Semi-analytical solution of an eddy current problem for a conducting medium with a surface flaw

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Abstract—The method of truncated eigenfunction expansions is used in the present paper to construct a semi-analytical solution for an eddy current problem containing a surface flaw. The flaw is represented by a cylindrical hole in the upper layer of a two-layer conducting medium. The axis of the cylindrical flaw coincides with the axis of an eddy current coil carrying alternating current. The obtained solution is semi-analytical since there are two steps in the solution procedure that require the use of numerical methods: (a) calculation of complex eigenvalues without apriori information on the location of the eigenvalues and (b) solution of a system of linear algebraic equations. Results of numerical calculations are presented.

Keywords—Eddy current testing, change in impedance, separation of variables, eigenvalues.

I. INTRODUCTION

ANALYTICAL solutions of eddy current testing problems for the cases of unbounded conducting media are well-known in the literature [1]-[3]. The presence of flaws in a conducting medium results in more complicated boundary value problems, which, in general, cannot be solved by analytical methods. Numerical methods (such as finite element methods [4]) are often used in such cases to calculate the electromagnetic field in the medium. If an eddy current problem is solved for an unbounded medium then it is assumed that the vector potential tends to zero as one of the geometrical coordinates tends to infinity. From a physical point of view the electromagnetic field is negligible at a sufficiently large distance from the source of current.

This idea is used in [5] where the authors assumed that the vector potential is exactly zero at a large radial distance $r = b$ from the axis of a cylindrical coil. Later [3] many problems are solved using the same assumption. It is shown in [3] that the computational error can easily be controlled by choosing the value of b. The proposed method [3] is usually called the TREE method where the abbreviation stands for the TRuncated Eigenfunction Expansions.

The approach based on the TREE method for a solution of eddy current testing problems with cylindrical symmetry is presented in [3]. Different types of axisymmetric flaws are recently considered in the literature: (a) an infinite cylindrical hole in a conducting half-space [3], (b) a cylindrical hole of finite length in a conducting half-space [6], (c) a cylindrical inclusion in a half-space [7], (d) a surface cylindrical flaw in a plate [8]. Direct solution of such axisymmetric problems can be used in practice in order to estimate electrical conductivity of plates and other objects with cylindrical symmetry [9], [10].

In the present paper the TREE method is used to solve an eddy current testing problem for the case where a cylindrical air-core coil is located above a two-layer conducting medium containing a surface flaw in the form of a cylindrical hole coaxial with the coil. The problem is solved by the method of separation of variables. The obtained solution is semi-analytical since there are two steps in the procedure that require the use of numerical methods: (a) calculation of complex eigenvalues without apriori information on the location of the eigenvalues and (b) solution of a system of linear algebraic equations. Such a model can be used to estimate the effect of corrosion in metal coatings.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider a cylindrical air-core coil of radii r_1 and r_2 ($r_2 > r_1$) located above a conducting two-layer medium (see Fig. 1). The upper layer of the conducting medium has the electrical conductivity σ_1 and is represented by regions R_1 and R_2 whereas σ_2 the conductivity of the lower half-space (region R_3). The coil carries an alternating current with frequency f. The lift-off is z_1 and the height of the coil is $z_2 - z_1$. The upper conducting layer (region R_1) contains a flaw in the form of a cylindrical hole of radius c and height d_1. The axis of the cylinder coincides with the axis of the coil. The thickness of the lower layer is d_2.
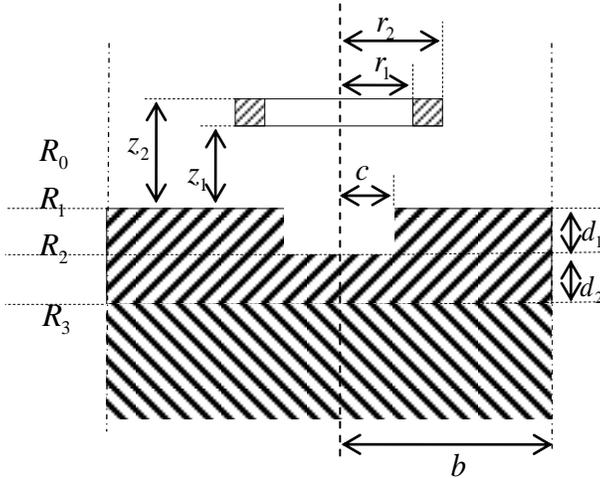
Fig. 1. A coil carrying an alternating current above a conducting medium with a flaw.

The solution of the problem shown in Fig. 1 can be obtained by a superposition principle if one knows the solution of the corresponding problem for a single-turn coil (see Fig. 2).
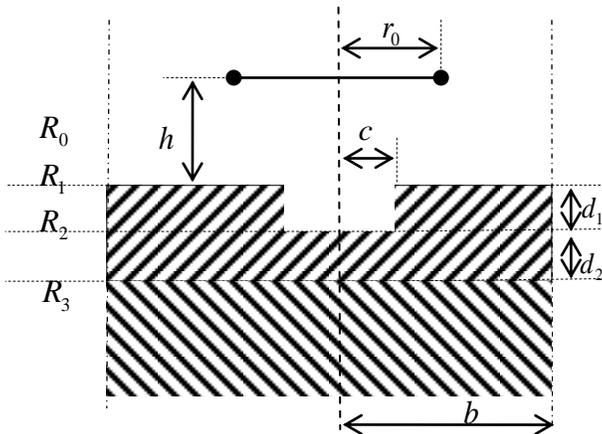
Fig. 2. A single-turn coil above a conducting medium with a flaw.

