Nonlinear thermodynamic model for reduced Cosserat continuum

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Abstract—In this paper we investigate a reduced Cosserat continuum. This model was suggested as possible model to describe granular materials. Here we get thermodynamic equations for the current and reference configuration.

Keywords—continuum, model, thermodynamic.

I. Introduction

In this work we are aiming to establish a system of thermodynamic equations for reduced Cosserat continuum. The idea of the reduced Cosserat continuum as an elastic medium is proposed as a model for granular medium as well. This type of medium and its behavior is very important in different branches of engineering and industrial applications such as mining, agriculture, construction and geological processes.

Most of the models suggest that the sizes of solid particles are negligible in comparison with typical distances between particles. Our model deals with granular materials where grain's size and nearest-neighbour distance are roughly comparable. In contrast to solid bodies in granular materials there is no "rotational springs" that keep rotations of neighbouring grains. For example in the simplest case, solids can be modeled as an array of point masses connected by Springs [1].

Originally an idea of an equal footing of rotational and translational degrees of freedom appeared in [2]. In that work authors obtained good correspondents between their theoretical results with experimental data. We used this work as in inspiration in our studies.

There are two well-known theories for described solids: moment theory of elasticity (Cosserat Continuum), moment theory of elasticity with constrained rotation (Cosserat Pseudocontinuum). There exists a vast amount of literature on these models, such as [3], [4], [5], [6], [7], [8]. A practical application of these models requires an experimental determination of a large number of additional constants in constitutive equations. These theories can still be applied to granular media although there are many other specific models describing this type of media [9], [10], [11], [12], [13]. In recent papers [14], [15] more advanced Reduced Cosserat Continuum was suggested as possible model to describe granular materials. In this continuum translations and rotations are independent, stress tensor is not symmetric and couple stresses tensor equal to zero. A feature of this model that it has a classical continuum as its static limit. More advanced studies of this model were performed quit recently in the works [16], [17], [18].

In this paper, we further develop results achieved in [19], [20], [21] for reduced Cosserat continuum as a suitable model for granular medium. In these works we have presented linear reduced Cosserat continuum equations, plane wave propagation and dispersion curves for an isotropic case. It is now very relevant study of thermal problem in the mechanics [22], [23], [24]. Here we present thermodynamic nonlinear reduced Cosserat continuum equations for the current configuration.

II. System of equation for the current configuration.

In reduced Cosserat continuum each particle has 6 degrees of freedom, in terms of kinematics its state is described by vector r and turn tensor P. The turn tensor is orthogonal tensor that is defined by 3 independent parameters with determinant equal to 1. Current position of the body at time t is called the current configuration (CC). Let us introduce a basis $\mathbf{r}_k(x^s,t)=\partial\mathbf{r}/\partial x^k$, a dual basis $\mathbf{r}^k(x^s,t)$ and a Hamiltonian for the CC $\nabla=\mathbf{r}^k\dfrac{\partial}{\partial x^k}$.

In this paper we obtain thermodynamic equations for the nonlinear reduced Cosserat continuum as Eulerian description for the CC.

Here we list equations that are necessary to establish the system of equations for the CC:

a linear momentum balance equation

$$\frac{d}{dt}\int_V \rho\mathbf{v}dV=\int_S \mathbf{n}\cdot\boldsymbol{\tau}dS \tag{1}$$

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a kinetic momentum balance equation

$$\frac{d}{dt}\int_V (\rho\mathbf{J}\cdot\boldsymbol{\omega}+\mathbf{r}\times\rho\mathbf{v})dV=\int_S \mathbf{r}\times(\mathbf{n}\cdot\boldsymbol{\tau})dS \quad (2)$$

an energy balance equation:

$$\frac{d}{dt}\int_V (\frac{1}{2}\rho\mathbf{v}^2+\frac{1}{2}\omega\cdot\rho\mathbf{J}\cdot\boldsymbol{\omega}+\rho\Pi)dV=\int_S \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{v}dS+Q \quad (3)$$

and the Reynolds transport theorem

$$\frac{d}{dt}\int_V \rho A dV=\int_V \rho\dot{A}dV \quad (4)$$

where ρ is density for the CC, $\boldsymbol{\tau}$ - stress tensor, v - velocity vector $(\mathbf{v}=\dot{\mathbf{r}})$, r - radius vector for the CC, ω - angular velocity vector $(\dot{\mathbf{P}}=\boldsymbol{\omega}\times\mathbf{P})$, n - an outward unit normal to the surface S, J - mass density of an inertia tensor, Π - mass density of the strain energy, $\dot{(...)}=\frac{\partial}{\partial t}(...)+\mathbf{v}\cdot\nabla(...)$ - material time derivative, A - arbitrary scalar, vector or tensor field, V - volume limited by a surface S, Q - thermal power. To simplify calculations we assume that the body forces are equal to zero.

