Linear Matrix Inequalities and Semidefinite Programming: Applications in Control

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***Abstract*—**This paper outlines the issues of Linear Matrix Inequalities (LMIs) and semidefinite programming with emphasis on their wide application potential in the field of automatic control. It presents the history and basic theory of LMIs, briefly introduces their possible solution by means of convex optimization, and overviews the selected problems from the control theory viewpoint. The final part of the work deals with the most popular LMI software solvers.

***Keywords*—**Linear Matrix Inequalities, Semidefinite Programming, Convex Optimization, Control Theory.

I. Introduction

LINEAR Matrix Inequalities (LMIs) represent elegant and effective tool for solving many optimization problems in the area of system and control theory, identification, and signal processing. Even though historically first LMI introduced Lyapunov as early as in about 1890, they have become popular not until the last decades when true "LMI-boom" has exploded – see just several examples in [1]–[4].

From the control point of view, the potential LMI usage is really wide and it includes various problems starting e.g. from stability analysis, going through H_2 and H_∞ issues, up to the synthesis of robust state-feedback, and many more as will be shown later. Obviously, considering such extensive range of problems in LMI way would not be useful unless they could be effectively solved. The key feature of LMI is that it defines convex constraint with respect to the variable vector. For that reason, the feasible set is convex and it can be found with the assistance of numerical algorithms of convex optimization, more specifically using the subfield of these optimization techniques which is known as semidefinite programming and which can be interpreted as a generalization of linear programming. Nowadays, the most frequently applied and implemented algorithms rely on interior point methods. The excellent sources for LMIs and semidefinite programming are e.g. [5]–[8].

Due to the fact that analytical tools generally do not exist, an array of powerful LMI software solvers based on numerical algorithms has been produced. The user can choose from the commercial packages such as LMI Lab which is a part of Robust Control Toolbox for Matlab and many high-performance freeware alternatives represented by YALMIP, SeDuMi, etc.

The paper is going to present recent developments in semidefinite programming and LMIs with emphasis on their possible applications in control. However, it is not intended to bring any novel theoretical knowledge nor application results. Its main purpose is to aggregate the basic theory of LMI and semidefinite programming and introduce its utilization in the field of control on the basis of literature from "References" section. The necessary theoretical foundations are going to be accompanied by the outline of selected control problems and overview of available software solvers.

The paper is organized as follows. In section 2, some historical aspects of LMIs and related issues are provided. The section 3 then presents basic forms, interpretations and properties of LMIs including several "tricks" for their modification. Next, the section 4 outlines the key facts on semidefinite programming. The selected LMI applications from the automatic control area are presented in section 5. Further, the section 6 overviews commercial as well as free software packages for solving the LMI problems. And finally, section 7 offers some conclusion remarks.

II. History of LMIs

Although the huge research on LMIs and their possible applications in control theory is the matter of the several last decades, the historically first LMI appeared as early as in about 1890 when Aleksandr Mikhailovich Lyapunov proved that the differential equation:

$$\frac{d}{dt}x(t) = Ax(t), \quad x(0) = x_0 \tag{1}$$

is stable if and only if there exists a solution of matrix inequality:

$$A^T P + PA < 0, \quad P = P^T > 0 \tag{2}$$

which is linear with respect to unknown symmetric positive-definite matrix P. The stability of equation (1) means that:

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$$x(t) \to 0 \quad \forall x_0 \tag{3}$$

i.e. all trajectories converge to zero.

Thus the stability of dynamical system (1) can be analyzed via the LMI (2) which is solvable analytically by means of a set of linear equations.

In the 1940's, the Lyapunov's method was applied to some practical control issues, especially from the realms of nonlinear control system stability, by Lur'e, Postnikov, etc.

Then, in the early 1960's Kalman, Yakubovich and Popov worked out the graphical solution of a class of LMIs which is known as Positive Real Lemma or KYP Lemma. Furthermore, the following years revealed that this LMI family can be solved also with assistance of an algebraic Riccati equation.

More recently, in 1984, Karmarkar presented a new algorithm for solution of linear programs in polynomial time which in 1988 consequently resulted in development of Interior Point Methods by Nesterov and Nemirovskii [9]. Their approach has been already directly applicable to convex problems involving LMIs and it has brought really efficient way of computer-based solving the LMIs by means of convex optimization.

Finally, in 1993, Gahinet and Nemirovskii released a high-performance package LMI Control Toolbox for Matlab [10]–[12].

Nice survey on historical aspects of LMIs in the area of control theory can be found at the beginning of the book [5]; some notes then e.g. in [13], [14].

