$\begin{cases}A(\vec{u},\vec{v})+B(p,\vec{v})=0\\ B(q,\vec{u})=0,\end{cases}\tag{3.8}$$

for all $\vec{v}\in H^1(\Omega)$ *and* $q\in L_0^2(\Omega)$.

IV. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

In this section we will study the existence and uniqueness of the solution of problem (3.8). For this, we need the following results:

Theorem 4.1 There are two strictly positive constants c_1 and c_2 such that:

$$\forall\vec{v}\in H^1(\Omega): c_1\|\vec{v}\|_{1,\Omega}\le\|\vec{v}\|_{J,\Omega}\le c_2\|\vec{v}\|_{1,\Omega}\tag{4.1}$$

Where

$$\|\vec{v}\|_{1,\Omega}=\left(\int_\Omega \nabla\vec{v}:\nabla\vec{v}+\int_\Omega \vec{v}.\vec{v}\right)^{\frac{1}{2}}=\left(\|\nabla\vec{v}\|_{0,\Omega}^2+\|\vec{v}\|_{0,\Omega}^2\right)^{\frac{1}{2}}$$

and

$$\|\vec{v}\|_{J,\Omega}==\left(A(\vec{v},\vec{v})\right)^{\frac{1}{2}}=\left(\int_\Omega \nabla\vec{v}:\nabla\vec{v}+\int_\Gamma \frac{a}{b}\vec{v}.\vec{v}\right)^{\frac{1}{2}}$$

Proof: The mapping $\gamma_0:H^1(\Omega)\to L^2(\Gamma)$, such that $\gamma_0(\vec{v})=\vec{v}/\Gamma$ is continuous [2], then there exists c > 0 such that:

$$\|\vec{v}\|_{0,\Gamma}\le c_1\|\vec{v}\|_{1,\Omega}\text{ for all }\vec{v}\in H^1(\Omega)$$

Using this result and the result (2.4) gives,

$$\begin{aligned}\|\vec{v}\|_{J,\Omega}^2&=\int_\Omega \nabla\vec{v}:\nabla\vec{v}d\Omega+\int_{\partial\Omega}\frac{a}{b}\vec{v}.\vec{v}d\gamma\\&\le\|\nabla\vec{v}\|_{0,\Omega}^2+\beta_1\|\vec{v}\|_{0,\Gamma}^2\\&\le\left(1+\beta_1c^2\right)\|\vec{v}\|_{1,\Omega}^2\end{aligned}$$

On the other hand. According to 5.55 in [1], there exists a constant $\rho\succ 0$ such that:

$$\|\vec{v}\|_{0,\Omega}^2=\rho\left(\|\nabla\vec{v}\|_{0,\Omega}^2+\|\vec{v}\|_{0,\Gamma}^2\right)$$

then

$$\begin{aligned}\|\vec{v}\|_{1,\Omega}^2&=\|\nabla\vec{v}\|_{0,\Omega}^2+\|\vec{v}\|_{0,\Omega}^2\\&\le(\rho+1)\left(\|\nabla\vec{v}\|_{0,\Omega}^2+\|\vec{v}\|_{0,\Gamma}^2\right)\\&\le(\rho+1)\max(1;\frac{1}{\alpha_1})\left(\|\nabla\vec{v}\|_{0,\Omega}^2+\int_\Gamma \frac{a}{b}\vec{v}\vec{v}d\gamma\right)\end{aligned}$$

then $c_1\|\vec{v}\|_{1,\Omega}\le\|\vec{v}\|_{n,\Omega}$,

where $c_1=\left((\rho+1)\max(1;\frac{1}{\alpha_1})\right)^{-\frac{1}{2}}$.

Theorem 4.2 : $\left(H^1(\Omega),\|.\|_{J,\Omega}\right)$ is a real Hilbert space.

Proof: $(H^1(\Omega),\|.\|_{1,\Omega})$ is a real Hilbert space [2, 21] and $\|.\|_{1,\Omega}$ and $\|.\|_{J,\Omega}$ are equivalents norms, then $(H^1(\Omega),\|.\|_{J,\Omega})$ is a real Hilbert space.

Theorem 4.3 The bilinear form B satisfies the inf-sup condition: there exists a constant $\beta\succ 0$ such that:

$$\sup_{\vec{v}\in H^1(\Omega)}\frac{B(\vec{v},q)}{\|\vec{v}\|_{J,\Omega}}\ge\beta\|q\|_{0,\Omega}\ \ for\ all\ q\in L_0^2(\Omega)$$

Proof : We have [1, 2, 21]

$\exists\beta\succ 0$ such that :

$$\sup_{\vec{v}\in H_0^1(\Omega)}\frac{B(\vec{v},q)}{|\vec{v}|_{1,\Omega}}\ge\beta\|q\|_{0,\Omega}$$

Since $H_0^1(\Omega)\subset H^1(\Omega)$ and $\|\vec{v}\|_{J,\Omega}=|\vec{v}|_{1,\Omega}$ in $H_0^1(\Omega)$, then

$$\begin{aligned}\sup_{\vec{v}\in H^1(\Omega)}\frac{B(\vec{v},q)}{\|\vec{v}\|_{J,\Omega}}&\ge\sup_{\vec{v}\in H_0^1(\Omega)}\frac{B(\vec{v},q)}{\|\vec{v}\|_{J,\Omega}}\\&=\sup_{\vec{v}\in H_0^1(\Omega)}\frac{B(\vec{v},q)}{|\vec{v}|_{1,\Omega}}\ge\beta\|q\|_{0,\Omega}\end{aligned}$$

We define the “big” symmetric bilinear form

$$C\left[(\vec{u},p);(\vec{v},q)\right]=A(\vec{u},\vec{v})+B(\vec{u},q)+B(\vec{v},p).\tag{4.5}$$

And the corresponding function $F(\vec{v},q)=L(\vec{v})$, choosing the successive test vectors $(\vec{v},0)$ and $(\vec{0},q)$ shows that the Stokes problem (3.8) can be rewritten in the form:

$$\begin{aligned}&\textit{find } (\vec{u}, p) \in H^1(\Omega) \times L_0^2(\Omega) \text{ such that}\\ &C[(\vec{u}, p); (\vec{v}, q)] = F(\vec{v}, q)\\ &\textit{for all } (\vec{v}, q) \in H^1(\Omega) \times L_0^2(\Omega).\end{aligned} \tag{4.6}$$

The bilinear form A is positive continuous and $H^1(\Omega)-$ elliptic and the bilinear form B is continuous and satisfies the inf-sup condition. Then the problem (3.8) is well-posed [1, 2, 4, 21] and C and A are bilinear forms satisfies the following propositions.

Proposition 4.4 ([1. 21])

For all $(\vec{w}, s) \in H^1(\Omega) \times L_0^2(\Omega)$, we have

$$\sup_{(\vec{v},q)\in H^1(\Omega)\times L_0^2(\Omega)} \frac{C[(\vec{w}, s); (\vec{v}, q)]}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \geq \gamma(\|\vec{w}\|_{J,\Omega} + \|s\|_{0,\Omega}) \tag{4.7}$$

where γ is a positive constant depends only on the shape of the domain Ω.

Proposition 4.5

For all $(\vec{w}, s) \in H^1(\Omega) \times L_0^2(\Omega)$, we have

$$\sup_{(\vec{v},q)\in H^1(\Omega)\times L_0^2(\Omega)} \frac{A(\vec{w}, \vec{v}) + d(s, q)}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \geq \frac{1}{2}(\|\vec{w}\|_{J,\Omega} + \|s\|_{0,\Omega}) \tag{4.8}$$

where $d(s, q) = \int_\Omega s.q$ for all $s, q \in L^2(\Omega)$.