We assume that a single-turn coil of radius r_0 is located at height h above the conducting medium shown in Fig. 2. The problem is axisymmetric so that the amplitude of the vector potential has only one non-zero component in the φ−direction (here (r, φ, z) is the system of cylindrical polar coordinates centered at O).

The system of equations for the amplitude of the vector potential in regions $R_i (i = 0,1,2,3)$ has the form (see Fig. 2):

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0)\delta(z - h), \tag{1}$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r}\frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} - j\omega\sigma\mu_0 A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \tag{2}$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} - j\omega\sigma_1\mu_0 A_2 + \frac{\partial^2 A_2}{\partial z^2} = 0, \tag{3}$$

$$\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r}\frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} - j\omega\sigma_2\mu_0 A_3 + \frac{\partial^2 A_3}{\partial z^2} = 0, \tag{4}$$

where $\delta(x)$ is the Dirac delta-function, $\sigma = 0$ if $0 \le r < c$ and $\sigma = \sigma_1$ if $c < r < b$, b is the distance from the axis of the coil where the electromagnetic field is assumed to be exactly zero and $\omega = 2\pi f$.

The boundary conditions are

$$A_i \mid_{r=b} = 0, \quad i = 0,1,2,3, \tag{5}$$

$$A_0 \mid_{z=0} = A_1^a \mid_{z=0}, \quad \frac{\partial A_0}{\partial z}\mid_{z=0} = \frac{\partial A_1^a}{\partial z}\mid_{z=0}, \ 0 \le r < c, \tag{6}$$

$$A_0 \mid_{z=0} = A_1^c \mid_{z=0}, \quad \frac{\partial A_0}{\partial z}\mid_{z=0} = \frac{\partial A_1^c}{\partial z}\mid_{z=0}, \ c < r < b, \tag{7}$$

$$A_1^c \mid_{z=-d_1} = A_2 \mid_{z=-d_1}, \frac{\partial A_1^c}{\partial z}\mid_{z=-d_1} = \frac{\partial A_2}{\partial z}\mid_{z=-d_1}, c < r < b, \tag{8}$$

$$A_1^a \mid_{z=-d_1} = A_2 \mid_{z=-d_1}, \frac{\partial A_1^a}{\partial z}\mid_{z=-d_1} = \frac{\partial A_2}{\partial z}\mid_{z=-d_1}, \ 0 \le r < c, \tag{9}$$

$$A_2 \mid_{z=-d_3} = A_3 \mid_{z=-d_3}, \frac{\partial A_2}{\partial z}\mid_{z=-d_3} = \frac{\partial A_3}{\partial z}\mid_{z=-d_3}, \tag{10}$$

where $d_3 = d_1 + d_2$ and the superscripts a and c correspond to air and conductive region, respectively.
The interface conditions at $r = c$ have the form

$$A_1^c \mid_{r=c} = A_1^a \mid_{r=c}, \quad \frac{\partial A_1^c}{\partial r}\mid_{r=c} = \frac{\partial A_1^a}{\partial r}\mid_{r=c}. \tag{11}$$

In addition, vector potential is bounded at infinity in regions R_0 and R_3:

$$A_0 \to 0 \text{ as } z \to +\infty, A_3 \to 0 \text{ as } z \to -\infty. \tag{12}$$

III. SOLUTION FOR A SINGLE-TURN COIL

In order to construct the solution to (1) we consider two sub-regions of region R_0, namely,

$R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$. The solutions in R_{00} and R_{01} are denoted by A_{00} and A_{01}, respectively. Using the principle of superposition we represent the solutions to (1) in R_{00} and R_{01} in the form

$$A_{01}(r, z) = \sum_{i=1}^{\infty} D_{1i} e^{-\lambda_i z} J_1(\lambda_i r), \tag{13}$$

$$A_{00}(r,z) = \sum_{i=1}^{\infty}(D_{2i}e^{-\lambda_i z} + D_{3i}e^{\lambda_i z})J_1(\lambda_i r), \qquad (14)$$

where D_{1i}, D_{2i} and D_{3i} are arbitrary constants, $\lambda_i = \alpha_i / b$, α_i are the roots of the equation

$$J_1(\alpha) = 0 \qquad (15)$$

and $J_1(x)$ is the Bessel's function of the first kind of order one.

