P-completions of lattices and its Applications to Formal Concept Analyses

A.Satekbayeva, A. Basheyeva, A. Nurakunov, and J.Tussupov

***Abstract*—** The main purpose of the paper is to present one approach of completion of a given lattice to lattice with some properties of Boolean lattices and its applications to concept lattices. We provide some proofs, examples and their applications. Also we construct a minimal P- completion of one concept lattice, that naturally arised in a study of the suppressors of plant viruses.

***Keywords*—** Formal Concept Analysis, a formal context, concept lattices, extension of lattice.

I. Introduction

In practice often the Formal concept lattices are very big and it is difficult to study and say about properties of such lattices. To reduce or correct the big formal concept lattices to lattices with well studied properties is usual problem of the Theory of Formal Concept Analysis [5].

From the point view of the Lattice Theory and Formal Concept Analysis it is natural to correct given lattice to lattice with a well studied properties without losing relations (order) between elements.

Doing this we will extend a given lattice to other lattice guiding by the following two principles of completions:

- extended lattice should have the "well-known" properties;
- extended lattice should preserve partial order of given lattice.

Under "well-known" property we mean a property that is natural defined in Lattice theory, as well as, a property that can be expressed by the first order formulas. For example, the "well-known" property is "to be complete lattice" or "to be distributive lattice".

Also for finite lattices we will use the next principle:

- The difference between given a lattice and the extended lattice should be "difficult to distinguish".

This work has been supported by grants of the Ministry of Education and Science of Republic Kazakhstan: "Contribution on two leading problems of the Universal algebra and Lattice theory" (Grant # 0112PK02245) and "The formal methods of the specification and their application in the development of programs".

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For given finite lattice and finite extended lattice which satisfies to condition 1) and 2) our measure of "difficult to distinguish" is a rational number that is equal to the number of added elements divided by the number of all elements of extendeded lattice.

Terminology corresponds to G. Birkhoff [1] on Lattice Theory, S. Burris - H. P. Sankappanavar [2] on Universal Algebra, H. M. MacNeille [7] on Partially ordered sets and B. Gunter - Willy [5] for formal concept analysis.

II. Preliminaries and definitions

The authors assume that the reader is familiar with the basic concepts of lattice theory, universal algebra and formal concept analysis.

Definition 1. Let P be a set of some properties of lattices and L a lattice. A latice L^* is called P- *completions* of L if L is embedding to L^* as poset and L^* satisfies to all properties from P.

We note that not for any finite lattice L and a set of properties P there are P- completions of L.

We will extend given lattice to lattice with some properties of Boolean lattices. The reason of such consideration is the following theorem that provides existence of P- completion:

Theorem 1. *Let P be a set of some properties of finite Boolean lattices. For any finite lattice L there is a lattice L^* such that P are valid on L^* and L is embedded into L^* as partially order set.*

From the above theorem and fact that every nontrivial universal Horn class contains the class of all distrinutive lattices, in particular, the class of all boolean lattices we get:

Corollary 1. *Let P be a consistent set of the universal Horn sentences. Then for any finite lattice L there are P-completions of L.*

We recall some main classes of lattices in which we will find completions.

- A lattice L is called *distributive* if it satisfies to distributive law:

$$\forall xyz[x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)];$$

- A lattice L is called *modular* if it satisfies to modular law:

$$\forall xyz[x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)];$$

- A lattice L is called *meet (join) semi-distributive* if it satisfies to meet (join) semi-distributive law:

$\forall xyz[x \wedge y = x \wedge z \Rightarrow x \wedge y = x \wedge (y \vee z)]$;
$(\forall xyz[x \vee y = x \vee z \Rightarrow x \vee y = x \vee (y \wedge z)])$.

An element a of the lattice L is called an *atom* if $0 < b \leq a$ implies $b = a$ for all $b \in L$. A lattice L is called *atomic (coatomic)* if every element of L is a sum of some atoms (an intersection of some coatoms) of L.

Let L be an arbitrary lattice and $\mathcal{B}(L)$ a set of all subsets of the set L. Obviously, $\mathcal{B}(L)$ is a boolean lattice for any lattice L. A lattices L that are isomorphic to the lattice $\mathcal{B}(X)$ for some set X is called a lattice of the set of subsets.

Further we will often use the following lattices: N_5 is a lattice with basic set $\{0, a, b, c, 1\}$ and partial order $0 < a < 1$, $0 < b < c < 1$, and M_3 is a lattice with basic set $\{0, a, b, c, 1\}$, and partial oder $0 < a < 1$, $0 < b < 1$, $0 < c < 1$ (see Figure 1). The lattice N_5 is called pentagon and M_3 is diamond.