Proof. Let $(\vec{w}, s) \in H^1(\Omega) \times L_0^2(\Omega)$,

We will take $(\vec{v}, q) = (\vec{w}, 0)$ in the first and $(\vec{v}, q) = (\vec{0}, s)$ in the second, we obtain

$$\sup_{(\vec{v},q)\in H^1(\Omega)\times L_0^2(\Omega)} \frac{A(\vec{w}, \vec{v}) + d(s, q)}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \geq \|\vec{w}\|_{J,\Omega}, \tag{4.9}$$

$$\sup_{(\vec{v},q)\in H^1(\Omega)\times L_0^2(\Omega)} \frac{A(\vec{w}, \vec{v}) + d(s, q)}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \geq \|s\|_{0,\Omega} \tag{4.10}$$

we gather (4.9) and (4.10) to get (4.8).

The bilinear form A is symmetric and continuous and semi positive definite on $H^1(\Omega)$ in this case we say the problem (3.8) is a type of saddle-point problem. The results (4.1)-(4.4) ensure the existence and uniqueness of the solution of the problem (3.8) (see the theorem 6. 2 in [1]). In the following section we will solve this problem by mixed finite element method.

V. Mixed finite element approximation

Let $T_h; h \succ 0$, be a family of rectangulations of Ω. For any $T \in T_h$, ω_T is of rectangles sharing at least one edge with element T, $\tilde{\omega}_T$ is the set of rectangles sharing at least one vertex with T. Also, for an element edge E, ω_E denotes the union of rectangles sharing E, while $\tilde{\omega}_E$ is the set of rectangles sharing at least one vertex whit E.

Next, ∂T is the set of the four edges of T. We denote by $\varepsilon(T)$ and N_T the set of its edges and vertices, respectively.

We let $\varepsilon_h = \cup_{T\in T_h} \varepsilon(T)$ denotes the set of all edges split into interior and boundary edges.

$$\varepsilon_h = \varepsilon_{h,\Omega} \cup \varepsilon_{h,\Gamma}$$

Where

$\varepsilon_{h,\Omega} = \{E \in \varepsilon_h : E \subset \Omega\}$ and $\varepsilon_{h,\Gamma} = \{E \in \varepsilon_h : E \subset \Gamma\}$

We denote by h_T the diameter of a simplex T, by h_E the diameter of a face E of T, and we set $h = Max\{h_T : T \in T_h\}$

A discrete weak formulation (3.8) is defined using finite dimensional spaces $X_h^1 \subset H^1(\Omega)$ and $M^h \subset L_0^2(\Omega)$ as:

$$\begin{cases} \textit{find } \vec{u}_h \in X_h^1 \textit{ and } p_h \in M^h \textit{ such that:} \\ A(\vec{u}_h, \vec{v}_h) + B(\vec{v}_h, p_h) = L(\vec{v}_h) \\ B(\vec{v}_h, p_h) = 0 \end{cases} \tag{5.1}$$

For all $\vec{v}_h \in X_h^1$ *and* $q_h \in M^h$.

We use a set of vector-valued basis functions $\{\vec{\varphi}_i\}_{i=1,\ldots,n_u}$, so that

$$\vec{u}_h = \sum_{j=1}^{n_u} u_j \vec{\varphi}_j \tag{5.2}$$

We introduce a set of pressure basis functions $\{\psi_k\}_{k=1,\ldots,n_p}$ and set

$$p_h = \sum_{k=1}^{n_p} p_k \psi_k \tag{5.3}$$

Where n_u and n_p are the numbers of velocity and pressure basis functions, respectively.

We find that the discrete formulation (5.1) can be expressed as a system of linear equations

$$\begin{pmatrix} A_0 & B_0^T \\ B_0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{5.4}$$

With

$$A_0 = [a_{i,j}]; a_{i,j} = \int_\Omega \nabla\vec{\varphi}_i : \nabla\vec{\varphi}_j + \int_{\partial\Omega} \frac{a}{b} \vec{\varphi}_i . \vec{\varphi}_j \tag{5.5}$$

$$B_0 = [b_{k,j}]; b_{k,j} = -\int_\Omega \psi_k \nabla . \vec{\varphi}_j \tag{5.6}$$

$$f = [f_i]; \ f_i = \int_\Omega \vec{f} . \vec{\varphi}_i + \int_{\partial\Omega} \frac{1}{b} \vec{t} \vec{\varphi}_i, \tag{5.7}$$

for $i; j = 1,\ldots, n_u$ and $k = 1,\ldots, n_p$. And the function pair $(\vec{u}_h, p_h)$ obtained by substituting the solution vectors $U \in IR^{n_u}$ and $p \in IR^{n_p}$ into (5.2) and (5.3) is the mixed finite element solution. The system (5.4)-(5.7) is henceforth referred to as the discrete stokes problem. We use the iterative methods Minimum Residual Method (MINRES) for solving the symmetric system.

VI. A POSTERIORI ERROR ESTIMATOR

In this section, we propose two types of a posteriori error indicator. The first one is the residual error estimator and the second one is the local Poisson problem estimator. Which are shown to give global error estimates that are equivalent to the true error.

A. *A Residual error estimator.*

The bubble functions on the reference element $\widetilde{T}=(0,1)\times(0,1)$ are defined as follows:

$$b_{\widetilde{T}}=2^4x(1-x)y(1-y)$$
$$b_{\widetilde{E}_1,\widetilde{T}}=2^2x(1-x)(1-y)$$
$$b_{\widetilde{E}_2,\widetilde{T}}=2^2y(1-y)x$$
$$b_{\widetilde{E}_3,\widetilde{T}}=2^2y(1-x)x$$
$$b_{\widetilde{E}_4,\widetilde{T}}=2^2y(1-y)(1-x)$$

Here $b_{\widetilde{T}}$ is the reference element bubble function, and $b_{\widetilde{E}_i,\widetilde{T}}, i=1:4$ are reference edge bubble functions. For any $T\in T_h$, the element bubble functions is $b_T=b_{\widetilde{T}}\circ F_T$ and the element edge bubble function is $b_{E_i,T}=b_{\widetilde{E}_i,\widetilde{T}}\circ F_T$, where F_T the affine map form $\widetilde{T}$ to T. For an interior edge $E\in\varepsilon_{h,\Omega}$, b_E is defined piecewise, so that $b_{E/T_i}=b_{E,T_i}, i=1:2$ where $\mathrm{E}=\overline{\mathrm{T}}_1\cap\overline{\mathrm{T}}_2$. For a boundary edge $\mathrm{E}\in\varepsilon_{h,\Gamma}$, $b_E=b_{E,T}$, where T is the rectangle such that $\mathbf{E}\in\partial\mathbf{T}$.

With these bubble functions, ceruse et al ([19], lemma 4.1] established the following lemma.

