The Approximate Solution of a Fredholm Integral Equation

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***Abstract*—**In this paper we present two algorithms to calculate the approximate solution of a Fredholm integral equation. These approximation algorithms were obtained under the conditions of theorem of existence and uniqueness of the solution of this integral equation in the sphere $\overline{B}(f;r) \subset C[a,b]$, that has been presented in [14]. The approximate solution of this integral equation was obtained by applying the successive approximations method and for the approximate calculation of integrals that appear in the terms of the successive approximations sequence were used two quadrature formulas: the trapezoids formula and the rectangles formula, respectively.

***Keywords*—**Approximate solution, approximation algorithm, Fredholm integral equation, method of successive approximations, quadrature formula.

I. INTRODUCTION

SOME of the approximation algorithms for the solution of several integral equations, established by the author, were published in the papers [1], [4], [5], [10], [12]. We mention the following three nonlinear Fredholm integral equations, for which were set approximation algorithms of the solution:

a) $$x(t) = \int_a^b K(t, s, x(s), x(g(s)))ds + f(t),$$

where $K\in C([a,b]\times[a,b]\times\boldsymbol{R}^2)$ or $K\in C([a,b]\times[a,b]\times J^2)$, $J\subset\boldsymbol{R}$ is a closed interval, $g\in C([a,b],[a,b])$ and $f\in C[a,b]$;

b) $$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t),$$

where $K\in C([a,b]\times[a,b]\times\boldsymbol{R}^3)$ or $K\in C([a,b]\times[a,b]\times J^3)$, $J\subset\boldsymbol{R}$ is a closed interval and $f\in C[a,b]$, ([1], [4]);

c) $$x(t) = \int_a^b K(t, s, x(s), x(g(s)), x(a), x(b))ds + f(t),$$

where $K\in C([a,b]\times[a,b]\times\boldsymbol{R}^4)$ or $K\in C([a,b]\times[a,b]\times J^4)$, $J\subset\boldsymbol{R}$ is a closed interval, $g\in C([a,b],[a,b])$, $f\in C[a,b]$, ([5], [10]).

In each of the three cases we have $a, b \in \boldsymbol{R}$, $a < b$.

Moreover, for the solution of these integral equations, but, also for others, such the integral equation of epidemics

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds\ ,$$

where $t \in\boldsymbol{R}$, $\tau > 0$ is a parameter, were studied the existence and uniqueness, lower and upper subsolutions, data dependence and differentiability with respect to a parameter (see [1], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]).

In this paper, the approximation algorithms were established for the following Fredholm integral equation:

$$x(t) = \int_a^b K(t, s)\cdot h(s, x(s), x(a), x(b))ds + f(t), \tag{1}$$

where a, $b \in \boldsymbol{R}$, $a < b$, $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times\boldsymbol{R}^3)$ or $h\in C([a,b]\times J^3)$, $J\subset\boldsymbol{R}$ is a closed interval and $f\in C[a,b]$.

The author studied the existence and uniqueness of the solution of this integral equation, in the space $C[a,b]$ and in the sphere $\overline{B}(f;r) \subset C[a,b]$, respectively, and he obtained two theorems that were published in [14]. We present below these theorems.

First of all, we denote by M_K a positive constant, such that

$|K(t,s)| \le M_K$, *for all* $t, s\in[a,b]$.

Theorem 1 (of existence and uniqueness in the space $C[a,b]$, [14])

For the Fredholm integral equation (1) *we assume that*:

(i) $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times\boldsymbol{R}^3)$, $f\in C[a,b]$;

(ii) *there exists* $\alpha, \beta, \gamma > 0$ *such that*

$|h(s, u_1, u_2, u_3)-h(s, v_1, v_2, v_3)| \le \alpha|u_1-v_1|+\beta|u_2-v_2|+\gamma|u_3-v_3|$, *for all* $s\in[a,b]$, $u_i, v_i\in\boldsymbol{R}$, $i = 1, 2, 3$;

(iii) $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a) < 1$.

Under these conditions the integral equation (1) *has a unique solution* $x^*\in C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in C[a,b]$.