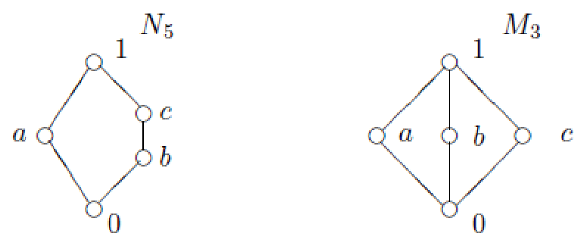

Fig. 1. Lattices N_5 and M_3

III. Minimal P- extensions of the finite lattices

Further we will consider finite lattices only. For a finite set A we sign $|A|$ the number of elements of the set A . Let L be a finite lattice, P a set of the properties of the lattices and $\mathcal{P}$ a set of all P- completions of L. A number

$$d_p(L) = \begin{cases} 1, & \text{if } \mathcal{P}(L) = \emptyset \\ \min\{\dfrac{|\bar{L}| - |L|}{|\bar{L}|} : \bar{L} \in \mathcal{P}(L)\}, & \text{if } \mathcal{P}(L) \neq \emptyset \end{cases}$$

is called a *P-defect* of the lattice L.

So, we have $0 \leq d_p(L) \leq 1$ for any finite lattice L. Furthermore, $d_p(L) = 0$ if and only if L satisfies to all properties from P and $d_p(L) = 1$ if and only if L does not have P- completions.

Definition 2. *A P- zCompletion L^* of finite lattice L is called a minimal P- completion if $d_p(L) = (|L^*| - |L|)/|L^*|$.*

We note that a lattice can have more than one nonisomorphic minimal completions.

In some cases, the problem of finding the minimal P-completions of certain lattices is quite simple. In particular, we study the lattices pentagon N_5 and diamond M_3 on distributive, modular and semidistributive completions.

Let D, M and $SD_\wedge$ be distributive, modular and meet semidistributive properties, respectively. And let N_5^c and M_3^b are lattices on the Figure 2. Note that both lattices are isomorphic, but fixed elements a, b, c have different partial orders.

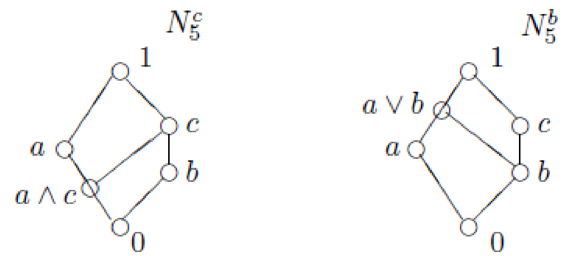

Fig. 2. Minimal D- completion and M- completion of N_5

Lemma 1. *The lattices N_5^c and N_5^b are all minimal D-completions and M-extention of the lattice N_5 and $d_D(N_5) = d_M(N_5) = 1/6$, $d_{SD\wedge}(N_5) = 0$.*

Proof. Obviously, if $|\bar{L}| - |L| = 1$ for some P- completion $\bar{L}$, then $\bar{L}$ - minimal P- completion. We use this fact to find all minimal D-, M- and $SD_\wedge$- completions of the lattice N_5. According to Dedekind Theorem [1], N_5 is the smallest nonmodular lattice and Birkhoff Theorem [1], N_5 and M_3 are the smallest nondistributive lattices. It is easy to see N_5^b and M_5^c are the 6-element lattices. Hence $|\bar{L}| - |L| = 1$ and, therefore, N_5^b, N_5^c are minimal D- и M-completions of the lattice N_5. So D-defect and M-defect of the lattice N_5 equal $1/6$, i.e., $d_D(N_5) = d_M(N_5) = 1/6$. As N_5 is semi-distributive lattice, then $d_{SD\wedge}(N_5) = 0$.

We show there are no other minimal completions. Let $\bar{L}$ be some minimal D-completion of N_5. As $d_D(N_5) = d_M(N_5) = 1/6$, then $\bar{L}$ consists of 6 elements, in particular, $a, b, c, 0{,}1 \in \bar{L}$. Due to the fact that $\bar{L}$ is distributive lattice, from the condition $a \vee b = a \vee c$ follows $c = c \wedge (a \vee b) = (c \wedge a) \vee b$. Hence b and $a \wedge c$ are incomparable. In this case we receive $\bar{L} = N_5^c$. In the case of $a \vee b < a \vee c$ elements $a \vee b$, c are incomparable. Consequently, $\bar{L} = N_5^b$.