Lemma 6.1. Let T be an arbitrary rectangle in T_h and $E\in\partial T$. For any $\vec{v}_T\in P_{k_0}(T)$ and $\vec{v}_E\in P_{k_0}(E)$, the following inequalities hold.

$$c_k\|\vec{v}_T\|_{0,T}\le\left\|\vec{v}_T b_T^{\frac{1}{2}}\right\|_{0,T}\le C_k\|\vec{v}_T\|_{0,T} \tag{6.1}$$
$$|\vec{v}_T b_T|_{1,T}\le C_k h_T^{-\frac{1}{2}}\|\vec{v}_T\|_{0,T} \tag{6.2}$$
$$c_k\|\vec{v}_E\|_{0,E}\le\left\|\vec{v}_E b_E^{\frac{1}{2}}\right\|_{0,E}\le C_k\|\vec{v}_E\|_{0,E} \tag{6.3}$$
$$\|\vec{v}_E b_E\|_{0,T}\le C_k h_E^{\frac{1}{2}}\|\vec{v}_E\|_{0,E} \tag{6.4}$$
$$|\vec{v}_E b_E|_{1,T}\le C_k h_E^{-\frac{1}{2}}\|\vec{v}_E\|_{0,E} \tag{6.5}$$

Where c_k and C_k are tow constants which only depend on the element aspect ratio and the polynomial degrees k_0 and k_1. Here, k_0 and k_1 are fixed and c_k and C_k can be associated with generic constants c and C In addition, $\vec{v}_E$ which is only defined on the edge E also denotes its natural extension to the element T. From the inequalities (6.4) and (6.5) we established the following lemma:

Lemma6.2. Let T be a rectangle and $E\in\partial T\cap\varepsilon_{h,\Gamma}$. for any $\vec{v}_E\in P_{k_0}(E)$, the following inequalities hold.

$$\|\vec{v}_E b_E\|_{J,T}\le C_k h_E^{-\frac{1}{2}}\|\vec{v}_E\|_{0,E} \tag{6.6}$$

Proof. Since $\vec{v}_E b_E=\vec{0}$ in the other three edges of rectangle T, it can be extended to the whole of Ω by setting $\vec{v}_E b_E=\vec{0}$ in $\Omega-\overline{T}$, then

$\|\vec{v}_E b_E\|_{1,T}=\|\vec{v}_E b_E\|_{1,\Omega}$ and $\|\vec{v}_E b_E\|_{J,T}=\|\vec{v}_E b_E\|_{J,\Omega}$.

Using the inequalities (6.4), (6.5) and (4.1) , gives

$$\begin{aligned}\|\vec{v}_E b_E\|_{J,T}&=\|\vec{v}_E b_E\|_{J,\Omega}\\&\le c_2\|\vec{v}_E b_E\|_{1,\Omega}\\&=c_2\|\vec{v}_E b_E\|_{1,T}\\&=c_2\left(\|\vec{v}_E b_E\|_{0,T}^2+|\vec{v}_E b_E|_{1,T}^2\right)^{\frac{1}{2}}\\&\le c_2C_k\left(h_E+h_E^{-1}\right)^{\frac{1}{2}}\|\vec{v}_E\|_{0,E}\\&\le c_2C_k\left(D^2+1\right)^{\frac{1}{2}}h_E^{-\frac{1}{2}}\|\vec{v}_E\|_{0,E}\\&\le Ch_E^{-\frac{1}{2}}\|\vec{v}_E\|_{0,E}\end{aligned}$$

With D is the diameter of Ω and $C=c_2C_k\left(D^2+1\right)^{\frac{1}{2}}$.

Lemma 6.3. Clement interpolation estimate: Given $\vec{v}\in H^1(\Omega)$, let $\vec{v}_h\in X_h^1$ be the quasi-interpolant of $\vec{v}$ defined by averaging as in [20]. For any $T\in T_h$, and any $E\in\partial T$, we have:

$$\|\vec{v}-\vec{v}_h\|_{0,T}\le Ch_T|\vec{v}|_{1,\widetilde{\omega}_T} \tag{6.7}$$
$$\|\vec{v}-\vec{v}_h\|_{0,E}\le Ch_E^{\frac{1}{2}}|\vec{v}|_{1,\widetilde{\omega}_E} \tag{6.8}$$

We let $(\vec{u},p)$ denote the solution of (3.8) and let denote $(\vec{u}_h,p_h)$ the solution of (5.1) with an approximation on a rectangular subdivision T_h.

Our aim is to estimate the velocity and the pressure errors $\vec{e}=\vec{u}-\vec{u}_h\in H^1(\Omega)$; $\varepsilon=p-p_h\in L^2(\Omega)$.

The element contribution $\eta_{R,T}$ of the residual error estimator η_R is given by

$$\eta_{R,T}^2=h_T^2\left\|\vec{R}_T\right\|_{0,T}^2+\|R_T\|_{0,T}^2+\sum_{E\in\partial T}h_E\left\|\vec{R}_E\right\|_{0,E}^2 \tag{6.9}$$

and the components in (6.9) are given by

$$\vec{R}_T=\left\{\vec{f}+\nabla^2\vec{u}_h-\nabla p_h\right\}/T$$
$$R_T=\left\{\nabla.\vec{u}_h\right\}/T$$
$$\vec{R}_E=\begin{cases}\frac{1}{2}[\![\nabla\vec{u}_h-p_hI]\!];\ E\in\varepsilon_{h,\Omega}\\ \frac{1}{b}\vec{t}-\left(\frac{a}{b}\vec{u}_h+(\nabla\vec{u}_h-p_hI)\vec{n}_{E,T}\right);\ E\in\varepsilon_{h,\Gamma}\end{cases}$$

With the key contribution coming from the stress jump associated with an edge E adjoining elements T and S:

$$[\![\nabla\vec{u}_h-p_hI]\!]=\left((\nabla\vec{u}_h-p_hI)/T-(\nabla\vec{u}_h-p_hI)/S\right)\vec{n}_{E,T}$$

The global residual error estimator is given by:

$$\eta_R = \sqrt{\sum_{T\in T_h} \eta_{R,T}^2}\ .$$

Our aim is to bound $\|\vec{u}-\vec{u}_h\|_X$ and $\|p-p_h\|_X$ with respect to the norm $\|.\|_J$ for velocity $\|.\|_X = \|.\|_{J,\Omega}$ and the quotient norm for the pressure $\|.\|_M = \|.\|_{0,\Omega}$.

For any $T\in T_h$, and $E\in\partial T$, we define the following two functions:

$$\vec{w}_T = \vec{R}_T b_T\ ;\ \vec{w}_E = \vec{R}_E b_E\ .$$

- $\vec{w}_T = \vec{0}$ on ∂T.
- If $E\in\partial T\cap\varepsilon_{h,\Omega}$ then $\vec{w}_E = \vec{0}$ in $\partial\omega_E$.
- If $E\in\partial T\cap\varepsilon_{h,\Gamma}$ then $\vec{w}_E = \vec{0}$ in the other edges of rectangle T.
- $\vec{w}_T$ and $\vec{w}_E$ can be extended to the whole of Ω by setting

$\vec{w}_T = \vec{0}$ in Ω - T ,

$\vec{w}_E = \vec{0}$ in Ω - $\overline{\omega}_E$ if $E\in\partial T\cap\varepsilon_{h,\Omega}$.

$\vec{w}_E = \vec{0}$ in Ω - $\overline{T}$ if $E\in\partial T\cap\varepsilon_{h,\Gamma}$.