Moreover, if x_m *is the m-th successive approximation, then we have the following estimation*:

$$\|x^* - x_m\|_{C[a,b]} \le \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\|x_1 - x_0\|_{C[a,b]}. \tag{2}$$

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Theorem 2 (of existence and uniqueness in the sphere $\overline{B}(f;r) \subset C[a,b]$, [14])

For the Fredholm integral equation (1) *we assume that*:

(i) $K\in C([a,b]\times[a,b])$, $h\in C([a,b]\times J^3)$, $J\subset \mathbf{R}$ *is a closed interval*, $f\in C[a,b]$;

(ii) *there exists* $\alpha, \beta, \gamma > 0$ *such that*

$|h(s, u_1, u_2, u_3)-h(s, v_1, v_2, v_3)| \le \alpha|u_1-v_1|+\beta|u_2-v_2|+\gamma|u_3-v_3|$,

for all $s\in[a,b]$, $u_i, v_i\in J\subset\mathbf{R}$, $i = 1, 2, 3$;

(iii) $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a) < 1$.

If there exists $r > 0$ *such that*

$$\left[x\in \overline{B}(f;r)\right] \Rightarrow \left[x(t)\in J\subset R\right] \tag{3}$$

and the following condition is fulfilled:

(iv) $M_K\cdot M_h\cdot(b-a) < r$,

where M_h *is a positive constant such that, for the restriction* $h\big|_{[a,b]\times J^3}$, $J\subset\mathbf{R}$ *a closed interval, we have*:

$|h(s, u, v, w)| \le M_h$, *for all* $s\in[a,b]$, $u, v, w\in J$,

then the integral equation (1) *has a unique solution* $x^*\in \overline{B}(f;r)\subset C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0\in \overline{B}(f;r)\subset C[a,b]$.

Moreover, if x_m *is the m-th successive approximation, then the estimation* (2) *is met.*

The integral equations of similar type have been studied in [2], [3], [5], [6], [7], [8], [9], [11], [13].

The numerical analysis of an integral equation consists in establishment of an algorithm for approximating the solution of the studied equation.

Under the conditions of theorem 2, presented above, the purpose of this paper is to develop two methods for approximating the solution of the integral equation (1), using the successive approximations method, and for approximate calculation of the integrals that appear in the terms of the successive approximations sequence were used the following two quadrature formulas: the trapezoids formula and the rectangles formula, respectively.

II. THE STATEMENT OF THE PROBLEM

To establish these procedures for approximating the solution of the integral equation (1) were used the results given by Gh Coman, I. Rus, G. Pavel and I. A. Rus [3], D. V. Ionescu [16], I. A. Rus [22], V. Mureșan [18] and Gheorghe Marinescu [17].

We suppose that the conditions of theorem 2 are fulfilled and therefore the integral equation (1) has a unique solution in the sphere $\overline{B}(f;r)\subset C[a,b]$.

We denote this solution by $x^*\in \overline{B}(f;r)\subset C[a,b]$ and it can be obtained by the successive approximations method, starting at any element $x_0\in \overline{B}(f;r)\subset C[a,b]$.

In addition, if x_m is the m-th successive approximation, then the estimation (2) is true.

Therefore, for the determination of x^* we apply the successive approximations method.

To get a better result, it is considered an equidistant division Δ of the interval $[a,b]$ through the points:

$$\Delta:\ a = t_0 < t_1 < \ldots < t_n = b .$$

Now, we have the sequence of successive approximations:

$$\begin{aligned}
&x_0(t_k) = f(t_k), \quad x_0\in \overline{B}(f;r)\subset C[a,b]\\
&x_1(t_k) = \int_a^b K(t_k,s)\cdot h(s,x_0(s),x_0(a),x_0(b))ds + f(t_k)\\
&x_2(t_k) = \int_a^b K(t_k,s)\cdot h(s,x_1(s),x_1(a),x_1(b))ds + f(t_k)\\
&\ldots\ldots\ldots\ldots\\
&x_{m+1}(t_k) = \int_a^b K(t_k,s)\cdot h(s,x_m(s),x_m(a),x_m(b))ds + f(t_k)\\
&\ldots\ldots\ldots\ldots
\end{aligned} \tag{4}$$

Next, in the following two sections we present two methods for approximating the solution of integral equation (1), obtained by applying the successive approximations method and using also, the trapezoids formula and the rectangles formula for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence.