The little more difficult minimal D- and $SD_\wedge$- completions of the lattice M_3. Let $\mathcal{B}(3)$ and $SD_\wedge(7)$ be lattices on *the* Figure 3, and $\mathcal{B}(3)^d$ is a lattice dual to the lattice $\mathcal{B}(3)$.

Lemma 2. *Let $\mathcal{B}(3)$, $SD_\wedge(3)$ and $SD_\wedge(3)^+$ are lattices on the Figure 3.*

1. *The lattices $\mathcal{B}(3)$ up to isomorphism is unique minimal D- extensions of the lattice M_3 and $d_D(M_3) = 3/8$;*

2. *The lattices $SD_\wedge(3)$ and $SD_\wedge(3)^+$ up to isomorphism are all the minimal $SD_\wedge$- extensions of the lattice M_3 and $d_{SD_\wedge}(M_3) = 2/7$.*

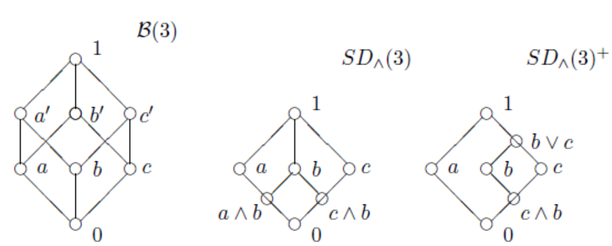

Fig. 3. Lattices$\mathcal{B}(3)$, $SD_\wedge(3)$и$SD_\wedge(3)^+$

Proof. We consider an arbitrary D-completion $\bar{M}_3$ of the lattice M_3. We assume $a \wedge b = a \wedge c = b \wedge c$. As $\bar{M}_3$ is distributive lattice, then

$$a = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$$

$$b = b \vee (a \wedge c) = (b \vee a) \wedge (b \vee c),$$
$$c = c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b).$$

Using the conditions a, b, c are incomparable, from the first equality we receive that the elements $(a \vee b)$, $(a \vee c)$ are incomparable. Similarly, from the second and third equality we receive $(a \vee b)$, $(b \vee c)$ are incomparable and $(c \vee b)$, $(a \vee c)$ are incomparable. Hence, $(a \vee b)$, $(a \vee c)$ and $b \vee c$ are incomparable. Consequently, any D-completion M_3 contains at least $|M_3| + 3 = 8$ elements. As $\mathcal{B}(3)$ is distributive lattice and contains exactly 8 elements, by definition of the minimal P- completion we get $\mathcal{B}(3)$ -minimal D- completion. Hence $d_D(M_3) = 3/8$, as M_3 is a modular lattice, then $d_M(M_3) = 0$.

From proved above also follows that any minimal D-completion of M_3 of exactly 8 elements and contains $\mathcal{B}(3)$ as a sublattice. Therefore, $\mathcal{B}(3)$ is single minimal D-completion of M_3 on condition $a \wedge b = a \wedge c = b \wedge c$.

In a dual way considered the completion on condition of $a \vee b = a \vee c = b \vee c$. In this case we get a lattice dual to the lattice $\mathcal{B}(3)$, this lattice is also D-completion of M_3.

2. Obviously, that the lattices $SD_\wedge(3)$ and $SD_\wedge(3)^+$are $SD_\wedge$-completions of lattice M_3. Therefore, any minimal $SD_\wedge$-completion is not more than 7 elements and $d_{SD_\wedge}(M_3) \leq 2/7$.

We consider arbitrary minimal $SD_\wedge$-completion $\bar{M}_3$ of the lattice M_3, generated by the elements a, b, c. Since $\bar{M}_3$ is meet semi-distributive lattice, in $\bar{M}_3$ are executed the following implications:

$$a \wedge b = a \wedge c \rightarrow a \wedge b = a \wedge (b \vee c),$$
$$b \wedge a = b \wedge c \rightarrow b \wedge a = b \wedge (a \vee c),$$
$$c \wedge a = c \wedge b \rightarrow c \wedge a = c \wedge (a \vee b)).$$

Let $a \wedge b = a \wedge c = b \wedge c$. Then, by meet semi-distributive law $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$ for any $x, y, z \in \{a, b, c\}$. Thus, the sublattice generated by the elements a, b, c are a distributive and have at least 8 elements. It contradicts a condition that any minimal $SD_\wedge$- completion has no more than 7 elements.