With these two functions we have the following lemmas:

Lemma 6.4. For any $T\in T_h$, we have:

$$\int_T \vec{f}.\vec{w}_T = \int_T(\nabla\vec{u}-pI):\nabla\vec{w}_T\ . \tag{6.10}$$

Proof. We apply the Green formula, $\vec{w}_T = \vec{0}$ in ∂T and (2.1), we obtain

$$\begin{aligned}\int_T(\nabla\vec{u}-pI):\nabla\vec{w}_T &= \int_{\partial T}(\nabla\vec{u}-pI)\vec{n}.\vec{w}_T\\ &\quad-\int_T(\nabla^2\vec{u}-\nabla p).\vec{w}_T\\ &= \int_T \vec{f}.\vec{w}_T\end{aligned}$$

Lemma 6.5.

i) If $E\in\partial T\cap\varepsilon_{h,\Omega}$, We have:

$$\int_{\omega_E}\vec{f}.\vec{w}_E = \int_{\omega_E}(\nabla\vec{u}-pI):\nabla\vec{w}_E \tag{6.11}$$

ii) If $E\in\partial T\cap\varepsilon_{h,\Gamma}$, We have:

$$\int_T\vec{f}.\vec{w}_E = \int_T(\nabla\vec{u}-pI):\nabla\vec{w}_E + \int_{\partial T}\left(\frac{a}{b}\vec{u}-\frac{1}{b}\vec{t}\right).\vec{w}_E \tag{6.12}$$

Proof. i) The same proof of (6.10).

ii) if $E\in\partial T\cap\varepsilon_{h,\Gamma}$. Using (2.1), gives

$$\int_T\vec{f}.\vec{w}_E = \int_T\left(-\nabla^2\vec{u}+\nabla p\right).\vec{w}_E$$

By applying the Green formula, we obtain

$$\int_T\vec{f}.\vec{w}_E = \int_T(\nabla\vec{u}-pI):\nabla\vec{w}_E - \int_{\partial T}(\nabla\vec{u}-pI)\vec{n}.\vec{w}_E$$

Since $\vec{w}_E = \vec{0}$ in the other three edges of rectangle T and we have by (2.3), $a\vec{u}+b(\nabla\vec{u}-pI)\vec{n} = \vec{t}$ in $E\subset\Gamma\cap\partial T$, then

$$\int_T\vec{f}.\vec{w}_E = \int_T(\nabla\vec{u}-pI):\nabla\vec{w}_E + \int_{\partial T}\left(\frac{a}{b}\vec{u}-\frac{1}{b}\vec{t}\right)\vec{w}_E\ .$$

Theorem 6.6. For any mixed finite element approximation (not necessarily inf-sup stable) defined on rectangular grids T_h, the residual estimator η_R satisfies:

$$\|\vec{e}\|_{J,T} + \|\varepsilon\|_{0,T} \le C_\Omega\eta_R$$

And
$$\eta_{R,T} \le C\left(\sum_{T'\in\omega_T}\left\{\|\vec{e}\|_{J,T'}^2 + \|\varepsilon\|_{0,T'}^2\right\}\right)^{\frac{1}{2}}$$

Note that the constant C in the local lower bound is independent of the domain.

Proof. We include this for completeness. To establish the upper bound we let $(\vec{v},q)\in H^1(\Omega)\times L^2(\Omega)$. and $\vec{v}_h\in X_h^1$ be the clement interpolant of $\vec{v}$ then

$$\begin{aligned}
&C[(\vec{e},\varepsilon);(\vec{v},q)]\\
&= C[(\vec{e},\varepsilon);(\vec{v}-\vec{v}_h,q)]\\
&= C[(\vec{u},p);(\vec{v}-\vec{v}_h,q)] - C[(\vec{u}_h,q_h);(\vec{v}-\vec{v}_h,q)]\\
&= L(\vec{v}-\vec{v}_h) - A(\vec{u}_h,\vec{v}-\vec{v}_h) - B(\vec{v}-\vec{v}_h,p_h) - B(\vec{u}_h,q)\\
&= \sum_{T\in T_h}\left\{\left(\vec{R}_T,\vec{v}-\vec{v}_h\right)_T - \sum_{E\in\partial T}\left(\vec{R}_E,\vec{v}-\vec{v}_h\right)_E + \left(R_T,q\right)_T\right\}\\
&\le \sum_{T\in T_h}\left\{\|\vec{R}_T\|_{0,T}\|\vec{v}-\vec{v}_h\|_{0,T} + \sum_{E\in\partial T}\|\vec{R}_E\|_{0,T}\|\vec{v}-\vec{v}_h\|_{0,E}\right.\\
&\qquad\left. + \|q\|_{0,T}\|R_T\|_{0,E}\right\}\\
&\le C\left(\sum_{T\in T_h}h_T^2\|\vec{R}_T\|_{0,T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in T_h}\frac{1}{h_T^2}\|\vec{v}-\vec{v}_h\|_{0,T}^2\right)^{\frac{1}{2}}\\
&\quad+\left(\sum_{T\in T_h}\sum_{E\in\partial T}h_E\|\vec{R}_E\|_{0,T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in T_h}\sum_{E\in\partial T}\frac{1}{h_E}\|\vec{v}-\vec{v}_h\|_{0,E}^2\right)^{\frac{1}{2}}\\
&\quad+\left(\sum_{T\in T_h}\|q\|_{0,T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in T_h}\|R_T\|_{0,T}^2\right)^{\frac{1}{2}}
\end{aligned}$$

Using (6.7) and (6.8), then gives:

$$C[(\vec{e},\varepsilon);(\vec{v},q)] \le C'\left(\sum_{T\in T_h}\left\{\|\vec{v}\|_{J,T}^2 + \|q\|_{0,T}^2\right\}\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{T\in T_h}\left\{h_T^2\|\vec{R}_T\|_{0,T}^2 + \|R_T\|_{0,T}^2 + \sum_{E\in\partial T}h_E\|\vec{R}_E\|_{0,E}^2\right\}\right)^{\frac{1}{2}} \tag{6.13}$$

Finally, using (4.7), gives:

$$\|\vec{e}\|_{J,T} + \|\varepsilon\|_{0,T} \le C_\Omega\left(\sum_{T\in T_h}\left\{h_T^2\|\vec{R}_T\|_{0,T}^2 + \|R_T\|_{0,T}^2 + \sum_{E\in\partial T}h_E\|\vec{R}_E\|_{0,E}^2\right\}\right)^{\frac{1}{2}} \tag{6.14}$$

This establishes the upper bound. Turning to the local lower bound. First, for the element residual part, we have:

$$\int_T \vec{R}_T.\vec{w}_T = \int_T (\vec{f} + \nabla^2\vec{u}_h - \nabla p_h).\vec{w}_T$$
$$= \int_T \vec{f}.\vec{w}_T - \int_T (\nabla\vec{u}_h - p_h I) : \nabla\vec{w}_T$$
$$+ \int_{\partial T} (\nabla\vec{u}_h - p_h I)\vec{n}.\vec{w}_T.$$

Using (6.10), (6.2) and $\vec{w}_T = \vec{0}$ in ∂T , gives:

$$\int_T \vec{R}_T.\vec{w}_T = \int_T (\nabla\vec{e} - \varepsilon I) : \nabla\vec{w}_T$$
$$\leq (|\vec{e}|_{1,T} + \|\varepsilon\|_{0,T})|\vec{w}_T|_{1,T}$$
$$\leq C(\|\vec{e}\|_{J,T}^2 + \|\varepsilon\|_{0,T}^2) h_T^{-1} \|\vec{R}_T\|_{0,T}$$

In addition, from the inverse inequality (6.1),

$$\int_T \vec{R}_T.\vec{w}_T = \left\|\vec{R}_T b_T^{\frac{1}{2}}\right\|_{0,\mathrm{T}}^2 \geq \mathrm{c}\left\|\vec{R}_T\right\|_{0,\mathrm{T}}^2$$

Thus,

$$h_T^2 \left\|\vec{R}_T\right\|_{0,\mathrm{T}}^2 \leq C(\|\vec{e}\|_{J,T}^2 + \|\varepsilon\|_{0,T}^2). \tag{6.15}$$