We shall combine equation (4), Gauss-Ostrogradskii theorem and equations (1) and (2). As a result we get motion equations for the CC:

$$\nabla\cdot\boldsymbol{\tau}=\rho\dot{\mathbf{v}} \quad (5)$$

$$\boldsymbol{\tau}_x=\rho(\boldsymbol{\omega}\times\mathbf{J}\cdot\boldsymbol{\omega}+\mathbf{J}\cdot\dot{\boldsymbol{\omega}}), \quad (6)$$

where $\boldsymbol{\tau}_x$ denotes a vector invariant of tensor $\boldsymbol{\tau}$. A definition of this vector invariant was given by Lurie [25].

Equations (5) and (6) are the first two equations in our system describing CC.

Heat Q is sum of a heat coming into the volume V through its surface and a heat distributed in volume and can expressed as a following formula

$$Q=\int_S -\mathbf{n}\cdot\mathbf{h}dS+\int_V \rho q dV=\int_V (\rho q-\nabla\cdot\mathbf{h})dV. \quad (7)$$

Here $\mathbf{h}$ is a heat flux vector, q is a heat source per unit mass.

After combining equations (3), (4), (7) and applying an identity $\nabla\cdot(A\cdot a)=\nabla\cdot A\cdot a+A^T\cdot\cdot\nabla a$ we arrive at

$$\int_V \rho(\dot{\mathbf{v}}\cdot\mathbf{v}+\dot{\boldsymbol{\omega}}\cdot\mathbf{J}\cdot\boldsymbol{\omega}+\dot{\Pi})dV=\int_V (\nabla\cdot\boldsymbol{\tau}\cdot\mathbf{v}+\boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v})+(\rho q-\nabla\cdot\mathbf{h})dV. \quad (8)$$

Since volume V is an arbitrary volume, than using (8) and equation (5) we obtain the following relation

$$\rho\dot{\Pi}=\boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v}-\rho(\mathbf{J}\cdot\dot{\boldsymbol{\omega}})\cdot\boldsymbol{\omega}+\rho q-\nabla\cdot\mathbf{h} \quad (9)$$

Now let us apply relation (6) for a second term in the right hand side of an equation above. The latter combined with $(\omega\times J\cdot\omega)\cdot\omega=0$ results

$$-\rho(\mathbf{J}\cdot\dot{\boldsymbol{\omega}})\cdot\boldsymbol{\omega}=\boldsymbol{\tau}_x^T\cdot\boldsymbol{\omega}=\boldsymbol{\tau}^T\cdot\cdot(\mathbf{I}\times\boldsymbol{\omega}).$$

The equality above was obtained with help of expression $(\mathbf{A}\cdot\mathbf{B})_x\cdot\mathbf{a}=\mathbf{A}\cdot\cdot(\mathbf{B}\times\mathbf{a})$ with B = I [26]. Now we can rewrite the equation (9) in the following form

$$\rho\dot{\Pi}=\boldsymbol{\tau}^T\cdot\cdot(\nabla\mathbf{v}+\mathbf{I}\times\boldsymbol{\omega})+\rho q-\nabla\cdot\mathbf{h}. \quad (10)$$

A strain state for reduced Cosserat continuum is described by a strain tensor $\mathbf{e}(x^k,t)$. We can define it for the CC as:

$$\mathbf{e}=\mathbf{I}-\mathbf{F}^{-T}\cdot\mathbf{P}^T, \quad (11)$$

where F should satisfy a relation $\mathbf{F}^{-1}=\nabla\mathbf{R}^T$ with R as a radius vector in the reference configuration.

Let us differentiate the expression (11) with respect to time, using $\dot{\mathbf{F}}^{-T}=-\nabla\mathbf{v}\cdot\mathbf{F}^{-T}$ and $\dot{\mathbf{P}}^T=-\mathbf{P}^T\times\boldsymbol{\omega}$ to obtain rhe

$$D\mathbf{e}=\dot{\mathbf{e}}+\mathbf{e}\times\boldsymbol{\omega}+\nabla\mathbf{v}\cdot\mathbf{e}=\nabla\mathbf{v}+\mathbf{I}\times\boldsymbol{\omega}. \quad (12)$$

From now we will use this short notation for $D\mathbf{A}=\dot{\mathbf{A}}+\mathbf{A}\times\omega+\nabla\mathbf{v}\cdot\mathbf{A}$ for an arbitrary tensor **A**. Value $D\mathbf{A}$ is shown to be an objective derivative [11].

The equation (12) is the compatibility equation for the CC. And it is number 3 in our system of equations.