The lattice $\bar{M}_3$ will also more than 7 elements if $a \wedge b$, $a \wedge c$, $b \wedge c$ are incomparable. Therefore, without loss of generality, put $b \wedge c > a \wedge c$. Hence $a \wedge c = 0$ and $a \wedge b \ngeq b \wedge c$, since otherwise $b \wedge c = 0$. We will assume $a \wedge b$ is incomparable with $b \wedge c$. Then by $|\mathrm{M}_3| \leq 7$, we get $a \vee b = a \vee c = b \vee c = 1$. This means $\bar{M}_3 = SD_\wedge(3)$. We will assume $a \wedge b < b \wedge c$. Then $a \wedge b = 0$. Consequently, $0 = a \wedge b = a \wedge c$ and by meet semi-distributive law $0 = a \wedge (b \vee c)$. Hence, $b \vee c \neq 1$. Thus the elements $b \wedge c$, $b \vee c$ different from $0, 1, a, b, c$ and by $|M_3| \leq 7$, $a \vee b = a \vee c = 1$. This means что $\bar{M}_3 = SD_\wedge(3)^+$. Consequently, $d_{SD_\wedge}(M_3) = 2/7$.

From the proof of this lemma implies that the lattices $SD_\wedge(3)^d$, $SD_\wedge(3)^+$ are minimal $SD_\vee$- completion of the lattice M_3, where $SD_\wedge(3)^d$ is lattice dual to the lattice $SD_\wedge(3)$. Obviously, $d_{SD_\vee}(M_3) = 2/7$. Also easy to see in the general case for an arbitrary lattice L number of defects $d_{SD_\wedge}(L)$ and $d_{SD_\vee}(L)$ do not necessarily coincide.

Note that the choice of the minimal completion (or set of properties P) may be dictated by practical considerations , and as will be shown in the next section , is consistent with the principle of the third replenishment of finite lattices.

IV. Concept lattice of suppressors of the plant viruses and extensions

In this section we will construct a minimal P- completion of one concept lattice, that naturally arised in a study of the suppressors of plant viruses.

We present the article from biologist[7]. In the review the current knowledge on activities and biochemical mechanisms of selected Virus-encoded suppressors of RNAi (VSRs) with regard to their biological role of suppressing RNAi in plantswas described.

There is a diagram which shows on Figure 4:

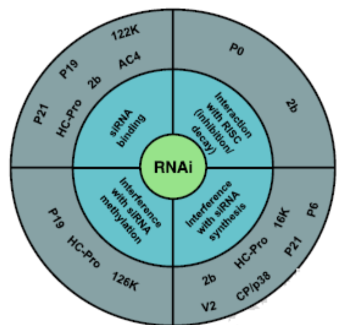

Fig. 4. Biological Chemistry of Virus-Encoded Suppressors of RNA Silencing

In the review paper [8] the current knowledge on activities and biochemical mechanisms of selected Virus-encoded suppressors of RNAi (VSRs) with regard to their biological role of suppressing RNAi in plants was described. The database of this paper allow us to get the context $\mathcal{S} =< Sup, Sil; I >$, where the set of objects Sup is the set of suppressors that are produced by plant viruses to suppress the immune system of the cells of plants; Sil is a set of attributes that are a variety of immune mechanisms of plant cells, the binary relation I consists of the pairs $(s, r) \in Sup \times Sil$ such that a pair (s, r) belongs to I if and only if suppressor $s \in Sup$ and suppress immune mechanism $r \in Sil$. The context $\mathcal{S}$ is represented as Context вЂ“ Table $\mathcal{S}$ on Figure 5.

$Sup \backslash Sil$	Interaction with RISC	siRNA binding	Interference with siRNA methylation	Interference with siRNA synthesis
P0	x			
2b	x	x		x
122K		x		
P19		x	x	
P21		x		x
AC4		x		
HC-		x	x	x

Pro				
126K			x	
16K				x
V2				x
CP/p38				x
P6				x

Sup (abbreviation of Suppressors) is a set consists suppressors P0, 2b, 122K, P19, P21, AC4, HC-Pro, 126K, 16K, V2, CP/p38, P6; and Sil (abbreviation RNA Silencing) is a set consists immune mechanisms Interaction with RISC, siRNA binding, Interference with siRNA methylation, Interference with siRNA synthesis.

Fig. 5. Context Table $\mathcal{S}$

From the context $\mathcal{S}$ we construct concept lattice $FCL(\mathcal{S})$ pictured on Figure 6.