Next comes the divergence part,

$$\|R_T\|_{0,T} = \|\nabla.\vec{u}_h\|_{0,T}$$
$$= \|\nabla.(\vec{u} - \vec{u}_h)\|_{0,T}$$
$$\leq \sqrt{2}|\vec{u} - \vec{u}_h|_{1,T}$$
$$\leq \sqrt{2}\|\vec{u} - \vec{u}_h\|_{J,T}$$
$$\leq \sqrt{2}\|\vec{e}\|_{J,T} \tag{6.16}$$

Finally, we need to estimate the jump term. For an edge if $E \in \partial T \cap \varepsilon_{h,\Omega}$, We have

$$2\int_E \vec{R}_E.\vec{w}_E = \sum_{i=1} \int_{\partial T_i} (\nabla\vec{u}_h - p_h I)\vec{n}.\vec{w}_E$$
$$= \int_{\omega_E} (\nabla\vec{u}_h - p_h I) : \nabla\vec{w}_E$$
$$+ \sum_{i=1:2} \int_{T_i} (\nabla^2\vec{u}_h - \nabla p_h).\vec{w}_E. \tag{6.17}$$

Using (6.11) and $\vec{w}_E = \vec{0}$ in $\partial\omega_{\mathrm{E}}$, gives :

$$2\int_E \vec{R}_E.\vec{w}_E = -\int_{\omega_E} (\nabla\vec{e} - \varepsilon I) : \nabla\vec{w}_E + \sum_{i=1:2} \int_{T_i} \vec{R}_{T_i} \vec{w}_E$$
$$\leq (|\vec{e}|_{1,\omega_E} + \|\varepsilon\|_{0,\omega_E})|\vec{w}_E|_{1,\omega_E} + \sum_{i=1:2} \left\|\vec{R}_{T_i}\right\|_{0,T_i} \|\vec{w}_E\|_{0,T_i}$$

Using (6.4) and (6.5) , gives

$$2\int_E \vec{R}_E.\vec{w}_E \leq C(|\vec{e}|_{1,\omega_E}^2 + \|\varepsilon\|_{0,\omega_E}^2)^{\frac{1}{2}} h_E^{-\frac{1}{2}} \left\|\vec{R}_E\right\|_{0,E}$$
$$+ \sum_{i=1:2} \left\|\vec{R}_{T_i}\right\|_{0,T_i} h_E^{\frac{1}{2}} \left\|\vec{R}_E\right\|_{0,E} \tag{6.18}$$

Using (6.15), gives

$$\int_E \vec{R}_E.\vec{w}_E \leq C(\|\vec{e}\|_{J,\omega_E}^2 + \|\varepsilon\|_{0,\omega_E}^2)^{\frac{1}{2}} h_E^{-\frac{1}{2}} \left\|\vec{R}_E\right\|_{0,E} \tag{6.19}$$

Using (6.3) gives

$$\int_E \vec{R}_E.\vec{w}_E = \left\|\vec{R}_E b_E^{\frac{1}{2}}\right\|_{0,E}^2 \geq c\left\|\vec{R}_E\right\|_{0,E}^2,$$

and thus using (6.19) gives,

$$h_E \left\|\vec{R}_E\right\|_{0,\mathrm{E}}^2 \leq C(\|\vec{e}\|_{J,\omega_E}^2 + \|\varepsilon\|_{0,\omega_E}^2). \tag{6.20}$$

We also need to show that (6.20) holds for boundary edges.

For an $E \in \partial T \cap \varepsilon_{h,\Gamma}$, We have

$$\int_E \vec{R}_E.\vec{w}_E = \int_{\partial T} \left[\frac{a}{b}\vec{u}_h + (\nabla\vec{u}_h - p_h I)\vec{n} - \frac{1}{b}\vec{t} \right].\vec{w}_E$$
$$= \int_{\partial T} \left(\frac{a}{b}\vec{u}_h - \frac{1}{b}\vec{t} \right).\vec{w}_E + \int_{\partial T} \left((\nabla\vec{u}_h - p_h I)\vec{n}\right).\vec{w}_E$$
$$= \int_{\partial T} \left(\frac{a}{b}\vec{u}_h - \frac{1}{b}\vec{t} \right).\vec{w}_E + \int_T \left((\nabla\vec{u}_h - p_h I)\right) : \nabla\vec{w}_E$$
$$+ \int_T \left((\nabla^2\vec{u}_h - \nabla p_h)\right).\vec{w}_E$$

Using (6.12) and (2.4), gives

$$\int_E \vec{R}_E.\vec{w}_E = -\int_T \left((\nabla\vec{e} - \varepsilon I)\right) : \nabla\vec{w}_E - \int_{\partial T} \frac{a}{b}\vec{e}.\vec{w}_E$$
$$+ \int_T \vec{R}_T.\vec{w}_E$$
$$\leq (|\vec{e}|_{1,T} + \|\varepsilon\|_{0,T})|\vec{w}_E|_{1,T} + \beta_1\|\vec{e}\|_{0,\partial T}\|\vec{w}_E\|_{0,\partial T}$$
$$+ \left\|\vec{R}_T\right\|_{0,T}\|\vec{w}_E\|_{0,T}$$
$$\leq C(\|\vec{e}\|_{J,T} + \|\varepsilon\|_{0,T})\|\vec{w}_E\|_{J,T} + \left\|\vec{R}_T\right\|_{0,T}\|\vec{w}_E\|_{0,T}.$$

Using (6.4) and (6.6) , gives

$$\int_E \vec{R}_E.\vec{w}_E \leq C''(\|\vec{e}\|_{J,T} + \|\varepsilon\|_{0,T}) h_E^{-\frac{1}{2}} \left\|\vec{R}_E\right\|_{0,E}$$
$$+ \left\|\vec{R}_T\right\|_{0,T} h_E^{\frac{1}{2}} \left\|\vec{R}_E\right\|_{0,E}$$

Using (6.15), gives

$$\int_E \vec{R}_E.\vec{w}_E \leq C(\|\vec{e}\|_{J,T}^2 + \|\varepsilon\|_{0,T}^2)^{\frac{1}{2}} h_E^{-\frac{1}{2}} \left\|\vec{R}_E\right\|_{0,E} \tag{6.21}$$

Using (6.3)

$$\int_E \vec{R}_E.\vec{w}_E = \left\|\vec{R}_E b_E^{\frac{1}{2}}\right\|_{0,E}^2 \geq c\left\|\vec{R}_E\right\|_{0,E}^2,$$

and this using (6.21), gives

$$h_E \left\|\vec{R}_E\right\|_{0,\mathrm{E}}^2 \leq C(\|\vec{e}\|_{J,T}^2 + \|\varepsilon\|_{0,T}^2). \tag{6.22}$$

Finally, combining (6.15), (6.16), (6.20) and (6.22) establishes the local lower bound.

Remark 6.7. Theorem 6.6 also holds for stable (and unstable) mixed approximations defined on a triangular subdivision if we take the obvious interpretation of ω_T. The Proof is identical except for the need to define appropriate element and edge bubble functions.