III. Fundamental Forms and Properties of LMIs

The basic canonical form of LMI is:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{4}$$

where $x \in \mathbb{R}^m$ is the vector of decision variables and $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, m$ are given symmetric constant matrices. The symbol ">" means that matrix $F(x)$ is positive definite.

Besides, the nonstrict LMIs also exist. They can be expressed as:

$$F(x) \geq 0 \tag{5}$$

where the symbol "$\geq$" indicates the matrix is positive semidefinite. In all cases, the LMI represents a convex constraint with respect to x.

The two-dimensional examples of graphical representation of LMIs with the shapes of linear plane, quadratic plane, and circle can be seen in figs. 1-3, respectively [13], [14].

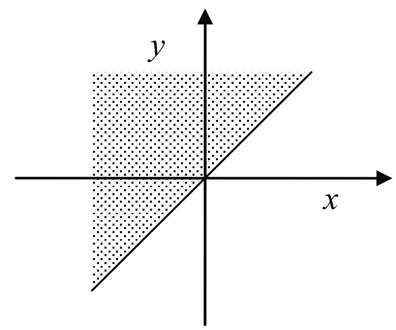

Fig. 1 linear plane

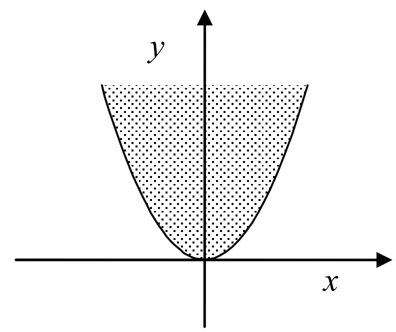

Fig. 2 quadratic plane

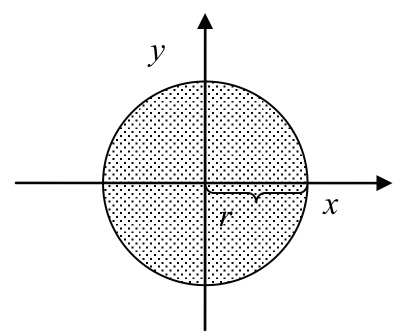

Fig. 3 circle

The LMI corresponding to the linear plane from fig. 1 can be simply derived as:

$$\begin{aligned} &y > x \\ &y - x > 0 \end{aligned} \tag{6}$$

The quadratic plane visualized in fig. 2 is described by:

$$\begin{aligned} &y > x^2 \\ &y - x^2 > 0 \\ &y - x \cdot 1^{-1} \cdot x > 0 \\ &\begin{bmatrix} y & x \\ x & 1 \end{bmatrix} > 0 \end{aligned} \tag{7}$$

And finally, the LMI for the circle (fig. 3) can be obtained as follows:

$$
\begin{aligned}
&x^2 + y^2 < r^2 \\
&r^2 - x^2 - y^2 > 0 \\
&r^2 - x \cdot 1^{-1} \cdot x - y \cdot 1^{-1} \cdot y > 0 \\
&\begin{bmatrix} r^2 & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} > 0
\end{aligned}
\tag{8}
$$

As can be noticed from the previous equations, there are number of common tools and "tricks" which extends the usability of LMIs. The simplest ones are for example:

$$F(x) > 0 \quad \Leftrightarrow \quad -F(x) < 0 \tag{9}$$

$$F_1(x) < F_2(x) \quad \Leftrightarrow \quad F_1(s) - F_2(x) < 0 \tag{10}$$

Then, the set of multiple LMIs can be expressed as the single LMI:

$$
\begin{aligned}
&\begin{matrix} F_1(x) < 0 \\ \vdots \\ F_k(x) < 0 \end{matrix} \quad \Leftrightarrow \quad F(x) = diag\left[F_1(x), \cdots, F_k(x)\right] = \\
&= \begin{bmatrix} F_1(x) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F_k(x) \end{bmatrix} < 0
\end{aligned}
\tag{11}
$$

Nonlinear (convex) inequalities are convertible to the form of LMI with the assistance of Schur complements [5], [14]. The principle is that the LMI:

$$F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{12}^T(x) & F_{22}(x) \end{bmatrix} > 0 \tag{12}$$

where $F_{11}(x) = F_{11}^T(x)$ and $F_{22}(x) = F_{22}^T(x)$ (the submatrices are symmetric), and moreover $F_{11}(x)$, $F_{22}(x)$, $F_{12}(x)$ depend affinely on x, is equivalent to:

$$F_{11}(x) > 0, \quad S_1(x) = F_{22}(x) - F_{12}^T(x) F_{11}^{-1}(x) F_{12}(x) > 0 \tag{13}$$

or

$$F_{22}(x) > 0, \quad S_2(x) = F_{11}(x) - F_{12}(x) F_{22}^{-1}(x) F_{12}^T(x) > 0 \tag{14}$$

where $S_1(x)$, $S_2(x)$ are Schur complements.