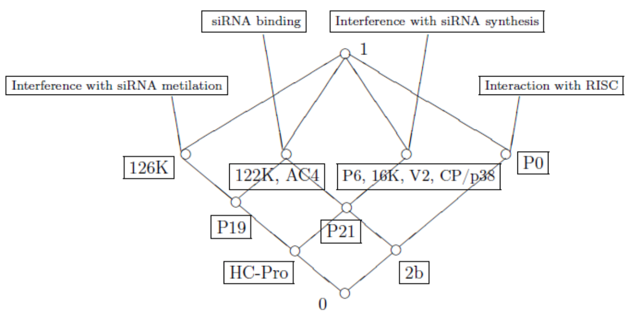

Fig 6. Concept lattice $FCL(\mathcal{S})$

Put $a = [P0]$, $b = [P6, 16K, V2, CP/p38]$ and $c = [122K, AC4]$. It is easy to see that $a \wedge b = a \wedge c$ and $a \wedge b \neq a \wedge (b \vee c)$. This means that the lattice is not meet semidistributive. We consider minimal meet semidistributive completions ($SD_\wedge$- completion of $FCL(\mathcal{S})$. Using the methods of completions we produce the lattice $FCL(\mathcal{S})^*$ pictured on Figure 7. We will omit detailed presentation and reasoning.

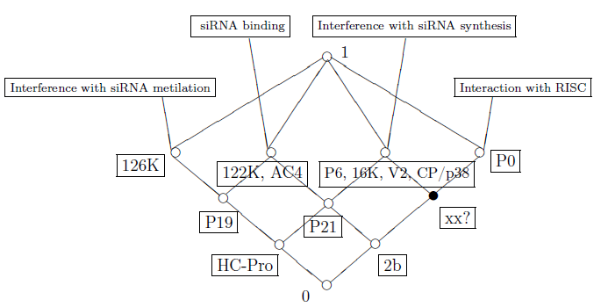

Fig 7. Concept lattice $FCL^*(\mathcal{S})$

The authors, being no experts in virology, have courage to assume the following:

- there is a suppressor [xx?] which participates in the process of suppression of immune Interference with siRNA synthesis and Interaction with RISC mechanisms, but doesn't participate in process of suppression of immune siRNA binding and Interference with siRNA methylation mechanisms;

- $[16K]$ or $[V2]$ or $[CP/p38]$ or $[P6]$ supressor participates in the process of suppression of the Interaction with RISC immune mechanism.

The authors would like to note that in the publication [6] have been shown that the suppressor of $P1$, which is not included in the *Sup*, participates in a locking of process of Interaction with RISC. Probably this known suppressor is an Interference with siRNA synthesis suppressor.

V. Conclusion

In the paper was introduce a definition of P-extension of lattice, where P is a set of some properties of lattices. The existence of such extensions for every finite lattice and any properties of Boolean lattices was shown. In particular, there were found all distributive and meet semi-distributive extensions of pentagon and diamond lattices. Using these results and database of suppressors we construct meet semi-distributive, co-atomistic lattice of suppressors of plant viruses.

Acknowledgment

The authors are grateful to Prof. R. T. Omarov for providing information and consultation on the virology of plants.

This work has been supported by grants of the Ministry of Education and Science of Republic Kazakhstan: "Contribution on two leading problems of the Universal algebra and Lattice theory" (Grant # 0112PK02245) and "The formal methods of the specification and their application in the development of programs".

References

[1] G. Birkhoff, *"Lattice theory"*, Third Edition, AMS Colloquium Publications 25, American Mathematical Society, Providence, RI, 1967.

[2] Stanley Burris, H. P. Sankappanavar (1981), *"A Course in Universal Algebra"*, Springer-Verlag, 1981, pp. 331.

[3] Richard Dedekind, *"Essays on the Theory of Number"*, Open Court Publishing Company, Chicago, 1901.

[4] G. Ganter, R. Wille, *"Formal Concept Analysis"*, Mathematical Foundations. Springer, 1999.- 314 p.

[5] Gillet, F.-X. et al. (2013), *"The RYMV-Encoded Viral Suppressor of RNA Silencing P1 Is a Zinc-Binding Protein with Redox-Dependent Flexibility"*, J. Mol. Biol. 425(2013), 2423-2435.

[6] H. M. MacNeil, *"Partially Ordered Sets"*, Trans. Amer. Math. Soc. 42(1937), 90-96.

[7] Rustem T. Omarov and Herman B. Scholtof (2012), *"Biological Chemistry of Virus-Encoded Suppressors of RNA Silencing: An Overview"*, Methods in Molecular Biology, Springer Science+Business Media, LLC, 2012.

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