III. USED QUADRATURE FORMULAS

A. Trapezoids formula

Let $g\in C^2[a,b]$ be a function and we present the trapezoids formula for calculating the approximate value of integral:

$$\int_a^b g(t)dt .$$

First, we consider an equidistant division of the interval $[a,b]$ through the points:

$$\Delta:\ a = t_0 < t_1 < \ldots < t_n = b .$$

The trapezoids formula for calculation of the approximate value of this integral has the expression:

$$\int_a^b g(t)dt = \frac{b-a}{2n}\left[g(a) + 2\sum_{i=1}^{n-1} g(t_i) + g(b)\right] + R^T(g) \tag{5}$$

and $R^T(g) = \sum_{i=1}^{n} R_i^T(g)$ is the rest of formula (5), having the estimation:

$$\left|R^T(g)\right| \le M^T \frac{(b-a)^3}{12n^2}. \qquad (6)$$

B. Rectangles formula

Let $g \in C^1[a,b]$ be a function and we present the rectangles formula for calculating the approximate value of integral:

$$\int_a^b g(t)dt.$$

For this, we consider an equidistant division of the interval $[a,b]$ through the points:

$$\Delta:\ a = t_0 < t_1 < \ldots < t_n = b.$$

The rectangles formula for calculation of the approximate value of this integral has two expressions:

a) if it consider the intermediary points of the division Δ of the interval $[a,b]$ on the left end of the partial intervals $\xi_I = t_i$, then the rectangles formula is defined by the relation:

$$\int_a^b g(t)dt = \frac{b-a}{n}\left[g(a) + \sum_{i=1}^{n-1} g(t_i)\right] + R^D(g), \qquad (7)$$

b) if it consider the intermediary points of the division of the interval $[a,b]$ on the right end of the partial intervals $\xi_I = t_{i+1}$, then the rectangles formula is defined by the relation:

$$\int_a^b g(t)dt = \frac{b-a}{n}\left[\sum_{i=1}^{n-1} g(t_i) + g(b)\right] + R^D(g). \qquad (7')$$

The rest of the formula $R^D(g) = \sum_{i=1}^{n} R_i^D(g)$ has the estimation:

$$\left|R^D(g)\right| \le M^D \frac{(b-a)^2}{n}. \qquad (8)$$

IV. Approximation of the Solution Using the Trapezoids Formula

To that effect we suppose that $K \in C^2([a,b]\times[a,b])$, $h \in C^2([a,b]\times J^3)$, $J \subset \mathbf{R}$ is a closed interval and $f \in C^2[a,b]$.

For the calculation of the integrals that appear in the terms of the successive approximations sequence, the trapezoids formula (5)+(6) is going to be applied.

In the general case for $x_m(t_k)$ we have:

$$x_m(t_k) = \frac{b-a}{2n}\left[K(t_k,a)\cdot h(a, x_{m-1}(a), x_{m-1}(a), x_{m-1}(b)) + \right.$$
$$+ 2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, x_{m-1}(t_i), x_{m-1}(a), x_{m-1}(b)) +$$
$$\left. + K(t_k,b)\cdot h(b, x_{m-1}(b), x_{m-1}(a), x_{m-1}(b))\right] +$$
$$+ f(t_k) + R_{m,k}^T,\quad k = \overline{0,n},\ m \in N \qquad (9)$$

with the estimation of the rest

$$\left|R_{m,k}^T\right| \le \frac{(b-a)^3}{12n^2}\cdot \max_{s\in[a,b]}\left|\left[K(t_k,s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))\right]_s''\right|. \qquad (10)$$

Since $K \in C^2([a,b]\times[a,b])$ and $h \in C^2([a,b]\times J^3)$, it results that $K\cdot h \in C^2([a,b]\times[a,b]\times J^3)$ and there exists the derivative of the function $K\cdot h$ from the expression of $R_{m,k}^T$, and therefore it has to be calculated. So, we have:

$$\frac{d(K\cdot h)}{ds} = \frac{\partial K}{\partial s}h + K\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial s}\right)$$

$$\frac{d^2(K\cdot h)}{ds^2} = \frac{\partial^2 K}{\partial s^2}\cdot h + 2\frac{\partial K}{\partial s}\cdot\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial s}\right) +$$
$$+ K\left(\frac{\partial^2 h}{\partial s^2} + 2\frac{\partial^2 h}{\partial x_{m-1}\partial s}\cdot\frac{\partial x_{m-1}}{\partial s} + \frac{\partial^2 h}{\partial x_{m-1}^2}\cdot\left(\frac{\partial x_{m-1}}{\partial s}\right)^2 + \right.$$
$$\left. + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial^2 x_{m-1}}{\partial s^2}\right)$$