The vector potential is continuous at $z = h$:

$$A_{00}|_{z=h} = A_{01}|_{z=h}. \qquad (16)$$

Integrating (1) with respect to z from $h - \varepsilon$ to $h + \varepsilon$ and considering the limit as $\varepsilon \to +0$ in the resulting equation we obtain

$$\frac{\partial A_{01}}{\partial z}|_{z=h} - \frac{\partial A_{00}}{\partial z}|_{z=h} = -\mu_0 I\delta(r - r_0). \qquad (17)$$

It follows from (13)-(17) that

$$\sum_{i=1}^{\infty}D_{1i}e^{-\lambda_i h}J_1(\lambda_i r)$$
$$= \sum_{i=1}^{\infty}(D_{2i}e^{-\lambda_i h} + D_{3i}e^{\lambda_i h})J_1(\lambda_i r), \qquad (18)$$

$$\sum_{i=1}^{\infty}\lambda_i D_{1i}e^{-\lambda_i h}J_1(\lambda_i r)$$
$$+ \sum_{i=1}^{\infty}(-\lambda_i D_{2i}e^{-\lambda_i h} + \lambda_i D_{3i}e^{\lambda_i h})J_1(\lambda_i r) \qquad (19)$$
$$= \mu_0 I\delta(r - r_0).$$

Using (18) we obtain

$$D_{1i}e^{-\lambda_i h} = D_{2i}e^{-\lambda_i h} + D_{3i}e^{\lambda_i h}. \qquad (20)$$

Multiplying (19) by $rJ_1(\lambda_j r)$, integrating the resulting equation with respect to r from 0 to b and using the orthogonality condition

$$\int_0^b rJ_1(\lambda_j r)J_1(\lambda_i r)dr = \begin{cases} 0, & i \neq j \\ \dfrac{b^2}{2}J_0^2(\lambda_j b), & i = j \end{cases} \qquad (21)$$

the following equation is obtained

$$\left(D_{1j}e^{-\lambda_j h} - D_{2j}e^{-\lambda_j h} + D_{3j}e^{\lambda_j h}\right)\lambda_j \frac{b^2}{2}J_0^2(\lambda_j b)$$
$$= \mu_0 Ir_0 J_1(\lambda_j r_0). \qquad (22)$$

Equations (20) and (22) can be rewritten in the form

$$D_{3j} = \frac{\mu_0 Ir_0 J_1(\lambda_j r_0)}{\lambda_j b^2 J_0^2(\lambda_j b)}e^{-\lambda_j h}, \qquad (23)$$

$$D_{1j} = D_{2j} + D_{3j}e^{2\lambda_j h}. \qquad (24)$$

Substituting (23) and (24) into (13) and (14) we obtain

$$A_{00}(r,z) = \sum_{i=1}^{\infty}D_{2i}e^{-\lambda_i z}J_1(\lambda_i r)$$
$$+ \frac{\mu_0 Ir_0}{b^2}\sum_{i=1}^{\infty}\frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)}e^{-\lambda_i(h-z)}J_1(\lambda_i r), \qquad (25)$$

$$A_{01}(r,z) = \sum_{i=1}^{\infty}D_{2i}e^{-\lambda_i z}J_1(\lambda_i r)$$
$$+ \frac{\mu_0 Ir_0}{b^2}\sum_{i=1}^{\infty}\frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)}e^{-\lambda_i(z-h)}J_1(\lambda_i r). \qquad (26)$$

Using the method of separation of variables and superposition principle the solution in region R_1 can be written in the form

$$A_1^{air}(r,z) = \sum_{i=1}^{\infty}J_1(p_i r)T_1(q_i c)[D_{4i}e^{p_i z} + D_{5i}e^{-p_i z}],$$
$$0 \leq r < c, \qquad (27)$$

$$A_1^{con}(r,z) = \sum_{i=1}^{\infty}J_1(p_i c)T_1(q_i r)[D_{4i}e^{p_i z} + D_{5i}e^{-p_i z}],$$
$$c < r < b, \qquad (28)$$

where

$$T_1(q_i r) = J_1(q_i r)Y_1(q_i b) - J_1(q_i b)Y_1(q_i r), \qquad (29)$$

$Y_1(x)$ is the Bessel's function of the second kind of order one, q_i are unknown eigenvalues and $p_i = \sqrt{q_i^2 + j\omega\sigma\mu_0}$.

The solutions in regions R_2 and R_3 are

$$A_2(r,z) = \sum_{i=1}^{\infty}(D_{6i}e^{p_{1i}z} + D_{7i}e^{-p_{1i}z})J_1(\lambda_i r), \qquad (30)$$

$$A_3(r,z) = \sum_{i=1}^{\infty}D_{8i}e^{p_{2i}z}J_1(\lambda_i r), \qquad (31)$$

where $p_{1i} = \sqrt{\lambda_i^2 + j\omega\sigma_1\mu_0}$ and $p_{2i} = \sqrt{\lambda_i^2 + j\omega\sigma_2\mu_0}$.