In other words, the matrix (12) is positive definite if and only if one of the statements (13), (14) hold true. Thus, the set of nonlinear inequalities (13) or (14) can be replaced by the LMI (12).

The useful application of Schur complements is e.g. matrix norm constraint expressed as the LMI:

$$\left\| F(x) \right\|_2 < 1 \quad \Leftrightarrow \quad \begin{bmatrix} I & F(x) \\ F^T(x) & I \end{bmatrix} > 0 \tag{15}$$

Another frequently used "trick" is the elimination of matrix variables [5], [13]. Suppose:

$$F_1(z) + F_2(z) X F_3^T(z) + F_3(z) X^T F_2^T(z) > 0 \tag{16}$$

where the vector z and the matrix X are independent variables, and $F_1(z)$, $F_2(z)$ and $F_3(z)$ do not depend on X. The statement (16) is equivalent to:

$$
\begin{aligned}
&\tilde{F}_2^T(z) F_1(z) \tilde{F}_2(z) > 0 \\
&\tilde{F}_3^T(z) F_1(z) \tilde{F}_3(z) > 0
\end{aligned}
\tag{17}
$$

for some X and $z = z_0$. The terms $\tilde{F}_2(z)$ and $\tilde{F}_3(z)$ represent orthogonal complements of $F_2(z)$ and $F_3(z)$ for every z, respectively.

The inequality (16) can be expressed also in different form – by means of Finsler's lemma [5]:

$$
\begin{aligned}
&F_1(z) - \sigma F_2(z) F_2^T(z) > 0 \\
&F_1(z) - \sigma F_3(z) F_3^T(z) > 0
\end{aligned}
\tag{18}
$$

for some $\sigma \in \mathbb{R}$. This version is useful in robust control [13].

Another tool suitable for control theory is so-called S-procedure which transfers the constraint on some quadratic functions (or quadratic forms) to the LMI form. For details see e.g. [5], [13].

It has been already indicated that the LMI represents a convex constraint with respect to x. Remind what the convexity means. The set $C \subseteq \mathbb{R}^m$ is said to be convex, if the line joining any two points c_1 and c_2 in C remains entirely within C [15], [16]. To state this in a different manner, it must hold true:

$$\lambda c_1 + (1 - \lambda) c_2 \in C \quad \forall c_1, c_2 \in C \quad \lambda \in \langle 0; 1 \rangle \tag{19}$$

The expression $\lambda c_1 + (1 - \lambda) c_2$ is then called a convex combination of c_1 and c_2. The main thought is illustrated in simple fig. 4. Typical examples of two-dimensional convex sets are regular polygons, nonconvex sets can be represented e.g. by stars.

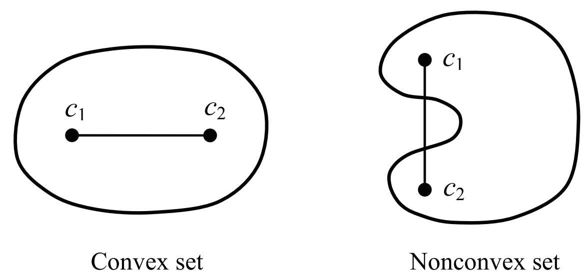

Fig. 4 examples of convex and nonconvex set [15]

Since the constraint of LMI is convex, it means that e.g. for $F(x) > 0$ and $F(y) > 0$ it holds true:

$$F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y) > 0 \tag{20}$$

for all $\lambda \in \langle 0,1 \rangle$.

Several interesting properties follow from this fact, e.g.:

- Feasible set is convex
- Solution can be found by means of convex optimization
- Generally, analytical solution does not exist
- There are numerical algorithms which are able to find the solution (if it exists)

In addition to the canonical form (4), the LMI can be expressed also in different ways. The frequent alternative form (similar to the linear programming) is a semidefinite one [13], [14], [6]:

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \in K \end{aligned} \tag{21}$$

where $c^T x$ represents a linear function, $Ax = b$ is a linear constraint, and $x \in K$ determines a positive semidefinite constraint with K representing the positive semidefinite cone (the set of all symmetric positive semidefinite matrices of particular dimension).

Actually, this is why the LMI optimization is sometimes called the semidefinite optimization and the problem (21) is called a semidefinite program. Semidefinite programming can be considered as a natural generalization of linear programming.