B. *The local Poisson problem estimator*

The local Poisson problem estimator:

$$\eta_P = \sqrt{\sum_{T\in T_h} \eta_{P,T}^2}$$

Where

$$\eta_{P,T}^2 = \left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2 \tag{6.23}$$

Let

$$\mathrm{Q_T} = L^2(T) \text{ and } \mathrm{V_T} = H^1(T) \tag{6.24}$$

$$A_T(\vec{e}_{P,T},\vec{v}) = \int_T \nabla\vec{e}_{P,T} : \nabla\vec{v} + \int_{\partial T} \frac{a}{b}\vec{e}_{P,T}.\vec{v} \tag{6.25}$$

$\vec{e}_{P,T} \in V_T$ satisfies the uncoupled Poisson problems

$$A_T(\vec{e}_{P,T},\vec{v}) = (\vec{R}_T,\vec{v})_T - \sum_{E\in\varepsilon(T)} (\vec{R}_E,\vec{v})_E \tag{6.26}$$

for any $\vec{v} \in V_T$.

And

$$(\varepsilon_{P,T}, q) = (\nabla.\vec{u}_h, q)/T \text{ for all } q \in Q_T \tag{6.27}$$

Theorem 6.8 The estimator $\eta_{P,T}$ is equivalent to the $\eta_{R,T}$ estimator: $c\eta_{P,T} \le \eta_{R,T} \le C\eta_{P,T}$.

Proof. For the upper bound, we first let $\vec{w}_T = \vec{R}_T b_T$ (b_T is an element interior bubble).

From (6.26),

$$(\vec{R}_T,\vec{w}_T)_T = \int_T \nabla\vec{e}_{P,T} : \nabla\vec{w}_T$$
$$\le \left|\vec{e}_{P,T}\right|_{1,T} \left|\vec{w}_T\right|_{1,T.}$$

Using (6.2), we get

$$(\vec{R}_T,\vec{w}_T)_T \le Ch_T^{-1}\left\|\vec{R}_T\right\|_{0,T}\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}} \tag{6.28}$$

In addition, from the inverse inequalities (6.1), $\left\|\vec{R}_T\right\|_{0,\mathrm{T}}^2 \le \mathrm{C}.(\vec{R}_T,\vec{w}_T)_T$ and using (6.28), to get

$$h_T^2\left\|\vec{R}_T\right\|_{0,T}^2 \le C\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}} \tag{6.29}$$

Next, we let $\vec{w}_E = \vec{R}_E b_E$ (b_E is an edge bubble function).

If $E \in \partial T \cap \varepsilon_{h,\Gamma}$, using (6.26), (6.4), (6.6) and (6.29), gives

$$(\vec{R}_E,\vec{w}_E)_E = -A_T(\vec{e}_{P,T},\vec{w}_E) + (\vec{R}_T,\vec{w}_E)_T$$
$$\le \left\|\vec{e}_{P,T}\right\|_{J,T}\left\|\vec{w}_E\right\|_{J,T} + \left\|\vec{R}_T\right\|_{0,T}\left\|\vec{w}_E\right\|_{0,T}$$
$$\le C\left\|\vec{w}_E\right\|_{J,T}\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}}$$
$$+ Ch_T^{-1}\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}}\left\|\vec{w}_E\right\|_{0,T}$$
$$\le Ch_E^{-\frac{1}{2}}\left\|\vec{R}_E\right\|_{0,E}\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}}$$

If $E \in \partial T \cap \varepsilon_{h,\Omega}$,

see that a and b defined just in Γ, then we can posed $a = 0$ in $\Omega - \Gamma$.

Using (6.26), (6.4), (6.5) and (6.29), gives

$$(\vec{R}_E,\vec{w}_E)_E = -(\nabla\vec{e}_{P,T},\nabla\vec{w}_E)_T + (\vec{R}_T,\vec{w}_E)_T$$
$$\le \left|\vec{e}_{P,T}\right|_{1,T}\left|\vec{w}_E\right|_{1,T} + \left\|\vec{R}_T\right\|_{0,T}\left\|\vec{w}_E\right\|_{0,T}$$
$$\le Ch_E^{-\frac{1}{2}}\left\|\vec{R}_E\right\|_{0,E}\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}}.$$

Finally, for any $\mathrm{T} \in T_h$ and any $E \in \partial T$, we have

$$(\vec{R}_E,\vec{w}_E)_E \le Ch_E^{-\frac{1}{2}}\left\|\vec{R}_E\right\|_{0,E}\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}} \tag{6.31}$$

From the inverse inequalities (6.3), $\left\|\vec{R}_E\right\|_{0,E}^2 \le \mathrm{C}.(\vec{R}_E,\vec{w}_E)_E$ and using (6.31), gives

$$h_E\left\|\vec{R}_E\right\|_{0,E}^2 \le C\left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right) \tag{6.32}$$

By (6.27), we have also

$$\left\|R_T\right\|_{0,T} = \left\|\nabla.\vec{u}_h\right\|_{0,T}$$
$$= \left\|\varepsilon_{P,T}\right\|_{0,T}$$
$$\le \left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}} \tag{6.33}$$

Combining (6.29), (6.31) and (6.32), establishes the upper bound in the equivalence relation.

For the lower , we need to use (4.8), (6.26) and (6.27):

$$\eta_{P,T} = \left(\left\|\vec{e}_{P,T}\right\|_{J,T}^2 + \left\|\varepsilon_{P,T}\right\|_{0,T}^2\right)^{\frac{1}{2}}$$
$$\le \left(\left\|\vec{e}_{P,T}\right\|_{J,T} + \left\|\varepsilon_{P,T}\right\|_{0,\mathrm{T}}\right)$$
$$\le 2\sup_{(\vec{v},q)\in V_T\times L_0^2(T)} \frac{A_T(\vec{e}_{P,T},\vec{v}) + d(\varepsilon_{P,T},q)}{\left\|\vec{v}\right\|_{J,T} + \left\|q\right\|_{0,T}}$$
$$\le 2\sup_{(\vec{v},q)\in V_T\times L_0^2(T)} \frac{(\vec{R}_T,\vec{v})_T - \sum_{E\subset\partial T}(\vec{R}_E,\vec{v})_E + (R_T,q)_T}{\left\|\vec{v}\right\|_{J,T} + \left\|q\right\|_{0,T}}$$
$$\le 2\sup_{(\vec{v},q)\in V_T\times L_0^2(T)} \frac{\left\|\vec{R}_T\right\|_{0,T}\left\|\vec{v}\right\|_{0,T} + \sum_{E\subset\partial T}\left\|\vec{R}_E\right\|_{0,E}\left\|\vec{v}\right\|_{0,E} + \left\|R_T\right\|_{0,T}\left\|q\right\|_{0,T}}{\left\|\vec{v}\right\|_{J,T} + \left\|q\right\|_{0,T}} \tag{6.34}$$

Now, since $\vec{v}$ is zero at the four vertices of T, a scaling argument and the usual trace theorem, see e.g. [14, Lemma 1.5], shows that $\vec{v}$ satisfies

$$\left\|\vec{v}\right\|_{0,E} \le Ch_E^{\frac{1}{2}}\left|\vec{v}\right|_{1,T} \tag{6.35}$$

$$\left\|\vec{v}\right\|_{0,T} \le Ch_T\left|\vec{v}\right|_{1,T} \tag{6.36}$$

Combining these two inequalities with (6.34) immediately gives the lower bound in the equivalence relation.

Theorem 6.9. For any mixed finite element approximation (not necessarily inf-sup stable) defined on rectangular grids T_h, the residual estimator η_P satisfies:

$$\|\bar{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \le C_{\Omega}\eta_P \tag{6.35}$$

and

$$\eta_{P,T} \le C\left(\sum_{T'\in\omega_T}\left\{\|\bar{e}\|^2_{J,T'} + \|\varepsilon\|^2_{0,T'}\right\}\right)^{\frac{1}{2}}.$$

Note that the constant C in the local lower bound is independent of the domain.