Using the first boundary conditions in (6) and (7) we obtain

$$\sum_{i=1}^{\infty}D_{2i}J_1(\lambda_i r) + \frac{\mu_0 Ir_0}{b^2}\sum_{i=1}^{\infty}\frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)}e^{-\lambda_i h}J_1(\lambda_i r)$$
$$= \sum_{i=1}^{\infty}J_1(p_i r)T_1(q_i c)(D_{4i} + D_{5i}), \; 0 \leq r < c, \qquad (32)$$

$$\sum_{i=1}^{\infty}D_{2i}J_1(\lambda_i r) + \frac{\mu_0 Ir_0}{b^2}\sum_{i=1}^{\infty}\frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)}e^{-\lambda_i h}J_1(\lambda_i r)$$
$$= \sum_{i=1}^{\infty}J_1(p_i c)T_1(q_i r)(D_{4i} + D_{5i}), \; c < r < b. \qquad (33)$$

Multiplying (32) by $rJ_1(\lambda_j r)$, integrating the obtained equation with respect to r from 0 to b and using (21) we

obtain

$$D_{2j}\frac{b^2}{2}J_0^2(\lambda_j b)+\frac{\mu_0 Ir_0}{b^2}\frac{J_1(\lambda_j r_0)}{\lambda_j J_0^2(\lambda_j b)}\frac{b^2}{2}J_0^2(\lambda_j b)e^{-\lambda_j h}$$

$$=\sum_{i=1}^{\infty}(D_{4i}+D_{5i})a_{ji},$$

(34)

where

$$a_{ij}=T_1(p_i c)\tilde{a}_{ij}+J_1(q_i c)\tilde{\tilde{a}}_{ij}.$$

(35)

The coefficients a_{ij} can be computed using the following formulas from [11]:

$$\int (k^2-l^2)tJ_1(kt)J_1(lt)dt$$

$$=t[kJ_2(kt)J_1(lt)-lJ_1(kt)J_2(lt)],$$

(36)

$$\int (k^2-l^2)tJ_1(kt)Y_1(lt)dt$$

$$=t[kJ_2(kt)Y_1(lt)-lJ_1(kt)Y_2(lt)].$$

(37)

Thus, using (36) and (37) we obtain

$$\tilde{a}_{ij}=\int_0^c rJ_1(\lambda_j r)J_1(q_i r)dr$$

$$=\frac{c}{\lambda_j^2-q_i^2}\left(\lambda_j J_2(\lambda_j c)J_1(q_i c)-q_i J_1(\lambda_j c)J_2(q_i c)\right)$$

$$\tilde{\tilde{a}}_{ij}=\int_c^b rJ_1(\lambda_j r)T_1(p_i r)dr$$

$$=Y_1(p_i b)\int_c^b rJ_1(\lambda_j r)J_1(p_i r)dr$$

$$-J_1(p_i b)\int_c^b rJ_1(\lambda_j r)Y_1(p_i r)dr$$

$$=\frac{1}{\lambda_j^2-p_i^2}\{bp_i J_1(\lambda_j b)[J_1(p_i b)Y_2(p_i b)$$

$$-J_2(p_i b)Y_1(p_i b)]+c\lambda_j J_2(\lambda_j c)[J_1(p_i b)Y_1(p_i c)$$

$$-J_1(p_i c)Y_1(p_i b)]+cp_i J_1(\lambda_j c)[J_2(p_i c)Y_1(p_i b)$$

$$-J_1(p_i b)Y_2(p_i c)]\}$$

Applying the same procedure to (33) we obtain

$$-\lambda_j D_{2j}\frac{b^2}{2}J_0^2(\lambda_j b)+\frac{\mu_0 Ir_0}{2}J_1(\lambda_j r_0)e^{-\lambda_j h}$$

$$=\sum_{i=1}^{\infty}p_i(D_{4i}+D_{5i})a_{ji}.$$

(38)

Eliminating D_{2j} from (34) and (38) we get

$$\sum_{i=1}^{\infty}[(\lambda_j+p_i)D_{4i}+(\lambda_j-p_i)D_{5i}]a_{ji}$$

(39)

$$=\mu_0 Ir_0 J_1(\lambda_j r_0)e^{-\lambda_j h}.$$

Using boundary conditions (8) and (9) we can express constants D_{6j} and D_{7j} in terms of D_{4i} and D_{5i}:

$$D_{6j}=\frac{e^{p_{1j}d_1}}{p_{1j}b^2 J_0(\lambda_j b)}\sum_{i=1}^{\infty}[(p_{1j}+p_i)e^{-p_i d_1}D_{4i}$$

(40)

$$+(p_{1j}-p_i)e^{p_i d_1}D_{5i}]a_{ji},$$

$$D_{7j}=\frac{e^{-p_{1j}d_1}}{p_{1j}b^2 J_0(\lambda_j b)}\sum_{i=1}^{\infty}[(p_{1j}-p_i)e^{-p_i d_1}D_{4i}$$

(41)

$$+(p_{1j}+p_i)e^{p_i d_1}D_{5i}]a_{ji}.$$

Similarly, applying the boundary conditions (10) and using (40), (41) we eliminate D_{6j},D_{7j} and D_{8j} to obtain

$$\sum_{i=1}^{\infty}\{[(p_{2j}-p_{1j})(p_{1j}+p_i)e^{(p_{1j}-p_i)d_1-p_{1j}d_3}$$

$$+(p_{2j}+p_{1j})(p_{1j}-p_i)e^{-(p_{1j}+p_i)d_1+p_{1j}d_3}]D_{4i}$$

$$+[(p_{2j}-p_{1j})(p_{1j}-p_i)e^{(p_{1j}+p_i)d_1-p_{1j}d_3}$$

$$+(p_{2j}+p_{1j})(p_{1j}+p_i)e^{-(p_{1j}-p_i)d_1+p_{1j}d_3}]D_{5i}\}a_{ji}=0.$$

(42)