In the field of control theory, the LMIs are rarely given directly in canonical (4) or semidefinite (21) form. The problems in which the variables are matrices are much more common. For example [5], [13], the Lyapunov inequality (2), where $P = P^T$ is the variable, can be rewritten to the canonical form (4) with:

$$\begin{aligned} &F_0 = 0 \\ &F_i - A^T P_i - P_i A \end{aligned} \tag{22}$$

where P_i for $i = 1, \cdots, \left[n(n+1)/2 \right]$ is a basis for symmetric $n \times n$ matrices.

Unfortunately, the software products for LMI problems usually suppose the canonical or semidefinite form. Thus, the LMIs with matrices as variables have to be preprocessed which can represent time and effort consuming task.

Basically, there are three main problems related to LMIs [5], [13], [14], [17], i.e. feasibility, linear objective minimization, and generalized eigenvalue minimization.

The aim of the feasibility is to find such x that:

$$F(x) < 0 \tag{23}$$

holds true.

The problem of linear objective minimization consists in searching for x which minimizes the linear criterion:

$$\min c^T x \tag{24}$$

under constraint (23).

And finally, the generalized eigenvalue minimization means the problem of minimizing the maximum generalized eigenvalue of a pair of matrices depending affinely on x, subject to LMI constraint, that is:

$$\min \lambda \tag{25}$$

under conditions:

$$\begin{aligned} &F_1(x) < \lambda F_2(x) \\ &F_2(x) > 0 \\ &F_1(x) < 0 \end{aligned} \tag{26}$$

General procedure of solving the LMI consists of two basic steps. The first one is the verification of LMI feasibility. If the result is positive, the process continues with the second part, i.e. finding the solution which differs from the global optimum by less than selected tolerance.

The LMIs can be solved effectively (in polynomial time) by means of Interior Point Methods [9].

IV. Semidefinite Programming

It has been already shown that solving the LMI problems mean minimization of the linear function under generally nonlinear and nonsmooth, but convex constraint with respect to vector of decision variables. Thus, the related convex optimization problems can be considered as semidefinite programs. As claimed e.g. in [6], semidefinite programming unifies standard issues such as linear and quadratic programming, and it has many useful applications in engineering and combinatorial optimization.

The semidefinite program (as already outlined) can be written as [6], [18]:

$$\begin{aligned} &\min c^T x \\ &F(x) = F_0 + \sum_{i=1}^{m} F_i x_i \geq 0 \end{aligned} \tag{27}$$

where $c \in \mathbb{R}^m$ is the vector and $F_0, \ldots, F_m \in \mathbb{R}^{n \times n}$ are symmetric constant matrices appearing in the constraining LMI.

Semidefinite programming is an extension of linear programming where the componentwise inequalities between vectors are replaced by matrix inequalities. For example, suppose the linear program [6]:

$$\begin{aligned} &\min c^T x \\ &Ax + b \geq 0 \end{aligned} \tag{28}$$

in which $Ax + b \geq 0$ is componentwise inequality. The linear program (28) can be expressed as the semidefinite one (27) with:

$$F(x) = \operatorname{diag}\left(Ax + b\right) \tag{29}$$

that is:

$$\begin{aligned} &F_0 = \operatorname{diag}\left(b\right) \\ &F_i = \operatorname{diag}\left(a_i\right), \quad i = 1, \dots, m \end{aligned} \tag{30}$$

where matrix $A = \left[a_1 \ \cdots \ a_m\right] \in \mathbb{R}^{n \times m}$.

However, the semidefinite programming is able to cope with problems which can not be specified by means of linear programs, i.e. the following nonlinear (but convex) optimization task [6]:

$$\begin{aligned} &\min \frac{\left(c^T x\right)^2}{d^T x} \\ &Ax + b \geq 0 \end{aligned} \tag{31}$$

where $d^T x > 0$ is supposed. Using an auxiliary variable *t*, the problem can be reformulated to:

$$\begin{aligned} &\min t \\ &Ax + b \geq 0 \\ &\frac{\left(c^T x\right)^2}{d^T x} \leq t \end{aligned} \tag{32}$$

The nonlinear (convex) objective function from (31) has constituted a nonlinear (convex) constraint in (32). Furthermore, the constraints can be expressed as LMI with the assistance of the "trick" (11) and Schur complements [6]:

$$\begin{aligned} &\min t \\ &\begin{bmatrix} \operatorname{diag}\left(Ax + b\right) & 0 & 0 \\ 0 & t & c^T x \\ 0 & c^T x & d^T x \end{bmatrix} \geq 0 \end{aligned} \tag{33}$$

An array of very important applications (not only from the field of control theory) as well as both practical and theoretical solving efficiency indicates the significance and high impact of the semidefinite programming. The set of application examples including quadratically constrained quadratic programming, minimum eigenvalue and matrix norm minimization, logarithmic Chebychev approximation, structural optimization, etc. can be found in [6].