and therefore

$$\left[K(t_k,s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))\right]_s'' =$$
$$= \frac{\partial^2 K}{\partial s^2}\cdot h + 2\frac{\partial K}{\partial s}\cdot\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}} x_{m-1}'(s)\right) +$$
$$+ K\cdot\left(\frac{\partial^2 h}{\partial s^2} + \frac{\partial^2 h}{\partial x_{m-1}\partial s} x_{m-1}'(s) + \frac{\partial^2 h}{\partial x_{m-1}^2}\left(x_{m-1}'(s)\right)^2 + \right.$$
$$\left. + \frac{\partial h}{\partial x_{m-1}} x_{m-1}''(s)\right)$$

and

$$x_{m-1}(t) = \int_a^b K(t,s)\cdot h(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b))ds + f(t)$$

$$x'_{m-1}(t) = \int_a^b \frac{\partial K(t,s)}{\partial t} h\left(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b)\right)ds + f'(t)$$

$$x''_{m-1}(t) = \int_a^b \frac{\partial^2 K(t,s)}{\partial t^2} h(s, x_{m-2}(s), x_{m-2}(a), x_{m-2}(b))ds + f''(t).$$

If we take into account the expression of the derivatives of $x_{m-1}(t)$ and we denote

$$M_1^T = \max_{\substack{|\alpha|\le 2 \\ t,s\in[a,b]}} \left| \frac{\partial^\alpha K(t,s)}{\partial t^{\alpha_1}\cdot \partial s^{\alpha_2}} \right| ,$$

$$M_2^T = \max_{\substack{|\alpha|\le 2 \\ s\in[a,b]}} \left| \frac{\partial^\alpha h(s,u,v,w)}{\partial s^{\alpha_1}\cdot \partial u^{\alpha_2}} \right| ,$$

$$M_3^T = \max_{\substack{\alpha\le 2 \\ t\in[a,b]}} \left| f^{(\alpha)}(t) \right| ,$$

then we obtain the following estimations for $x_{m-1}(t)$ and its derivatives:

$$|x_{m-1}(t)| \le M_1^T M_2^T (b-a) + M_3^T ,$$

$$|x'_{m-1}(t)| \le M_1^T M_2^T (b-a) + M_3^T ,$$

$$|x''_{m-1}(t)| \le M_1^T M_2^T (b-a) + M_3^T ,$$

while for the derivative of function $K\cdot h$ we have the estimation:

$$\left| \left[K(t_k,s)\cdot h(s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b)) \right]''_s \right| \le$$
$$\le 4M_1^T M_2^T + 5M_1^T M_2^T \left[M_1^T M_2^T (b-a) + M_3^T \right] +$$
$$+ M_1^T M_2^T \left[M_1^T M_2^T (b-a) + M_3^T \right]^2 = M_0^T .$$

It is obvious that M_0^T doesn't depend on m and k, therefore the estimation of the rest $R_{m,k}^T$ is:

$$\left| R_{m,k}^T \right| \le M_0^T \cdot \frac{(b-a)^3}{12n^2} , \tag{11}$$

where $M_0^T = M_0^T(K, D^\alpha K, h, D^\alpha h, f, D^\alpha f), |\alpha| \le 2$, and thus, we obtain a formula for the approximate calculation of the integrals that appear in the terms of the successive approximations sequence.

Using the method of successive approximations and the formula (9) with the estimation of the rest resulted from (11), we suggest further on an algorithm, in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence. Thus, we have:

$x_0(t_k) = f(t_k)$

$$x_1(t_k) = \int_a^b K(t_k,s)\cdot h(s, f(s), f(a), f(b))ds + f(t_k) =$$
$$= \frac{b-a}{2n} \left[K(t_k,a)\cdot h(a, f(a), f(a), f(b)) + \right.$$
$$+ 2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, f(t_i), f(a), f(b)) +$$
$$\left. + K(t_k,b)\cdot h(b, f(b), f(a), f(b)) \right] + f(t_k) + R_{1,k}^T =$$
$$= \tilde{x}_1(t_k) + R_{1,k}^T \quad , \quad k = \overline{0,n}$$

$$x_2(t_k) = \int_a^b K(t_k,s)\cdot h(s, x_1(s), x_1(a), x_1(b))ds + f(t_k) =$$
$$= \frac{b-a}{2n} \left[K(t_k,a)\cdot h(a, x_1(a), x_1(a), x_1(b)) + \right.$$
$$+ 2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, x_1(t_i), x_1(a), x_1(b)) +$$
$$\left. + K(t_k,b)\cdot h(b, x_1(b), x_1(a), x_1(b)) \right] + f(t_k) + R_{2,k}^T =$$
$$= \frac{b-a}{2n} \left[K(t_k,a)\cdot h(a, \tilde{x}_1(a) + R_{1,0}^T, \tilde{x}_1(a) + R_{1,0}^T, \tilde{x}_1(b) + R_{1,0}^T) + \right.$$
$$+ 2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, \tilde{x}_1(t_i) + R_{1,i}^T, \tilde{x}_1(a) + R_{1,i}^T, \tilde{x}_1(b) + R_{1,i}^T) +$$
$$\left. + K(t_k,b)\cdot h(b, \tilde{x}_1(b) + R_{1,n}^T, \tilde{x}_1(a) + R_{1,n}^T, \tilde{x}_1(b) + R_{1,n}^T) \right] +$$
$$+ f(t_k) + R_{2,k}^T = \frac{b-a}{2n} \left[K(t_k,a)\cdot h(a, \tilde{x}_1(a), \tilde{x}_1(a), \tilde{x}_1(b)) + \right.$$
$$+ 2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, \tilde{x}_1(t_i), \tilde{x}_1(a), \tilde{x}_1(b)) +$$
$$\left. + K(t_k,b)\cdot h(b, \tilde{x}_1(b), \tilde{x}_1(a), \tilde{x}_1(b)) \right] + f(t_k) + \tilde{R}_{2,k}^T =$$
$$= \tilde{x}_2(t_k) + \tilde{R}_{2,k}^T , \quad k = \overline{0,n} ,$$

where

$$\left| \tilde{R}_{2,k}^T \right| \le \frac{b-a}{2n} M_K(\alpha+\beta+\gamma) \left(\left| R_{1,0}^T \right| + 2\sum_{i=1}^{n-1} \left| R_{1,i}^T \right| + \left| R_{1,n}^T \right| \right) + \left| R_{2,k}^T \right| \le$$
$$\le M_K(\alpha+\beta+\gamma)(b-a) M_0^T \frac{(b-a)^3}{12n^2} + M_0^T \frac{(b-a)^3}{12n^2} =$$
$$= \frac{(b-a)^3}{12n^2} \cdot M_0^T \left[M_K(\alpha+\beta+\gamma)\cdot(b-a) + 1 \right] .$$

This reasoning continues for $m = 3$ and through induction we obtain:

$$x_m(t_k) = \frac{b-a}{2n}\left[K(t_k,a)\cdot h(a,\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b)) + \right.$$

$$+2\sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i,\tilde{x}_{m-1}(t_i),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b)) +$$

$$\left. + K(t_k,b)\cdot h(b,\tilde{x}_{m-1}(b),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b))\right] + f(t_k) + \tilde{R}^T_{m,k}$$

$$= \tilde{x}_m(t_k) + \tilde{R}^T_{m,k}, \quad k = \overline{0,n},$$

and

$$\left|\tilde{R}^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2}\cdot M_0^t \cdot \left\{\left[M_K(\alpha+\beta+\gamma)\cdot(b-a)\right]^{m-1} + \cdots + 1\right\},$$

$k = \overline{0,n}$.

Since the conditions of theorem 2 of existence and uniqueness of the solution of integral equation (1), are fulfilled, it results that $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a) < 1$, and we have the estimation:

$$\left|\tilde{R}^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2\left[1-M_K(\alpha+\beta+\gamma)\cdot(b-a)\right]}\cdot M_0^T. \tag{12}$$

Thus we have obtained a new sequence $(\tilde{x}_m(t_k))_{m\in N}$, $k = \overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in N}$ using an equidistant division of the interval $[a,b]$, $\Delta: a = t_0 < t_1 < ... < t_n = b$, with the following error in calculation:

$$\left|x_m(t_k) - \tilde{x}_m(t_k)\right| \le \frac{(b-a)^3}{12n^2\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}M_0^T \tag{13}$$

which, using the Chebyshev norm, becomes:

$$\left\|x_m - \tilde{x}_m\right\|_{C[a,b]} \le \frac{(b-a)^3}{12n^2\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}M_0^T. \tag{14}$$

Now, using the estimates (2) and (14) it is obtain the following result.