In order to solve the system (39), (42) numerically we have to truncate the number of terms in the series (in other words, the upper summation index in (39) and (42) should be replaced by n). Recommendations on the selection of the value of n are given, for example, in [12]. Thus, the system to be solved has the form

$$\sum_{i=1}^{n}[(\lambda_j+p_i)D_{4i}+(\lambda_j-p_i)D_{5i}]a_{ji}$$

(43)

$$=\mu_0 Ir_0 J_1(\lambda_j r_0)e^{-\lambda_j h},$$

$$\sum_{i=1}^{n}\{[(p_{2j}-p_{1j})(p_{1j}+p_i)e^{(p_{1j}-p_i)d_1-p_{1j}d_3}$$

$$+(p_{2j}+p_{1j})(p_{1j}-p_i)e^{-(p_{1j}+p_i)d_1+p_{1j}d_3}]D_{4i}$$

$$+[(p_{2j}-p_{1j})(p_{1j}-p_i)e^{(p_{1j}+p_i)d_1-p_{1j}d_3}$$

$$+(p_{2j}+p_{1j})(p_{1j}+p_i)e^{-(p_{1j}-p_i)d_1+p_{1j}d_3}]D_{5i}\}a_{ji}=0.$$

(44)

Using interface conditions (11) we obtain the following equation

$$\phi(p_i)=0,$$

(45)

where

$$\phi(p_i)=p_i J_1^{'}(p_i c)T_1(q_i c)-q_i T_1^{'}(q_i c)J_1(p_i c).$$

Equation (45) is used to determine complex eigenvalues p_i and the corresponding values q_i. The method of computation of the eigenvalues p_i is based on the two papers [13] and [14] published in 1967and uses the Cauchy's theorem. The number of zeros, s_0, of an analytic function $\phi(z)$ inside a closed contour C is equal to the following contour integral

$$s_0 = \frac{1}{2\pi i} \oint_C \frac{\phi'(z)}{\phi(z)} dz. \tag{46}$$

Another interesting result from the theory of complex variables is the following [13]. If C is a closed curve in the complex plane which does not pass through a zero of $\phi(z)$ and G is the interior of C, then

$$s_m = \frac{1}{2\pi i} \oint_C z^m \frac{\phi'(z)}{\phi(z)} dz = \sum_{i=1}^{k} \zeta_i^m, \tag{47}$$

where $\zeta_i, i = 1,2,...,k$ are all zeros of $\phi(z)$ which lie in G and $m = 1,2,...,k$. A multiple zero is counted according to its multiplicity in (47). Using (47) we can construct a polynomial of degree k which has the same zeros $\zeta_i, i = 1,2,...,k$ as the function $\phi(z)$. Note that in this case the polynomial is not an approximation to $\phi(z)$, it simply has the same roots as the function $\phi(z)$. At this stage several possible strategies are available in order to find ζ_i. If the domain G is small enough and there is only one zero of $\phi(z)$ inside C, then $k = 1$ and the value of s_1 from (47) gives us the only zero, ζ_1, of the function $\phi(z)$ inside C. However, if there is more than one root of $\phi(z)$ inside C then a polynomial is constructed using (47) and the roots of this polynomial can be calculated (for example, by means of Mathematica command **Roots**). As it is mentioned before, the roots of the corresponding polynomial are exactly the same as the zeros of $\phi(z)$ inside C. It is known that the roots of higher order polynomials can be quite sensitive to the variation of the coefficients of the polynomial. Thus, the number of zeros of $\phi(z)$ inside C should not be too large (k should not exceed 5). As a result, if $k > 5$ then the domain G should be divided into smaller sub-domains so that the number of zeros of $\phi(z)$ in each sub-domain be smaller than 5.

The next practical issue is the shape of the contour C. Two basic shapes are recommended for calculations in [13]: circles and rectangles. The choice of the shape of the contour C is dictated by calculation of the contour integrals in (46) and (47). It is known that contour integrals can be conveniently calculated if C is a circle or rectangle. We choose rectangle as the basic shape of C since any rectangle can be easily sub-divided into smaller rectangles.

As it has been already pointed out, there are two steps in the solution process where numerical calculations are necessary. First, one has to solve (45) and find the complex eigenvalues p_i (and the corresponding values of $q_i, i = 1,2,..,n)$. Thus, the value of n in (43) and (44) represents the number of eigenvalues p_i. Second, the system of linear equations (43), (44) has to be solved and all the constants $D_{4i}, D_{5i}, i = 1,2,...,n$ have to be determined. Next, we compute the coefficients $D_{2j}, j = 1,2,...,n$ from (34).