After having introduced strain and stress, it is necessary to establish a relation between them. This was done through constitutive equations. Let us substitute (12) in (10) and obtain

$$\rho\dot{\Pi}=\boldsymbol{\tau}^T\cdot\cdot D\mathbf{e}+\rho q-\nabla\cdot\mathbf{h} \quad (13)$$

One can easily recognize the first law of thermodynamics in (13).

As we know from [25], the second law of thermodynamics can be expressed as

$$\rho\theta\dot{\eta}-(\rho q-\nabla\cdot\mathbf{h})-\mathbf{h}\cdot\nabla\ln\theta\ge 0, \quad (14)$$

where $\eta=\eta(\mathbf{e},\theta)$ is a unit entropy, θ is a temperature.

Recombination of terms in the expression (13) gives us

$$\rho q-\nabla\cdot\mathbf{h}=\rho\dot{\Pi}-\boldsymbol{\tau}^T\cdot\cdot D\mathbf{e}.$$

Let us substitute this into (14) to get

$$\rho\theta\dot{\eta}-\rho\dot{\Pi}+\boldsymbol{\tau}^T\cdot\cdot D\mathbf{e}-\mathbf{h}\cdot\nabla\ln\theta\ge 0$$

Further, we use $f=\Pi-\theta\eta$ - the Helmholtz free-energy function. With equation above this results in

$$\boldsymbol{\tau}^T\cdot\cdot D\mathbf{e}-\rho(\dot{f}+\dot{\theta}\eta)-\mathbf{h}\cdot\nabla\ln\theta\ge 0. \quad (15)$$

Applying a techniques describing in [26] we can get

$$\dot{f}=\frac{\partial f}{\partial\mathbf{e}}^T\cdot\cdot D\mathbf{e}+\frac{\partial f}{\partial\theta}\dot{\theta}. \quad (16)$$

Substituting (16) into (15) we obtain

$$(\boldsymbol{\tau}-\rho\frac{\partial f}{\partial\mathbf{e}})^T\cdot\cdot D\mathbf{e}-\rho(\frac{\partial f}{\partial\theta}+\eta)\dot{\theta}-\mathbf{h}\cdot\nabla\ln\theta\ge 0. \quad (17)$$

It is shown in [25] that equation (17) holds only if

$$\boldsymbol{\tau}-\rho\frac{\partial f}{\partial\mathbf{e}}=0,\quad \rho(\frac{\partial f}{\partial\theta}+\eta)=0,\quad -\mathbf{h}\cdot\nabla\ln\theta\ge 0,$$

which leads us to the following result

$$\boldsymbol{\tau}=\rho\frac{\partial f}{\partial\mathbf{e}} \quad (18)$$

$$-\frac{\partial f}{\partial\theta}=\eta \quad (19)$$

In our case a mass density Π is a function of two arguments: a strain state $\mathbf{e}$ and a temperature θ. A partial derivative of Π with respect to $\mathbf{e}$ can be transformed in the following way

$$\frac{\partial \Pi}{\partial \mathbf{e}} = \eta \frac{\partial \theta}{\partial \mathbf{e}} + \frac{\partial f}{\partial \mathbf{e}} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial \mathbf{e}} = \frac{\partial f}{\partial \mathbf{e}} + \eta\left(\frac{\partial \theta}{\partial \mathbf{e}} - \frac{\partial \theta}{\partial \mathbf{e}}\right) = \frac{\partial f}{\partial \mathbf{e}}, \quad (20)$$

Here we used definition of f and expression (19). Using this expression (20) and the equation (18) arrive at

$$\boldsymbol{\tau} = \rho \frac{\partial \Pi}{\partial \mathbf{e}}. \quad (21)$$

The expression (21) is the constitutive equation for the CC.

In order to obtain next equation for the CC we need to refer back to the expression (13). For that we introduce $\varphi = \boldsymbol{\tau}^T \cdot\cdot D\mathbf{e} - \rho(\dot f + \dot\theta \eta)$ - as unit energy dissipation.

In the elastic medium $\varphi = 0$, thus the first law of thermodynamics has a following simple form

$$\rho\theta\dot\eta = \rho q - \nabla\cdot\mathbf{h}. \quad (22)$$

Since we consider isotropic media $\mathbf{h} = -k\nabla\theta$, where k denotes the coefficient of thermal conductivity.

Time derivative of $\eta = \eta(\mathbf{e},\theta)$ in combination with technique describing in [26] results in

$$\dot\eta = \frac{\partial \eta}{\partial \theta}\dot\theta + \frac{\partial \eta}{\partial \mathbf{e}}\cdot\cdot(D\mathbf{e})^T.$$

And with help of formula (21) we arrive at

$$\rho\theta\frac{\partial \eta}{\partial \theta}\dot\theta + \rho\theta\frac{\partial \eta}{\partial \mathbf{e}}\cdot\cdot(D\mathbf{e})^T = \rho q + \nabla\cdot k\nabla\theta. \quad (23)$$

Combining formulas (18) and (19) results in

$$\frac{\partial \eta}{\partial \mathbf{e}} = -\frac{1}{\rho}\frac{\partial \boldsymbol{\tau}}{\partial \theta}. \quad (24)$$

Now we introduce $\chi = \theta\frac{\partial \eta}{\partial \theta}$ - the specific heat capacity of constant deformation.