V. Selected Control Applications

The brief outline of possible control problems which can be solved via LMIs is going to be presented within this section.

A. Asymptotic Stability Analysis

The classical problem is represented by asymptotic stability investigation. Continuous-time time-invariant dynamic system (1) is asymptotically stable if and only if there exists a quadratic Lyapunov function [8], [14]:

$$V(x) = x^T P x \tag{34}$$

such that:

$$\begin{aligned} &V(x(t)) > 0 \\ &\dot{V}(x(t)) < 0 \end{aligned} \tag{35}$$

along trajectories of the system. Equivalently, matrix A from the system (1) must satisfy:

$$\max_i \operatorname{Re} \lambda_i(A) < 0 \tag{36}$$

Since:

$$V(x) = x^T P x = x^T \left(P + P^T\right) x / 2 \tag{37}$$

one can choose the Lyapunov matrix P as the symmetric one without loss of generality [8].

The condition of asymptotic stability in the LMI form (2) follows from the derivation:

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x \tag{38}$$

and from inequalities (35). Thus investigation of asymptotic stability is assumed a feasibility problem of LMI (2) or equivalently of LMI:

$$\begin{bmatrix} -A^T P - PA & 0 \\ 0 & P \end{bmatrix} > 0 \tag{39}$$

B. Computation of Norms

Another potential application of LMIs consists in computation of norms. Continuous-time single-input single-output linear time-invariant system with transfer function $G(s)$ has the H_2 norm defined as:

$$\left\|G\right\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|G(j\omega)\right|^2 d\omega} \tag{40}$$

For system described by:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{41}$$

with transfer function:

$$G(s) = C\left(sI - A\right)^{-1} B \tag{42}$$

the H_2 norm can be expressed according to [8], [14]:

$$\|G\|_2^2 = \mathrm{trace}\left(B^T P_0 B\right) \tag{43}$$

where P_0 is the observability gramian:

$$P_0 = \int_0^\infty e^{A^T t} C^T C e^{At} dt \tag{44}$$

given by solution of Lyapunov equation:

$$A^T P_0 + P_0 A + C^T C = 0 \tag{45}$$

with $P_0 > 0$ for observable $\left(A, C\right)$.

Thus, the H_2 norm can be computed via solving the LMI:

$$\begin{aligned}&\|G\|_2^2 = \min\ \mathrm{trace}\left(B^T P B\right)\\ &A^T P + PA + C^T C \le 0\\ &P > 0\end{aligned} \tag{46}$$

Further, the H_∞ norm of a system given by $G(s)$ is:

$$\|G\|_\infty = \sup_\omega \left|G(j\omega)\right| \tag{47}$$

Assume the continuous-time linear system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{48}$$

with transfer function:

$$G(s) = C\left(sI - A\right)^{-1} B + D \tag{49}$$

The H_∞ norm is limited according to:

$$\|G\|_\infty < \gamma \tag{50}$$

if and only if:

$$R = \gamma^2 I - D^T D > 0 \tag{51}$$

and Hamiltonian matrix [8], [14]:

$$\begin{bmatrix} A + BR^{-1}D^T C & BR^{-1}B^T \\ -C^T\left(I + DR^{-1}D^T\right)C & -\left(A + BR^{-1}D^T C\right)^T \end{bmatrix} \tag{52}$$

has no eigenvalues placed on the imaginary axis. Then, according to [8] a bisection algorithm with guaranteed quadratic convergence for finding the minimum γ such that (52) has no eigenvalues on the imaginary axis.

Consequently, the H_∞ norm can be determined through solving the LMI [8], [14]:

$$\begin{aligned}&\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} < 0\\ &P > 0\end{aligned} \tag{53}$$

which can be expanded to:

$$\begin{aligned}&\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0\\ &P > 0\end{aligned} \tag{54}$$

C. *Positive Real Lemma and Bounded Real Lemma*

Another important LMI-based result from control theory field is known as the Positive Real Lemma or KYP Lemma. It brings a frequency-domain interpretation for some LMI problems and moreover, in some special cases, a technique for numerical solution via Riccati equations [5]. The main idea of the Positive Real Lemma can be briefly formulated with the assistance of the following five statements which are equivalent to each other [13], [14]:

1. Linear system (48) is passive, i.e.:

$$\int_0^\tau u^T(t) y(t) dt \ge 0 \tag{55}$$

2. The transfer matrix (49) is positive real, i.e.:

$$G(s) + G^T(s) \ge 0 \quad \forall\, \mathrm{Re}\, s > 0 \tag{56}$$

3. The LMI:

$$\begin{aligned}&\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -\left(D^T + D\right) \end{bmatrix} \le 0\\ &P = P^T > 0\end{aligned} \tag{57}$$

is feasible.