Theorem 3 *Suppose that the conditions of theorem* 2 *are fulfilled. In addition, we assume that the exact solution* x^* *of the integral equation* (1) *is approximated by the sequence* $(\tilde{x}_m(t_k))_{m\in N}$, $k = \overline{0,n}$, *on the nodes* t_k, $k = \overline{0,n}$, *of the equidistant division* Δ *of the interval* $[a,b]$, *using the successive approximations method* (4) *and the trapezoids formula* (5)+(6).

Under these conditions, the error of approximation is given by the following evaluation:

$$\left\|x^* - \tilde{x}_m\right\|_{C[a,b]} \le \frac{\left[M_K(\alpha+\beta+\gamma)(b-a)\right]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\left\|x_1 - x_0\right\|_{C[a,b]} +$$

$$+\frac{(b-a)^3}{12n^2\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}M_0^T. \tag{15}$$

V. Approximation of the Solution Using the Rectangles Formula

To that effect we suppose that $K\in C^1([a,b]\times[a,b])$, $h\in C^1([a,b]\times J^3)$, $J\subset \mathbf{R}$ is a closed interval, $f\in C^1[a,b]$.

For the calculation of the integrals that appear in the terms of the successive approximations sequence, the rectangles formula (7)+(8) is going to be applied.

In the general case for $x_m(t_k)$ we have:

$$x_m(t_k) = \frac{b-a}{n}\left[K(t_k,a)\cdot h(a,x_{m-1}(a),x_{m-1}(a),x_{m-1}(b)) + \right.$$

$$\left. + \sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i,x_{m-1}(t_i),x_{m-1}(a),x_{m-1}(b))\right] +$$

$$+ f(t_k) + R^D_{m,k}, \quad k = \overline{0,n}, \quad m\in N \tag{16}$$

with the estimation of the rest

$$\left|R^D_{m,k}\right| \le \frac{(b-a)^2}{n}\cdot \max_{s\in[a,b]}\left|\frac{dK(t_k,s)\cdot h(s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))}{ds}\right|. \tag{17}$$

Since $K\in C^1([a,b]\times[a,b])$ and $h\in C^1([a,b]\times J^3)$, it results that $K\cdot h\in C^1([a,b]\times[a,b]\times J^3)$ and there exists the derivative of the function $K\cdot h$ from the expression of $R^D_{m,k}$, and therefore it has to be calculated. So, we have:

$$\frac{d(K\cdot h)}{ds} = \frac{\partial K}{\partial s}h + K\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial s}\right),$$

and therefore

$$\left[K(t_k,s)\cdot h(s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))\right]'_s =$$

$$= \frac{\partial K}{\partial s}\cdot h + K\cdot\left(\frac{\partial h}{\partial s} + \frac{\partial h}{\partial x_{m-1}}x'_{m-1}(s)\right)$$

and

$$x_{m-1}(t) = \int_a^b K(t,s)\cdot h(s,x_{m-2}(s),x_{m-2}(a),x_{m-2}(b))ds + f(t)$$

$$x'_{m-1}(t) = \int_a^b \frac{\partial K(t,s)}{\partial t}h(s,x_{m-2}(s),x_{m-2}(a),x_{m-2}(b))ds + f'(t).$$

If we take into account the expression of the derivative of $x_{m-1}(t)$ and we denote

$$M_1^D = \max_{\substack{|\alpha|\le 1\\ t,s\in[a,b]}} \left| \frac{\partial^\alpha K(t,s)}{\partial t^{\alpha_1}\cdot \partial s^{\alpha_2}} \right| ,$$

$$M_2^D = \max_{\substack{|\alpha|\le 1\\ s\in[a,b]}} \left| \frac{\partial^\alpha h(s,u,v,w)}{\partial s^{\alpha_1}\cdot \partial u^{\alpha_2}} \right| ,$$