After substitution of χ and (24) into (23) we arrive to the following equation

$$\rho\chi\dot\theta = \rho q + \nabla\cdot k\nabla\theta + \theta\frac{\partial \boldsymbol{\tau}}{\partial \theta}\cdot\cdot(D\mathbf{e})^T. \quad (25)$$

The last equation is a heat conductivity equation for the CC and it is the fifth equation in our system.

The system of equation for the CC will not be full without the mass conservation law [25]

$$\dot\rho + \rho\nabla\cdot\mathbf{v} = 0. \quad (26)$$

III. SYSTEM OF EQUATION FOR THE REFERENCE CONFIGURATION.

Now let us obtain the system of equation for the reference configuration (RC). Usually RC is selected as a known position of the body at the initial time t = 0. Let $\mathbf{r}(x^s,0) = \mathbf{R}(x^s)$. We introduce the basis $\mathbf{R}_k(x^s) = \partial\mathbf{R}/\partial x^k$, the dual basis $\mathbf{R}^k(x^s)$ and the Hamiltonian in the RC $\overset{\circ}{\nabla} = \mathbf{R}^s\frac{\partial}{\partial x^s}$.

To write down the system of equation for the RC we need to use tensors and vectors in a basis for the RC.

For the RC we need to use “rotated” velocities vector

$$\mathbf{V} = \mathbf{P}^T\cdot\mathbf{v} \quad (27)$$

$$\boldsymbol{\Omega} = \mathbf{P}^T\cdot\boldsymbol{\omega} \quad (28)$$

The vector $\boldsymbol{\Omega}$ is used in rigid body dynamic [27]. There it was called the right angular velocity vector and is defined by the equation

$$\dot{\mathbf{P}} = \mathbf{P}\times\boldsymbol{\Omega}, \quad (29)$$

where $\dot{(...)} = \frac{\partial}{\partial t}(...)$.

For the CC tensor **J** was defined as follows $\mathbf{J} = \mathbf{P}\cdot\mathbf{J}_0\cdot\mathbf{P}^T$ [27], where $\mathbf{J}_0$ is the known mass density of an inertia tensor for the RC. The stress state for the RC for the reduced Cosserat continuum is described by the stress tensor [26]

$$\mathbf{T} = \frac{\rho_0}{\rho}\mathbf{F}^{-1}\cdot\boldsymbol{\tau}\cdot\mathbf{P}, \quad (30)$$

where ρ_0 is a density in RC, $\mathbf{F} = \overset{\circ}{\nabla}\mathbf{r}^T$.

The strain state for the RC is described by the strain tensor [26]

$$\mathbf{E} = \mathbf{F}^T\cdot\mathbf{P} - \mathbf{I} \quad (31)$$

This strain tensor is identically equal to zero when body moves as rigid.

Define V_0 as a volume for the RC which changes in V for the CC. Volumes V and V_0 consist of the same particles.

The Nanson’s formula is necessary to establish the system of equation for the RC [25]:

$$\mathbf{n}dS = \frac{\rho_0}{\rho}\mathbf{N}\cdot\mathbf{F}^{-1}dS_0, \quad (32)$$

where **N** is an outward unit normal to the surface S_0.

So we shall combine equation (1), (32) and Gauss-Ostrogradskii theorem.

Because V_0 is arbitrary, we get

$$\rho_0\dot{\mathbf{v}} = \overset{\circ}{\nabla}\cdot\mathbf{T}\cdot\mathbf{P}^T + \mathbf{R}^s\cdot\mathbf{T}\cdot\frac{\partial\mathbf{P}^T}{\partial x^s}. \quad (33)$$

To be able to write down motion equations and compatibility equations we need an additional tensor $\mathbf{K}$.

$$\overset{\circ}{\nabla}\mathbf{P}^T = -\mathbf{K}\times\mathbf{P}^T. \quad (34)$$

Hence $\frac{\partial P^T}{\partial x^s} = -\mathbf{K}_s\times\mathbf{P}^T$, (35)

because $\mathbf{K}$ satisfies a relation $\mathbf{K} = \mathbf{r}^s\mathbf{K}_s$.