4. There exists a real matrix $P = P^T$ satisfying the Riccati equation:

$$A^T P + PA + \left(PB - C^T\right)\left(D^T + D\right)^{-1}\left(PB - C^T\right)^T \le 0 \tag{58}$$

5. Under some additional presumptions [5], there exists a real matrix $P = P^T$ satisfying the algebraic Riccati equation:

$$A^T P + PA + \left(PB - C^T\right)\left(D^T + D\right)^{-1}\left(PB - C^T\right)^T = 0 \tag{59}$$

The similar result appearing during H_∞ minimization is represented by the Bounded Real Lemma. Analogically to the previous case, the key features will be formulated using five equivalent statements [13], [5]:

1. Linear system (48) is nonexpansive, i.e.:

$$\int_0^\tau y^T(t)y(t)dt \le \int_0^\tau u^T(t)u(t)dt \tag{60}$$

2. The transfer matrix (49) is bounded real, i.e.:

$$G(s)G^T(s) \le I \quad \forall \operatorname{Re} s > 0 \tag{61}$$

3. The transfer matrix (49) has bounded H_∞ norm, i.e.:

$$\|G(s)\|_\infty \le 1 \tag{62}$$

4. The LMI:

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I \end{bmatrix} \le 0 \qquad P = P^T > 0 \tag{63}$$

is feasible.

5. Under some additional presumptions [5], there exists a real matrix $P = P^T$ satisfying the algebraic Riccati equation:

$$\begin{aligned} &A^T P + PA + C^T C + \cdots \\ &+\left(PB + C^T D\right)\left(I - D^T D\right)^{-1}\left(PB + C^T D\right)^T = 0 \end{aligned} \tag{64}$$

D. Other Problems

Certainly, this overview of is just a preliminary survey because the LMIs represent elegant and very efficient instrument for an array of problems in the control theory area. According to [5], [14], the range of possible solvable problems contains, e.g.:

- Matrix scaling problems
- Construction of quadratic Lypunov functions for stability and performance analysis of linear differential inclusions
- Joint synthesis of state-feedback and quadratic Lypunov functions for linear differential inclusions
- Synthesis of state-feedback and quadratic Lyapunov functions for stochastic and delay systems
- Synthesis of Lur'e-type Lyapunov functions for nonlinear systems
- Optimization over an affine family of transfer matrices
- Optimal system realization
- Interpolation problems
- Multicriterion LQG/LQR
- Inverse problem of optimal control
- H_2/H_∞ control problems
- Positivity of polynomials
- Robust stability analysis

VI. LMI Solvers

This Section offers an overview of the popular software solvers for LMI problems.

A. LMI Lab

The well known commercial high-performance package for LMI problems distributed by The MathWorks, Inc. is the LMI Lab [17] for Matlab. Formerly, it was the separate LMI Control Toolbox [10]–[12], but since Matlab Release 14 with Service Pack 1 it has been included as a part of Robust Control Toolbox [19]. Generally, the LMI Lab offers tools to [17]:

- Specify LMI systems either symbolically with the LMI Editor or incrementally with the "lmivar" and "lmiterm" commands
- Retrieve information about existing systems of LMIs
- Modify existing systems of LMIs
- Solve the three generic LMI problems (feasibility problem – "feasp" command, linear objective minimization – "mincx" command, and generalized eigenvalue minimization – "gevp" command)
- Validate results

Just for illustration, the fig. 5 shows the LMI Editor which can be launched by typing "lmiedit" command and which allows to define LMI systems in a symbolic manner by means of graphical user interface.