$$M_3^D = \max_{\substack{\alpha\le 1\\ t\in[a,b]}} \left| f^{(\alpha)}(t) \right| ,$$

then we obtain the following estimations for $x_{m-1}(t)$ and its derivative:

$$|x_{m-1}(t)| \le M_1^D M_2^D (b-a) + M_3^D ,$$

$$|x'_{m-1}(t)| \le M_1^D M_2^D (b-a) + M_3^D ,$$

while for the derivative of function $K\cdot h$ we have the estimation:

$$\left| \left[K(t_k,s)\cdot h(s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b)) \right]'_s \right| \le$$
$$\le M_1^D M_2^D \left[2 + M_1^D M_2^D (b-a) + M_3^D \right] = M_0^D .$$

It is obvious that M_0^D doesn't depend on m and k, so the estimation of the rest is:

$$\left| R_{m,k}^D \right| \le M_0^D \cdot \frac{(b-a)^3}{12n^2} , \tag{18}$$

where $M_0^D = M_0^D(K, D^\alpha K, h, D^\alpha h, f, D^\alpha f)$, $|\alpha| \le 1$ and thus, we obtain a formula for the approximate, calculation of the integrals that appear in the terms of the successive approximations sequence.

Using the method of successive approximations and the formula (16) with the estimation of the rest resulted from (18), we suggest further on an algorithm in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and we obtain:

$x_0(t_k) = f(t_k)$

$$x_1(t_k) = \frac{b-a}{n}\left[K(t_k,a)\cdot h(a, f(a), f(a), f(b)) + \right.$$
$$\left. + \sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i, f(t_i), f(a), f(b)) \right] + f(t_k) + R_{1,k}^D =$$
$$= \tilde{x}_1(t_k) + R_{1,k}^D , \quad k = \overline{0,n}$$

$$x_2(t_k) = \int_a^b K(t_k,s)\cdot h(s,x_1(s),x_1(a),x_1(b))ds + f(t_k) =$$
$$= \frac{b-a}{n}\left[K(t_k,a)\cdot h(a,x_1(a),x_1(a),x_1(b)) + \right.$$
$$\left. + \sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i,x_1(t_i),x_1(a),x_1(b)) \right] + f(t_k) + R_{2,k}^D =$$
$$= \frac{b-a}{n}\cdot \left[K(t_k,a)\cdot h(a,\tilde{x}_1(a)+R_{1,0}^D,\tilde{x}_1(a)+R_{1,0}^D,\tilde{x}_1(b)+R_{1,0}^D) + \right.$$
$$\left. + \sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i,\tilde{x}_1(t_i)+R_{1,i}^D,\tilde{x}_1(a)+R_{1,i}^D,\tilde{x}_1(b)+R_{1,i}^D) \right] +$$
$$+ f(t_k) + R_{2,k}^D =$$
$$= \frac{b-a}{n}\left[K(t_k,a)\cdot h(a,\tilde{x}_1(a),\tilde{x}_1(a),\tilde{x}_1(b)) + \right.$$
$$\left. + \sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i,\tilde{x}_1(t_i),\tilde{x}_1(a),\tilde{x}_1(b)) \right] + f(t_k) + \tilde{R}_{2,k}^D =$$
$$= \tilde{x}_2(t_k) + \tilde{R}_{2,k}^D , \quad k = \overline{0,n} ,$$

where

$$\left| \tilde{R}_{2,k}^D \right| \le \frac{b-a}{n} M_K(\alpha+\beta+\gamma)\left(\left| R_{1,0}^D \right| + \sum_{i=1}^{n-1} \left| R_{1,i}^D \right| \right) + \left| R_{2,k}^D \right| \le$$
$$\le (b-a) M_K(\alpha+\beta+\gamma) M_0^D \frac{(b-a)^2}{n} + M_0^D \frac{(b-a)^2}{n} =$$
$$= \frac{(b-a)^2}{n}\cdot M_0^D \left[M_K(\alpha+\beta+\gamma)(b-a) + 1 \right].$$

This reasoning continues for $m = 3$ and through induction we obtain

$$x_m(t_k) = \frac{b-a}{n}\left[K(t_k,a)\cdot h(a,\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b)) + \right.$$
$$\left. + \sum_{i=1}^{n-1} K(t_k,t_i)\cdot h(t_i,\tilde{x}_{m-1}(t_i),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b)) \right] +$$
$$+ f(t_k) + \tilde{R}_{m,k}^D = \tilde{x}_m(t_k) + \tilde{R}_{m,k}^D , \quad k = \overline{0,n} ,$$

and

$$\left| \tilde{R}_{m,k}^D \right| \le \frac{(b-a)^2}{n} M_0^D \left\{ \left[M_K(\alpha+\beta+\gamma)\cdot(b-a) \right]^{m-1} + \cdots + 1 \right\},$$

$k = \overline{0,n}$.