Now let us apply relation (35) for a second term in the right hand side of an equation (33). The latter combined with $\mathbf{A}_x = -\mathbf{A}_x^T$ and $(\mathbf{A}^T\cdot\mathbf{K})_x = \mathbf{R}^k\cdot\mathbf{A}\times\mathbf{K}_k$, that is valid for any tensor **A**, results

$$\mathbf{R}^s \cdot \mathbf{T} \cdot \frac{\partial \mathbf{P}^T}{\partial x^s} = -\mathbf{R}^s \cdot \mathbf{T} \cdot (\mathbf{K}_s \times \mathbf{P}^T) = = -(\mathbf{R}^s \cdot \mathbf{T} \times \mathbf{K}_s) \cdot \mathbf{P}^T = (\mathbf{K}^T \cdot \mathbf{T})_x \cdot \mathbf{P}^T \tag{36}$$

Then we shall transform term in the left hand side of an equation (33).

$$\dot{\mathbf{v}} = (\mathbf{P} \cdot \mathbf{V})^{\cdot} = (\mathbf{P} \times \mathbf{\Omega}) \cdot \mathbf{V} + \mathbf{P} \cdot \dot{\mathbf{V}} = (\dot{\mathbf{V}} + \mathbf{\Omega} \times \mathbf{V}) \cdot \mathbf{P}^T \tag{37}$$

Let us return to the equation (33) and multiply it by the tensor **P** on the right. After combining equations (33), (36) and (37) we obtain a local form of the linear momentum balance equation for the RC

$$\overset{\circ}{\nabla} \cdot \mathbf{T} + (\mathbf{K}^T \cdot \mathbf{T})_x = \rho_0 (\dot{\mathbf{V}} + \mathbf{\Omega} \times \mathbf{V}) \tag{38}$$

The equation (38) is the first motion equation for the RC.

Further we shall combine equation (2), (28), (30), (32) and Gauss-Ostrogradskii theorem. Since volume V_0 is an arbitrary volume, than we get

$$\rho_0 ((\mathbf{P} \cdot \mathbf{J}_0 \cdot \Omega)^{\cdot} + (\mathbf{r} \dot{\times} \mathbf{v})) = -\overset{\circ}{\nabla} \cdot (\mathbf{T} \cdot \mathbf{P}^T \times \mathbf{r}) - \mathbf{R}^k \cdot \mathbf{T} \cdot \mathbf{P}^T \times \mathbf{r}_k \tag{39}$$

Now let us transform a second term in the left hand side of an equation above.

$$(\mathbf{r} \times \mathbf{v})^{\cdot} = \mathbf{r} \times (\dot{\mathbf{V}} + \mathbf{\Omega} \times \mathbf{V}) \cdot \mathbf{P}^T . \tag{40}$$

Than we transform the first term in the left hand side of an equation (39).

$$(\mathbf{P} \cdot \mathbf{J}_0 \cdot \mathbf{\Omega})^{\cdot} = \mathbf{P} \cdot (\mathbf{J}_0 \cdot \dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{J}_0 \cdot \mathbf{\Omega}) \tag{41}$$

Now let us apply relation (36) for the first term in the right hand side of an equation (39).

$$\overset{\circ}{\nabla} \cdot (\mathbf{T} \cdot \mathbf{P}^T) \times \mathbf{r} = -\mathbf{r} \times (\overset{\circ}{\nabla} \cdot \mathbf{T} + (\mathbf{K}^T \cdot \mathbf{T})_x) \cdot \mathbf{P}^T . \tag{42}$$

Then we apply equation (31) for a second term in the right hand side of an equation (39).

$$\mathbf{R}^k \cdot \mathbf{T} \cdot \mathbf{P}^T \times \mathbf{r}_k = -\mathbf{P} \cdot ((\mathbf{E} + \mathbf{I})^T \cdot \mathbf{T})_x . \tag{43}$$

So after combining equations (38), (40), (41), (42, (43) we arrive at a local form of the kinetic moment balance equation for the RC.

$$((\mathbf{E} + \mathbf{I})^T \cdot \mathbf{T})_x = \rho_0 (\mathbf{J}_0 \cdot \dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{J}_0 \cdot \mathbf{\Omega}) . \tag{44}$$

The equation (44) is a second motion equation for the RC.

Equations (38) and (44) are the first two equations in our new system describing RC.

Now let us obtain compatibility equations for the RC.

For the RC basis does not depend on time. So after using the opportunity to reshuffle $\partial / \partial t$ and $\partial / \partial x$ we get

$$\dot{\mathbf{F}}^T = \overset{\circ}{\nabla} \mathbf{v} . \tag{45}$$

We shall differentiate with respect to time equation (31).