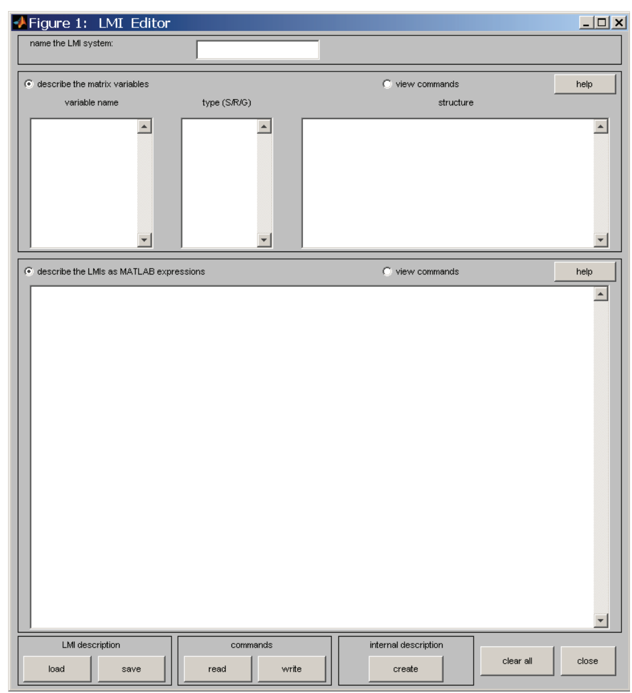

Fig. 5 The LMI Editor

B. Free Toolboxes

In addition to the commercial LMI Lab, there are also free alternatives available. The powerful tool popular in automatic control community is YALMIP [20], [21] which represents package for Matlab environment focused on advanced modelling and solution of convex and nonconvex optimization problems and which has been gradually developed and improved since the first public release in early 2001 [20]. The principal concept of YALMIP consists in the application of an array of the external solvers (e.g. SeDuMi, SDPT3, CSDP, SDPA, MOSEK, and many others) for low-level numerical computations related to optimization problems. On the other hand, the YALMIP itself aims especially at efficient modelling and high-level algorithms. The list of the employed codes can be found at [21].

Another well known product is SeDuMi [22]–[24]. Besides, it has been utilized also as one of the external solvers for YALMIP as mentioned. The SeDuMi package itself implements interior point methods for solving the optimization problems over symmetric cones. It can be combined also with an alternative SeDuMi Interface [25], but the development of this extension was stopped in 2002, thus the YALMIP is now the recommended option from the interface viewpoint.

The comparison of several LMI solvers can be found e.g. in [26]. Moreover, [27], [28] provide also benchmark results for many codes.

VII. Conclusion

This paper has been focused on fundamentals of LMIs and semidefinite programming and on their application potential in the field of system and control theory. Nevertheless, it has not been intended to bring any novel theoretical knowledge nor application results. The work has presented basics of effective work with LMIs and some tools and "tricks" for extending their usability. Moreover, it has described several typical control-based problems which can be effectively solved by means of LMIs such as analysis of asymptotic stability, computation of and norms, Positive Real Lemma and Bounded Real Lemma, and outline of an array of the other issues. Naturally, consideration of such extensive range of problems in LMI way can be useful only if they are effectively solvable. As it has been shown, the critical feature of LMI is that it defines convex set and thus they can be advantageously applied to description of constraints for optimization tasks. This leads to solution of so-called semidefinite programs which can be relatively effortlessly numerically solved using interior point methods. From the practical point of view, the LMI software solvers are irreplaceable nowadays. The typical and probably the most popular commercial and free packages have been introduced within this paper.

Moreover, some problems can lead to Bilinear Matrix Inequalities (BMIs):

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j F_{ij} > 0 \tag{65}$$

where, analogously to LMIs, $x \in \mathbb{R}^m$ is the vector of decision variables and $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $F_{ij} = F_{ij}^T \in \mathbb{R}^{n\times n}$, $i, j = 0,\ldots,m$ are given symmetric constant matrices.

The main disadvantage of BMIs is that they are, unlike the LMIs, not convex and so they lead to very complicated optimization algorithms with generally limited ability of convergence to global extremes. Some BMI problems can be trans-formed to the LMI description. However, this is already out of the scope of this work.

References

[1] O. Najat, and E. Noureddine, "Exponential Stability and Stabilization of T-S System for Synchronous Machine without Damper" *International Journal of Mathematical Models and Methods in Applied Sciences*, vol. 5, no. 8, pp. 1406-1413, 2011.

[2] J. Ezzine, and F. Tedesco, "H-infinity Approach Control for Regulation of Active Car Suspension" *International Journal of Mathematical Models and Methods in Applied Sciences*, vol. 3, no. 3, pp. 309-316, 2009.

[3] A. Ait Kaddour, N. Elalami, and E. Elmazoudi, "Dynamic Output Feedback Fault Tolerant Controller for an Exploited Polynomial Fish Population Model" *International Journal of Mathematics and Computers in Simulation*, vol. 7, no. 5, pp. 430-437, 2013.

[4] W. Assawinchaichote, "A New Approach to Non-fragile H-infinity Fuzzy Filter of Uncertain Markovian Jump Nonlinear Systems" *International Journal of Mathematics and Computers in Simulation*, vol. 4, no. 2, pp. 21-33, 2010.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia, Pennsylvania, USA: Society for Industrial and Applied Mathematics (SIAM), 1994.

[6] L. Vandenberghe, and S. Boyd, "Semidefinite Programming" *SIAM Review*, vol. 38, no. 1, pp. 49-95, 1996.