Since the conditions of theorem 2 of existence and uniqueness of the solution of integral equation (1), are fulfilled, it results that $M_K\cdot(\alpha+\beta+\gamma)\cdot(b-a) < 1$, and we have the estimation:

$$\left|\tilde{R}^D_{m,k}\right| \le \frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D \ . \tag{19}$$

Thus, we have obtained the sequence, $(\tilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in N}$ using an equidistant division of the interval $[a,b]$, Δ: $a=t_0<t_1<...<t_n=b$, with the following error in calculation:

$$\left|x_m(t_k)-\tilde{x}_m(t_k)\right| \le \frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D \tag{20}$$

which, using the Chebyshev norm, becomes:

$$\left\|x_m-\tilde{x}_m\right\|_{C[a,b]} \le \frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D \ . \tag{21}$$

Now, using the estimates (2) and (21) it is obtain the following result.

Theorem 4 *Suppose that the conditions of theorem* 2 *are fulfilled. In addition, we assume that the exact solution* x^* *of the integral equation* (1) *is approximated by the sequence* $(\tilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$ *on the nodes* t_k, $k=\overline{0,n}$, *of the equidistant division* Δ *of the interval* $[a,b]$, *using the successive approximations method* (4) *and the rectangles formula* (7)+(8).

Under these conditions, the error of approximation is given by the evaluation:

$$\left\|x^*-\tilde{x}_m\right\|_{C[a,b]} \le \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\left\|x_1-x_0\right\|_{C[a,b]} +$$

$$+\frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D \ . \tag{22}$$

VI. CONCLUSION

Regarding to these two algorithms presented in this paper, we observe the following:

a) In both cases were used the method of successive approximations;

b) The terms of successive approximations sequence were approximated using two quadrature formulas: the trapezoids formula (5)+(6), and the rectangles formula (7)+(8), respectively.

c) In order to obtain a better result, in both cases were used an equidistant division, Δ, of the interval $[a,b]$ through the points $a=t_0<t_1<...<t_n=b$.

d) In the both cases, were obtained a new sequence $(\tilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in N}$.

d) We obtained the following estimates of the error of approximation, on nodes, of the terms of successive approximations sequence:

- using the trapezoids formula:

$$\left|\tilde{R}^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2\left[1-M_K(\alpha+\beta+\gamma)\cdot(b-a)\right]}\cdot M_0^T$$

- using the rectangles formula:

$$\left|\tilde{R}^D_{m,k}\right| \le \frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D$$

and both values M_0^T and M_0^D, from evaluation of rests, are independent of *m* and *k*.

e) The approximation error of the exact solution x^* of the integral equation (1) through the terms of the new sequence $(\tilde{x}_m(t_k))_{m\in N}$, $k=\overline{0,n}$, is given by the relation (15):

$$\left\|x^*-\tilde{x}_m\right\|_{C[a,b]} \le \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\left\|x_1-x_0\right\|_{C[a,b]} +$$

$$+\frac{(b-a)^3}{12n^2\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}M_0^T \ ,$$

when we used the trapezoids formula and by the relation (22):

$$\left\|x^*-\tilde{x}_m\right\|_{C[a,b]} \le \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^m}{1-M_K(\alpha+\beta+\gamma)(b-a)}\left\|x_1-x_0\right\|_{C[a,b]} +$$

$$+\frac{(b-a)^2}{n\left[1-M_K(\alpha+\beta+\gamma)(b-a)\right]}\cdot M_0^D \ .$$

when we used the rectangles formula.

f) Of the two estimates of remainder, from above, we deduce that the approximation error of the solution obtained by applying the successive approximations method is less if the trapezoids formula is used, than if the rectangles formula is used.

Finally, it should be noted that all articles and books, respectively, from references constituted an important research material in preparation of this article.

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Fields of work:

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