As we know $\dot{\mathbf{I}} = 0$. Using equations (27), (29), (45) we obtain

$$\dot{\mathbf{E}} = \overset{\circ}{\nabla} \cdot \mathbf{P} + \mathbf{F}^T \cdot \mathbf{P} \times \mathbf{\Omega} . \tag{46}$$

Now let us apply relation (27), (34) for the first term in the right hand side of an equation (46).

$$\overset{\circ}{\nabla} \cdot \mathbf{P} = \overset{\circ}{\nabla} (\mathbf{v} \cdot \mathbf{P}) - \overset{\circ}{\nabla} \mathbf{P}^T \cdot \mathbf{v} = \overset{\circ}{\nabla} \mathbf{V} + \mathbf{K} \times \mathbf{P}^T \cdot \mathbf{v} = = \overset{\circ}{\nabla} \mathbf{V} + \mathbf{K} \times \mathbf{V} \tag{47}$$

Further let us apply relation (31) for a second term in the right hand side of an equation (46). So we get

$$\dot{\mathbf{E}} = \overset{\circ}{\nabla} \mathbf{V} + \mathbf{K} \times \mathbf{V} + (\mathbf{E} + \mathbf{I}) \times \mathbf{\Omega} . \tag{48}$$

The equation (48) is the first compatibility equation for the RC. And it is number 3 in our new system of equations.

Now let us obtain equation relating **K** and **Ω**. We shall transpose both sides of the equation (29). We arrive at

$$\overset{\circ}{\nabla} \dot{\mathbf{P}}^T = -\overset{\circ}{\nabla} \mathbf{\Omega} \times \mathbf{P}^T - \mathbf{R}^s \mathbf{\Omega} \times \frac{\partial \mathbf{P}^T}{\partial x^s} = = -\overset{\circ}{\nabla} \mathbf{\Omega} \times \mathbf{P}^T + \mathbf{R}^s \mathbf{\Omega} \times (\mathbf{K}_s \times \mathbf{P}^T) \tag{49}$$

Now let us transform a second term in the right hand side of an equation (49). The latter combined with $\mathbf{a} \times (\mathbf{b} \times \mathbf{A}) = \mathbf{b} \times (\mathbf{a} \times \mathbf{A}) + (\mathbf{a} \times \mathbf{b}) \times \mathbf{A}$, which is valid for any **a**, **b**, **A** [27] results

$$\overset{\circ}{\nabla} \dot{\mathbf{P}}^T = -\overset{\circ}{\nabla} \mathbf{\Omega} \times \mathbf{P}^T + \mathbf{K} \times (\mathbf{\Omega} \times \mathbf{P}^T) - (\mathbf{K} \times \mathbf{\Omega}) \times \mathbf{P}^T \tag{50}$$

Than let us differentiate with respect to time expression (33), so we get

$$(\overset{\circ}{\nabla} \mathbf{P}^T)^{\cdot} = -\dot{\mathbf{K}} \times \mathbf{P}^T + \mathbf{K} \times (\mathbf{\Omega} \times \mathbf{P}^T) \tag{51}$$

For the RC basis does not depend on time, which leads to $(\overset{\circ}{\nabla} \mathbf{P}^T)^{\cdot} = \overset{\circ}{\nabla} (\mathbf{P}^T)^{\cdot}$.

As we can see the left hand side of an equation (50) is equal to the left hand side of an equation (51). We obtain the following relation

$\dot{\mathbf{K}} \times \mathbf{P}^T = (\overset{\circ}{\nabla} \mathbf{\Omega} + \mathbf{K} \times \mathbf{\Omega}) \times \mathbf{P}^T$. Hence

$$\dot{\mathbf{K}} = \overset{\circ}{\nabla} \mathbf{\Omega} + \mathbf{K} \times \mathbf{\Omega} . \tag{52}$$

The equation (52) is the second compatibility equation for the RC. And it is number 4 in our new system of equations.

Further let us return to an energy balance equation (3). For RC it can be expressed as a following formula:

$$\frac{d}{dt} \int_{V_0} (\frac{1}{2} \rho_0 \mathbf{V}^2 + \frac{1}{2} \mathbf{\Omega} \cdot \rho_0 \mathbf{J}_0 \cdot \mathbf{\Omega} + \rho_0 \Pi) dV_0 = \int_{S_0} \mathbf{N} \cdot \mathbf{T} \cdot \mathbf{V} dS_0 + Q \tag{53}$$

Thermal power is $Q = \int_{V_0} (\rho_0 q - \overset{0}{\nabla} \cdot \mathbf{H}) dV_0$ for the RC, where **H** is a heat flux vector for RC and $\mathbf{H} = \mathbf{J} \mathbf{F}^{-1} \cdot \mathbf{h}$.