[7] C. Scherer, and S. Weiland, *Linear Matrix Inequalities in Control*, Material for a graduate course on the subject of LMI's in systems and control at Dutch Institute of Systems and Control, Netherlands, http://www.isa.ewi.tudelft.nl/~roos/courses/WI4218/lmi052.pdf. Accessed 17 August 2011.

[8] D. Henrion, *LMI, Optimization and Polynomial Methods*, Material for course on LMI Optimization with Applications in Control, Czech Technical University in Prague, Czech Republic, http://homepages.laas.fr/henrion/courses/lmi11/. Accessed 17 August 2011.

[9] Y. Nesterov, and A. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming*, Philadelphia, Pennsylvania, USA: Society for Industrial and Applied Mathematics (SIAM), 1994.

[10] P. Gahinet, and A. Nemirovskii, *LMI-Lab: a Package for User-Friendly LMI Design*, Reference Manual, 1993.

[11] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, "The LMI Control Toolbox" in *Proceedings of the 33rd Conference on Decision and Control*, Lake Buena Vista, Florida, USA, 1994.

[12] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox for Use with Matlab*, User's Guide, The MathWorks, Inc. http://www.mathworks.com.au/help/releases/R13sp2/pdf_doc/lmi/lmi.pdf. Accessed 18 August 2011.

[13] M. Šebek, *Robustní řízení*, (*Robust Control*), PDF slides for the course "Robust Control", Czech Technical University in Prague, Czech Republic, http://www.polyx.com/_robust/. (In Czech). Accessed 3 September 2011.

[14] M. Bakošová, *Lineárne maticové nerovnosti a ich využitie pri riešení problému stabilizácie systémov*, (*Linear matrix inequalities and their application to problem of stabilization of systems*), PDF presentation slides from 5 November 2010, Faculty of Applied Informatics, Tomas Bata University in Zlín, Czech Republic. (In Slovak).

[15] R. Matušů, *Systems with Parametric Uncertainty: An Algebraic Approach to Robust Control Design*, Köln, Germany: Lambert Academic Publishing, 2009.
[16] B. R. Barmish, *New Tools for Robustness of Linear Systems*, New York, USA: Macmillan, 1994.
[17] The MathWorks, Inc., *LMI Lab (Robust Control Toolbox)*, http://www.mathworks.com/help/toolbox/robust/ug/f9-2776.html. Accessed 11 October 2011.
[18] Z. Hurák, and M. Šebek, "Řešení lineárních maticových nerovností v Matlabu. (Solution of linear matrix inequalities in Matlab)" in *Proceedings of the International Conference Technical Computing Prague 2003*, Prague, Czech Republic, 2003. (In Czech).
[19] The MathWorks, Inc., *Products in Transition with Release 14*, http://www.math-works.com/products/new_products/R14_transition.html. Accessed 11 October 2011.
[20] J. Löfberg, "YALMIP : A toolbox for modeling and optimization in MATLAB" in *Proceedings of the 2004 IEEE International Symposium on Computer Aided Control Systems Design*, Taipei, Taiwan, 2004.
[21] J. Löfberg, *YALMIP Wiki*, http://users.isy.liu.se/johanl/yalmip/. Accessed 11 October 2011.
[22] J. F. Sturm, "Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones" *Optimization Methods and Software*, vol. 11, no. 1-4, pp. 625-653, 1999.
[23] I. Pólik, T. Terlaky, and Y. Zinchenko, "SeDuMi: a package for conic optimization" in *IMA Workshop Optimization and Control*, University of Minnesota, Minnesota, USA, 2004, http://imre.polik.net/wp-content/uploads/sedumi_ima_poster.pdf. Accessed 12 October 2011.
[24] SeDuMi, http://sedumi.ie.lehigh.edu/. Accessed 11 October 2011.
[25] D. Peaucelle, and K. Taitz, *SeDuMi Interface*, http://homepages.laas.fr/peaucell/soft-ware/SeDuMiInt.html. Accessed 12 October 2011.
[26] D. Arzelier, D. Peaucelle, and D. Henrion, *Some notes on standard LMI solvers*, Czech Technical University in Prague, Czech Republic, http://homepages.laas.fr/arzelier/publis/2002/prague102.pdf. Accessed 12 October 2011.
[27] H. D. Mittelmann, "An independent benchmarking of SDP and SOCP solvers" *Mathematical Programming*, vol. 95, no. 2, pp. 407-430, 2002.
[28] H. D. Mittelmann, *Decision Tree for Optimization Software*, http://plato.asu.edu/guide.html. Accessed 12 October 2011.

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