VII. NUMERICAL SIMULATION

In this section we propose two test problems, the first is a classic problem used in fluid dynamics, known as driven cavity flow [14, 17]. For this latter, we compute the solution and the two errors estimators η_R and η_P. The second is a test problem with an exact solution is solved in order to compare the affectivity of two error estimation strategies. The latter approach is frequently used and is generally considered by practitioners to be one the best error estimation strategies in terms of its simplicity and reliability, especially when used as a refinement indicator in a self-adaptive refinement setting.

Example 1. It is a model of the flow in a square cavity $\Omega =]0,1[\times]0,1[$ with the lid moving from left to right. Let the computational model:

$$\begin{cases} \vec{u} = (1-x^2, 0)\ sur\ \Gamma_1 = \{y=1;\ -1\le x\le 1\} \\ \vec{u} = (0,0)\ on\ \ the\ \ other\ tree\ \ edges. \end{cases}$$

See that the our $C_{a,b}$ boundary condition is satisfied, just take, just take a and b two real number strictly positive such that $a \succ\succ b$, $\vec{t} = (a(1-x^2);0)$ on Γ_1 and $\vec{t} = (0;0)$ on the other three boundary of the square domain.

The streamlines are computed from the velocity solution by solving the Poisson equation numerically subject to a zero Dirichlet boundary condition.

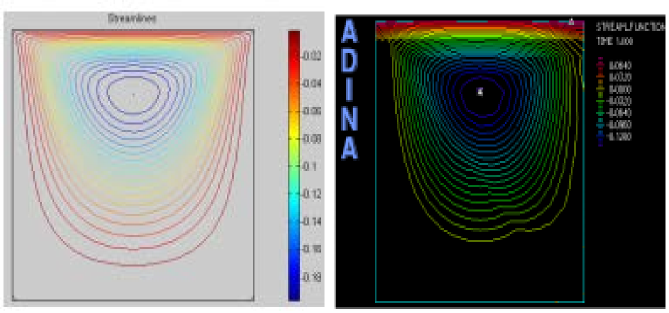

Figure 1. Uniform streamline plot by MFE (left) associated with a 64-64 square grid, Q2-Q1 approximation, and uniform streamline plot (right) computed with ADINA System.

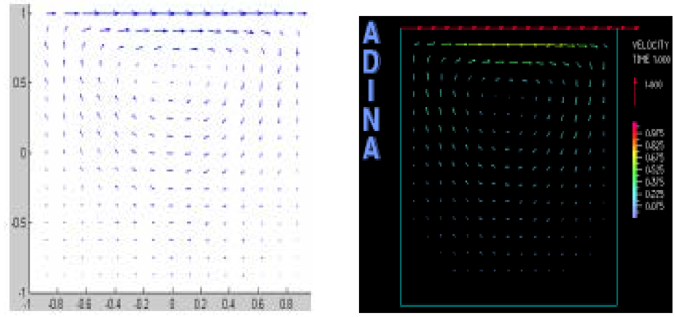

Figure 2. Velocity vectors solution by MFE (left) associated with a 64-64 square grid, Q2-Q1 approximation and Velocity vectors solution (right) computed with ADINA System.

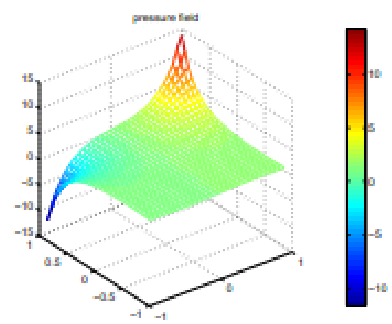

Figure 3. Pressure plot for the flow with a 64×64 square grid.

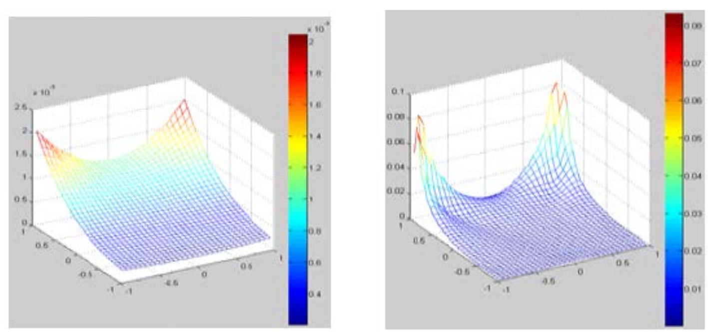

Figure 4. Estimator $\eta_{P,T}$ (left) and estimator $\eta_{R,T}$ (right) for the test problem with 32×32 square grid and $\nu = 1/500$.

Table1. Estimated errors η_R and η_P for the test problem.

Grid	η_R	η_P
8×8	1.4433e+000	7.1916e - 001
16×16	7.7582e -001	4.2234e - 001
32×32	3.9279e -001	2.1872e - 001
64×64	1.9692e -001	1.1023e - 001

The computational results (Figure 4, Table 1) suggest that all two estimators seem to beable to correctly indicate the structure of the error, but what is the most close to the exact error? We will answer this question in the following test.

Example 2. It's a test problem with an exact solution is solved in order to compare the affectivity of two error estimation strategies: the residual estimator η_R and the Poisson estimators η_P. The latter approach is frequently used and is generally considered by practitioners to be one the best error estimation strategies in terms of its simplicity and reliability, especially when used as a refinement indicator in a self-adaptive refinement setting. This analytic test problem is associated with the following solution of the Stokes equation system:

$$u_x = 20xy^3; u_y = 20x^4 - 5y^4; \qquad (7.1)$$
$$p = 60x^2y - 20y^3 + \text{constant}.$$

It is a simple model of colliding flow, and a typical solution of streamline is illustrated in Figure 4. To solve this problem numerically, the finite element interpolant of the velocity in (7.1) is specified everywhere on $\partial\Omega$. The Dirichlet boundary condition for the stream function calculation is the interpolant of the exact stream function: $\psi(x, y) = 5xy^4 - x^5$.

It is clear that the $C_{a,b}$ condition is satisfied with $a \succ\succ b$ and $\vec{t} = \left(20axy^3; (20x^4 - 5y^4)a\right)$ on Γ.

The flow problem is solved on a square domain $]-1,1[\times]-1,1[$ using a nested sequence of uniformly refined square grids.

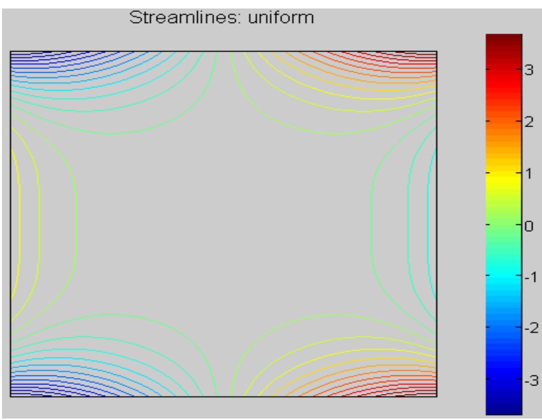

Figure 5. Uniform streamline plot by MFE associated with a 64×64 square grid.