Applying a techniques describing above for CC we can get

$$\rho_0 \dot{\Pi} = (\overset{0}{\nabla} \cdot \mathbf{T} - \rho_0 \mathbf{V}) \cdot \mathbf{V} + \mathbf{T}^T \cdot\cdot \overset{0}{\nabla} \mathbf{V} - \rho_0 (\mathbf{J}_0 \cdot \dot{\Omega}) \cdot \mathbf{\Omega} + \rho_0 q - \overset{0}{\nabla} \cdot \mathbf{H}$$

and than

$$\rho_0 \dot{\Pi} = \mathbf{T}^T \cdot\cdot (\mathbf{K} \times \mathbf{V} + \overset{\mathbf{0}}{\nabla} \mathbf{V} + (\mathbf{E} + \mathbf{I}) \times \mathbf{\Omega}) + \rho_0 q - \overset{0}{\nabla} \cdot \mathbf{H} . \tag{54}$$

An expression in brackets in the right hand side of an equation (54) is $\dot{\mathbf{E}}$. So we can easily get

$$\rho_0 \dot{\Pi} = \mathbf{T}^T \cdot\cdot \dot{\mathbf{E}} + \rho_0 q - \overset{0}{\nabla} \cdot \mathbf{H}. \quad (55)$$

The second law of thermodynamics for RC can be expressed as

$$\rho_0 \theta \dot{\eta} - (\rho_0 q - \overset{0}{\nabla} \cdot \mathbf{H}) - \mathbf{H} \cdot \overset{0}{\nabla} \ln \theta \geq 0. \quad (56)$$

Let us substitute (55) to (56) and use $f = f(\mathbf{E}, \theta)$ - the Helmholtz free-energy function to get

$$\mathbf{T}^T \cdot\cdot \dot{\mathbf{E}} - \rho_0 (\dot{f} + \dot{\theta} \eta) - \mathbf{H} \cdot \overset{0}{\nabla} \ln \theta \geq 0 \quad (57)$$

The latter combined with $\dot{f} = \dfrac{\partial f}{\partial \mathbf{E}}^T \cdot\cdot \dot{\mathbf{E}} + \dfrac{\partial f}{\partial \theta} \dot{\theta}$ results

$$(\mathbf{T} - \rho_0 \frac{\partial f}{\partial \mathbf{E}})^T \cdot\cdot \dot{\mathbf{E}} - \rho_0 (\frac{\partial f}{\partial \theta} + \eta) \dot{\theta} - \mathbf{H} \cdot \overset{0}{\nabla} \ln \theta \geq 0. \quad (58)$$

As we say earlier equation (58) holds only if

$$\mathbf{T} - \rho_0 \frac{\partial f}{\partial \mathbf{E}} = 0, \; \rho_0 (\frac{\partial f}{\partial \theta} + \eta) = 0, \; -\mathbf{H} \cdot \overset{0}{\nabla} \ln \theta \geq 0,$$

which leads us to the following result

$$\mathbf{T} = \rho_0 \frac{\partial f}{\partial \mathbf{E}}. \quad (59)$$

One can easily recognize that an equation (20) can be expressed as

$$\frac{\partial \Pi}{\partial \mathbf{E}} = \eta \frac{\partial \theta}{\partial \mathbf{E}} + \frac{\partial f}{\partial \mathbf{E}} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial \mathbf{E}} = \frac{\partial f}{\partial \mathbf{E}} + \eta (\frac{\partial \theta}{\partial \mathbf{E}} - \frac{\partial \theta}{\partial \mathbf{E}}) = \frac{\partial f}{\partial \mathbf{E}} \quad (60)$$

for RC.

Using this expression (60) and the equation (59) arrive at

$$\mathbf{T} = \rho_0 \frac{\partial \Pi}{\partial \mathbf{E}}. \quad (61)$$

The expression (61) is the constitutive equation for the RC. Now let us introduce unit energy dissipation for the RC

$$\varphi = \mathbf{T}^T \cdot\cdot \mathbf{E} - \rho_0 (\dot{f} + \dot{\theta} \eta). \quad (62)$$

In the elastic medium $\varphi = 0$, thus the first law of thermodynamics for the RC has a following simple form

$$\rho_0 \theta \dot{\eta} = \rho_0 q - \overset{0}{\nabla} \cdot \mathbf{H}. \quad (63)$$

Applying a techniques describing for the CC we can get

$$\rho_0 \chi \dot{\theta} = \rho_0 q - \overset{0}{\nabla} \cdot \mathbf{H} + \theta \frac{\partial \mathbf{T}}{\partial \theta} \cdot\cdot \dot{\mathbf{E}}^T. \quad (64)$$

Since we consider isotropic media $\mathbf{H} = -k \mathbf{G}^{-1} \cdot \overset{0}{\nabla} \theta$, where $\mathbf{G} = \mathbf{J}^{-1} (\mathbf{E}\mathbf{E}^T + \mathbf{E}^T + \mathbf{E} + \mathbf{I})$.