It is seen from (25) and (26) that the vector potential in region R_0 is the sum of the two terms. The second term on the right-hand side of (25) and (26) represents the vector potential of a single-turn coil located in an unbounded free space where the conducting medium is absent. The first term on the right-hand side of (25) and (26) is the induced vector potential due to the presence of the conducting medium. Hence, the induced vector potential in region R_0 is given by the formula

$$A_0^{ind}(r,z) = \sum_{i=1}^{\infty} D_{2i} e^{-\lambda_i z} J_1(\lambda_i r). \tag{48}$$

IV.   CALCULATION OF THE CHANGE IN IMPEDANCE OF A SINGLE-TURN COIL

The induced change in impedance of the coil shown in Fig. 2 is given by the formula

$$Z_0^{ind} = \frac{j\omega}{I} 2\pi r_0 A_0^{ind}(r_0, h). \tag{49}$$

Calculations are performed with Mathematica using (48) and (49). The following values of the parameters of the problem are chosen: $r_0 = 4.5\,\text{mm}$, $h = 0.2\,\text{mm}$, $\sigma_1 = 3\,\text{Ms/m}$, $\sigma_2 = 5\,\text{Ms/m}$, $c = 2.2\,\text{mm}$, $d_1 = 0.5\,\text{mm}$, $d_2 = 10\,\text{mm}$. The results are shown in Fig. 3 for seven values of frequency f from 1 kHz to 7 kHz (from top to bottom) with the step size of 1 kHz.
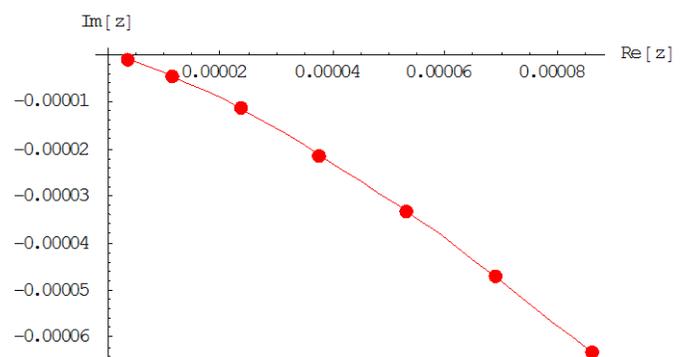
Fig. 3. The change in impedance of the single-turn coil for seven frequencies from 1 kHz to 7 kHz ($d_1 = 0.7\,\text{mm}$, $d_2 = 0.3\,\text{mm}$).

Computations are also performed for the case $d_1 = 0.7$ mm, $d_2 = 0.3$ mm (the other parameter values are the same as in Fig. 3). The change in impedance is shown in Fig. 4.
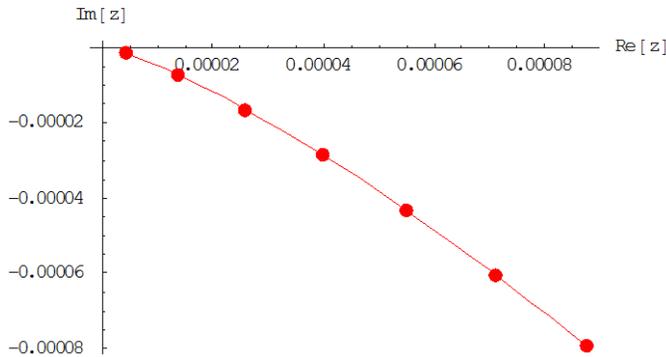
Fig. 4. The change in impedance of the single-turn coil shown in Fig. 2 for seven frequencies from 1 kHz to 7 kHz ($d_1 = 0.5$ mm, $d_2 = 10$ mm).

V. SOLUTION FOR A COIL OF FINITE DIMENSIONS

The induced vector potential in region R_0 for the case of a coil of finite dimensions shown in Fig. 1 is given by the formula

$$A_{0coil}^{ind}(r,z) = \int_{r_1}^{r_2}\int_{z_1}^{z_2} A_0^{ind}(r,z,r_0,h)dr_0 dh . \qquad (50)$$

Note that numerical solution of the system (43), (44) for a single-turn coil can be found by substituting the values of r_0 and h into the right-hand side of (43). For a coil of finite dimensions such an approach cannot be used since we have to integrate the change in impedance over the cross-section of the coil (with respect to r_0 and h). In order to perform calculations for the coil of finite size we rewrite (43), (44) in the matrix form

$$A\vec{X} = \vec{B}, \qquad (51)$$

where the coefficient matrix A is

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad (52)$$

and the elements of the block matrices A_{11}, A_{12}, A_{21} and A_{22} are given by

$$A_{11}(i,j) = (\lambda_j + p_i)a_{ji}, \quad A_{12}(i,j) = (\lambda_j - p_i)a_{ji},$$

$$A_{21}(i,j) = [(p_{2j} - p_{1j})(p_{1j} + p_i)e^{(p_{1j} - p_i)d_1 - p_{1j}d_3}$$

$$+ (p_{2j} + p_{1j})(p_{1j} - p_i)e^{-(p_{1j} + p_i)d_1 + p_{1j}d_3}]a_{ji}, \qquad (53)$$

$$A_{22}(i,j) = [(p_{2j} - p_{1j})(p_{1j} - p_i)e^{(p_{1j} + p_i)d_1 - p_{1j}d_3}$$

$$+ (p_{2j} + p_{1j})(p_{1j} + p_i)e^{-(p_{1j} + p_i)d_1 + p_{1j}d_3}]a_{ji}.$$

The matrices $\vec{X}$ and $\vec{B}$ in (51) are

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}, \vec{B} = \begin{pmatrix} \vec{b} \\ 0 \end{pmatrix}, \qquad (54)$$

where

$$\vec{X}_1 = \begin{pmatrix} D_{41} \\ D_{42} \\ ... \\ D_{4n} \end{pmatrix}, \ \vec{X}_2 = \begin{pmatrix} D_{51} \\ D_{52} \\ ... \\ D_{5n} \end{pmatrix}, \vec{b} = \begin{pmatrix} \mu_0 I r_0 J_1(\lambda_1 r_0)e^{-\lambda_1 h} \\ \mu_0 I r_0 J_1(\lambda_2 r_0)e^{-\lambda_2 h} \\ ... \\ \mu_0 I r_0 J_1(\lambda_n r_0)e^{-\lambda_n h} \end{pmatrix}.$$