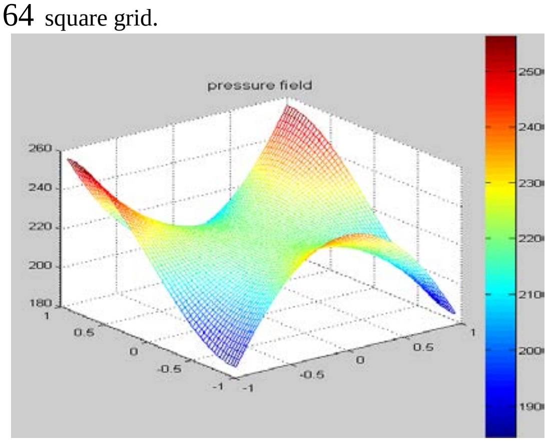

Figure 6. Pressure plot for the flow with a 64×64 square grid.

To interpret the results that are presented some notation will be needed:

$$e = \sqrt{\|\vec{u} - \vec{u}_h\|_{J,\Omega} + \|P - P_h\|_{0,\Omega}} \qquad (7.2)$$
$$e_T = \sqrt{\|\vec{u} - \vec{u}_h\|_{J,T} + \|P - P_h\|_{0,T}} \qquad (7.3)$$
$$e_T = \sqrt{\|\vec{u} - \vec{u}_h\|_{J,T} + \|P - P_h\|_{0,T}} \qquad (7.4)$$

The figure 5 shows the uniform streamline, figure 6 shows the pressure plot and figure 7 shows the estimated error $\eta_{P,T}$ associated with 64 × 64 square grid.

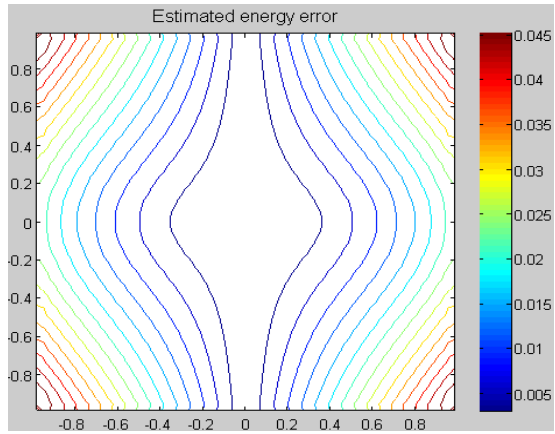

Figure 7. Estimated error $\eta_{P,T}$ associated with 64×64 square grid.

Table 2 Comparison of error estimator affectivity

Grid	e	e/η_R	e/η_p
8×8	1.4433e+00	3.1206e- 01	1.1916e + 00
16×16	7.7582^e-01	3.0597e -01	1.0234e + 01
32×32	3.9279e 01	2.9134e -01	9.2872e - 01
64×64	1.9692^e-01	2.9082e -01	9.1723e - 01

Looking at Table 2, we see that the global error e is decreasing and e/η_p is very close to 1, e/η_R is very close to $1/3$, then the Poisson problem estimator η_p provides the most accurate estimate of the global error and the local estimates $\eta_{p,T}$ is quantitatively close to the exact error and the estimates η_R is about three times larger than exact error.

Depending on Theorems 6.6 and 6.9, we see that the local error estimator η_T ($\eta_T = \eta_{P,T}$ or $\eta_{R,T}$) satisfied

$$e \le C_\Omega \eta \ \text{ and } \eta = \sqrt{\sum\nolimits_{T\in T_h} \eta_T^2} \tag{7.5}$$

$$\eta_T \le C\left(\sum_{T'\in T_h} \left\{ \|\bar{e}\|_{J,T'}^2 + \|\varepsilon\|_{0,T'}^2 \right\} \right)^{\frac{1}{2}}. \tag{7.6}$$

Here, the generic constant C_Ω is independent of the mesh size and the exact solution but may depend on the domain and the element aspect ration. Then the estimators η_T is likely to be effective if it is used to drive an adaptive refinement process. In general, if an error estimator is to be efficient then the constant on the right hand side of (7.6) should be bounded. An estimate of this constant $(e.g \max_{T\in T_h} \frac{\eta_T}{e_{\omega_T}})$ is provided in Table 3, where we also estimate this constant for the exact error $(e.g \max_{T\in T_h} \frac{e_T}{e_{\omega_T}})$.

Table 3 Comparison of affectivity indices

grid	e	$\max_{T\in T_h} \frac{e_T}{e_{\omega_T}}$	$\max_{T\in T_h} \frac{\eta_{R.T}}{e_{\omega_T}}$	$\max_{T\in T_h} \frac{\eta_{P.T}}{e_{\omega_T}}$
8×8	1.4433e+000	5.8923e-001	2.1659e+000	6:2529e - 001
16×16	7.7582e -001	6.1997e -001	2.2577e +000	5:5183e - 001
32×32	3.9279e -001	5.9143e-001	2.2743e +000	5:3152e - 001
64×64	1.9692e -001	5.3092e -001	2.2162e +000	6:3092e - 001

From the Table 3, $\max_{T\in T_h} \frac{\eta_{R.T}}{e_{\omega_T}}$ and $\max_{T\in T_h} \frac{\eta_{P.T}}{e_{\omega_T}}$, seem to be bounded. In addition $\max_{T\in T_h} \frac{\eta_{P.T}}{e_{\omega_T}}$ is closed to $\max_{T\in T_h} \frac{e_T}{e_{\omega_T}}$.

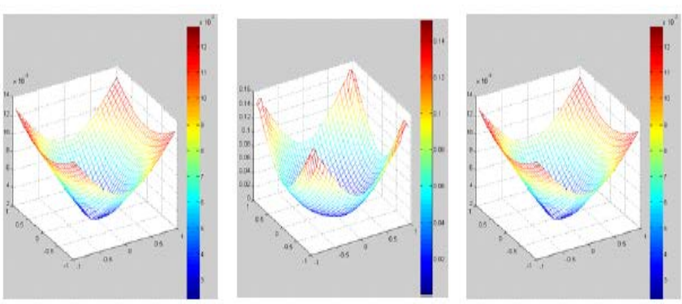

Figure 8. Exact e_T (left), estimator $\eta_{R.T}$ (middle) and estimator $\eta_{P.T}$ (right) for the problem with 32×32 square grid.

The local affectivity indices $\frac{\eta_T}{e_{\omega_T}}$ will be bounded above and below across the whole domain, so that elements with large errors can be singled out for local mesh refinement. This is assessed in Figure 8. Looking at the distribution of these indices it is clear that the our two estimators give a very different picture. Once again, $\eta_{P.T}$ is closely aligned with the exact error but $\eta_{R.T}$ is not.

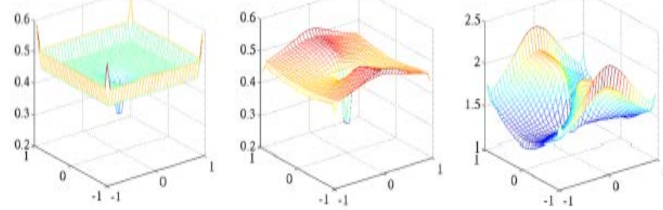

Figure 9. Exact affectivity $\frac{e_T}{e_{\omega_T}}$ (left), estimator affectivity $\frac{\eta_{p.T}}{e_{\omega_T}}$ (middle) and estimator affectivity $\frac{\eta_{R.T}}{e_{\omega_T}}$ (right) with a 32 × 32 square grid and $\nu = 1/500$.

VII. Conclusion

We were interested in this work in the numeric solution for two dimensional partial differential equations modeling (or arising from) model steady incompressible fluid flow. It includes algorithms for discretization by mixed finite element methods and a posteriori error estimation of the computed solutions. Two types of a posteriori error indicator are introduced and are shown to give global error estimates that are equivalent to the true discretization error. The computational results suggest that all two estimators seem to be able to correctly indicate the structure of the error.

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