Hence, (51) can be written in the form

$$\begin{cases} A_{11}\vec{X}_1 + A_{12}\vec{X}_2 = \vec{b} \\ A_{21}\vec{X}_1 + A_{22}\vec{X}_2 = 0 \end{cases} \qquad (55)$$

Solving the second equation in (55) we obtain

$$\vec{X}_2 = -A_{22}^{-1}A_{21}\vec{X}_1. \qquad (56)$$

Substituting (56) into the first equation in (55) gives

$$\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)\vec{X}_1 = \vec{b}. \qquad (57)$$

It follows from (56) and (57) that

$$\begin{cases} \vec{X}_1 = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}\vec{b} \\ \vec{X}_2 = -A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}\vec{b} \end{cases} \qquad (58)$$

Equation (34) should be written in matrix form:

$$\vec{D}_2 = Y\vec{b}, \qquad (59)$$

where

$$\vec{D}_2 = \begin{pmatrix} D_{21} \\ D_{22} \\ ... \\ D_{2n} \end{pmatrix},$$

$$Y = B_{11}\left(E - A_{22}^{-1}A_{21}\right)\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} - C_{diag},$$

$$\qquad (60)$$

$$B_{11} = \begin{pmatrix} \dfrac{2a_{11}}{b^2 J_0^2(\lambda_1 b)} & \dfrac{2a_{12}}{b^2 J_0^2(\lambda_1 b)} & \cdots & \dfrac{2a_{1n}}{b^2 J_0^2(\lambda_1 b)} \\ \dfrac{2a_{21}}{b^2 J_0^2(\lambda_2 b)} & \dfrac{2a_{22}}{b^2 J_0^2(\lambda_2 b)} & \cdots & \dfrac{2a_{2n}}{b^2 J_0^2(\lambda_2 b)} \\ .... \\ \dfrac{2a_{n1}}{b^2 J_0^2(\lambda_n b)} & \dfrac{2a_{n2}}{b^2 J_0^2(\lambda_n b)} & \cdots & \dfrac{2a_{nn}}{b^2 J_0^2(\lambda_n b)} \end{pmatrix},$$

$$C_{diag} = \begin{pmatrix} \dfrac{1}{\lambda_1 b^2 J_0^2(\lambda_1 b)} & 0 & 0 & \dots & 0 \\ 0 & \dfrac{1}{\lambda_2 b^2 J_0^2(\lambda_2 b)} & 0 & 0 & \dots & 0 \\ & & \dots\dots\dots\dots\dots & & \\ 0 & 0 & \dots & 0 & \dfrac{1}{\lambda_n b^2 J_0^2(\lambda_n b)} \end{pmatrix}. \tag{61}$$

Formula (48) for the induced vector potential of a single-turn coil can be rewritten in the form

$$A_0^{ind}(r,z,r_0,h) = \vec{D}_2^T \vec{f}, \tag{62}$$

where

$$\vec{f} = \begin{pmatrix} J_1(\lambda_1 r)e^{-\lambda_1 z} \\ J_1(\lambda_2 r)e^{-\lambda_2 z} \\ \dots \\ J_1(\lambda_n r)e^{-\lambda_n z} \end{pmatrix}.$$

Substituting (62) into (50) and using the formulas

$$\int_{z_1}^{z_2} e^{-\lambda_j h} dh = -\frac{1}{\lambda_j}\left(e^{-\lambda_j z_2} - e^{-\lambda_j z_1}\right)$$

$$\int_{r_1}^{r_2} r_0 J_1(\lambda_j r_0) dr_0 = \Big| \xi = \lambda_j r_0 \Big| = \frac{1}{\lambda_i^2} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \tag{63}$$

we obtain the induced vector potential in air due to the presence of the conducting medium (the current amplitude I in this case is replaced by the current density $\dfrac{NI}{(r_2 - r_1)(z_2 - z_1)}$):

$$A_{0coil}^{ind}(r,z) = \frac{\mu_0 NI}{(r_2 - r_1)(z_2 - z_1)}$$

$$\times \sum_{j=1}^{n} f_j \sum_{i=1}^{n} Y_{ji} \frac{\left(e^{-\lambda_i z_1} - e^{-\lambda_i z_2}\right)}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi. \tag{64}$$

The integral with respect to ξ in (64) can be computed in terms of the Bessel and Struve functions [11] as follows

$$\int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi = \left\{ \frac{\pi}{2}\xi\big[J_0(\xi)H_1(\xi) - J_1(\xi)H_0(\xi)\big] \right\} \Bigg|_{\xi = \lambda_i r_1}^{\xi = \lambda_i r_2}$$

The induced change in impedance of a coil of finite dimensions is calculated by means of the following formula [12]:

$$Z^{ind} = \frac{j\omega}{I^2} \iiint_V \vec{A}\vec{I} dv = \frac{j\omega}{I^2} \int_0^{2\pi} d\varphi \int_{r_1}^{r_2} r\,dr \int_{z_1}^{z_2} A_{0coil}^{ind} dz \tag{65}$$

Using (64) and (65) we obtain the induced change in

impedance of the coil in the form

$$Z^{ind} = \frac{2j\omega\pi\mu_0 N^2}{(r_2 - r_1)^2 (z_2 - z_1)^2}$$

$$\times \sum_{j=1}^{n} \frac{\left(e^{-\lambda_j z_1} - e^{-\lambda_j z_2}\right)}{\lambda_j^3} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \tag{66}$$

$$\times \sum_{i=1}^{n} Y_{ji} \frac{\left(e^{-\lambda_i z_1} - e^{-\lambda_i z_2}\right)}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} \xi J_1(\xi) d\xi$$

VI. CALCULATION OF THE CHANGE IN IMPEDANCE FOR A COIL OF FINITE DIMENSIONS

Calculations are done using formula (66) for the following values of the parameters of the problem shown in Fig. 2: $\sigma_1 = 3\,\text{Ms/m}$, $\sigma_2 = 5\,\text{Ms/m}$, $c = 2.2\,\text{mm}$, $r_1 = 3.5\,\text{mm}$, $r_2 = 5\,\text{mm}$, $z_1 = 0.3\,\text{mm}$, $z_2 = 2.6\,\text{mm}$, $N = 200$. Seven values of frequencies from 1 kHz to 7 kHz with the step size of 1 kHz are used. Computational results are shown in Figs. 5-6. Fig. 5 plots the change in impedance of the coil for the following values of the parameters $d_1 = 0.5\,\text{mm}$ and $d_2 = 10\,\text{mm}$.
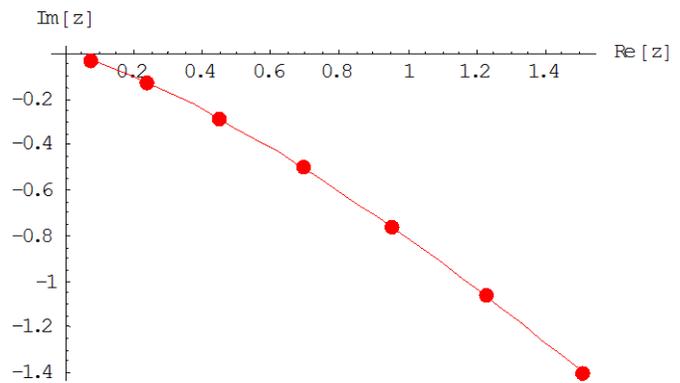
Fig. 5. The change in impedance of the coil shown in Fig.2 for seven different frequencies from 1 kHz (top point) to 7 kHz (bottom point) for the case $d_1 = 0.5\,\text{mm}$ and $d_2 = 10\,\text{mm}$.

The graph of the change in impedance for the values of the parameters $d_1 = 0.7\,\text{mm}$ and $d_2 = 0.3\,\text{mm}$ is shown in Fig. 6 for the same seven frequencies.
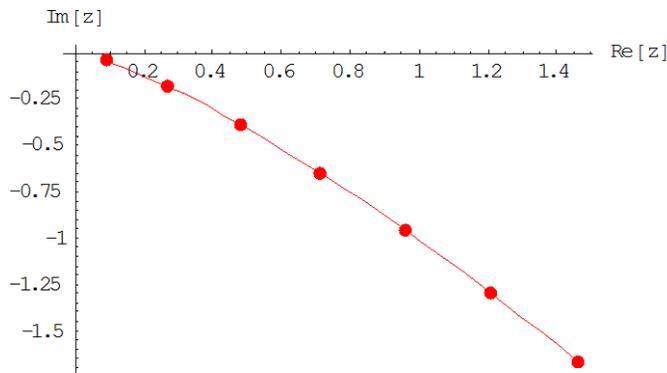
Fig. 6. The change in impedance of the coil shown in Fig.2 for seven different frequencies from 1 kHz (top point) to 7 kHz (bottom point) for the case $d_1 = 0.7\,\text{mm}$ and $d_2 = 0.3\,\text{mm}$.

VII. CONCLUSION AND DIRECTIONS OF FUTURE WORK

The method of truncated eigenfunction expansions is used in the present paper to compute the change in impedance of a coil due to a cylindrical flaw in a conducting medium. The problem is solved by the method of separation of variables. The obtained solution is semi-analytical since the method of separation of variables is combined in the paper with numerical methods in order to compute complex eigenvalues and solve systems of linear algebraic equations. The method can be generalized for other problems with axial symmetry.

Recently a similar asymmetric problem is solved in [15] where a coil is located above a conducting half-space with an infinite vertical cylindrical hole. The axis of the coil does not coincide with the axis of the hole. The problem is solved using the second order vector potential formulation of the Maxwell's equations. Such an approach can also be generalized for the case considered in the present paper.

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