Now we arrive to the following equation

$$\rho_0 \chi \dot{\theta} = \rho_0 q + \overset{0}{\nabla} \cdot k \mathbf{G}^{-1} \cdot \overset{0}{\nabla} \theta + \theta \frac{\partial \mathbf{T}}{\partial \theta} \cdot\cdot \dot{\mathbf{E}}^T. \quad (65)$$

The last equation is a heat conductivity equation for the RC and it is the sixth equation in our system.

IV. Conclusion

Here we gather all the equation in our system.

For the CC:

motion equations

$$\nabla \cdot \boldsymbol{\tau} = \rho \dot{\mathbf{v}},$$

$$\boldsymbol{\tau}_x = \rho (\boldsymbol{\omega} \times \mathbf{J} \cdot \boldsymbol{\omega} + \mathbf{J} \cdot \dot{\boldsymbol{\omega}}),$$

the compatibility equation

$$\dot{\mathbf{e}} + \mathbf{e} \times \boldsymbol{\omega} + \nabla \mathbf{v} \cdot \mathbf{e} = \nabla \mathbf{v} + \mathbf{I} \times \boldsymbol{\omega},$$

the constitutive equation

$$\boldsymbol{\tau} = \rho \frac{\partial \Pi}{\partial \mathbf{e}},$$

the heat conductivity equation

$$\rho \chi \dot{\theta} = \rho q + \nabla \cdot k \nabla \theta + \theta \frac{\partial \boldsymbol{\tau}}{\partial \theta} \cdot\cdot (D\mathbf{e})^T,$$

the mass conservation law

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0.$$

The system for the CC consists of 26 equation and depends on the following unknown functions: 9 stresses $\boldsymbol{\tau}$, 9 strains $\mathbf{e}$, 6 velocities $\mathbf{v}$, $\boldsymbol{\omega}$, temperature θ and density ρ. As a result we have 26 unknown functions. The problem becomes fully set after adding the boundary and initial conditions.

For the RC:

motion equations

$$\overset{\circ}{\nabla} \cdot \mathbf{T} + (\mathbf{K}^T \cdot \mathbf{T})_x = \rho_0 (\dot{\mathbf{V}} + \boldsymbol{\Omega} \times \mathbf{V}),$$

$$((\mathbf{E} + \mathbf{I})^T \cdot \mathbf{T})_x = \rho_0 (\mathbf{J}_0 \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{J}_0 \cdot \boldsymbol{\Omega}),$$

the compatibility equation

$$\dot{\mathbf{E}} = \overset{\circ}{\nabla} \mathbf{V} + \mathbf{K} \times \mathbf{V} + (\mathbf{E} + \mathbf{I}) \times \boldsymbol{\Omega},$$

$$\dot{\mathbf{K}} = \overset{\circ}{\nabla} \boldsymbol{\Omega} + \mathbf{K} \times \boldsymbol{\Omega},$$

the constitutive equation

$$\mathbf{T} = \rho_0 \frac{\partial \Pi}{\partial \mathbf{E}},$$

the heat conductivity equation

$$\rho_0 \chi \dot{\theta} = \rho_0 q + \overset{0}{\nabla} \cdot k \mathbf{G}^{-1} \cdot \overset{0}{\nabla} \theta + \theta \frac{\partial \mathbf{T}}{\partial \theta} \cdot\cdot \dot{\mathbf{E}}^T.$$

The system for the RC consists of 34 equation and depends on the following unknown functions: 9 stresses **T**, 9 strains **E**, 6 velocities **V**, $\boldsymbol{\Omega}$, temperature θ and 9 components of the additional tensor **K**. As a result we have 34 unknown functions.

The main advantage of our work is that our description as for the CC as for the RC does not contain kinematic unknown r, P as well as strain gradient F. Unknowns r, P can be found by integrating equations $\mathbf{v} = \dot{\mathbf{r}}$, $\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}$ after solving the system of equation.

References

[1] Ashenai Ghasemi, F.,Shokuhfar, A., Khalili, S.M.R., Payganeh, G.H.,Malekzadeh, K. “The effect of some physical and geometrical parameters on improvement of the impact response of smart composite structures” in *International Journal of Mathematical Models and Methods in Applied Sciences*, V. 3, Issue 3, 2009. Pp 266-274

[2] Schwartz L.M., Johnson D.L., Feng S. "Vibrational modes in granular materials" in *Physical review letters*, v. 52, №10, 1984. pp.831-834.

[3] Neff P., Jeong J. "A new paradigm: the linear isotropic Cosserat model with conformally invariant curvature energy" in *ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik*. vol.89, №2, February 2009, pp 107-122.

[4] Jasiuk I., Ostoja-Starzewski M. "On the reduction of constants in planar cosserat elasticity with eigenstrains and eigencurvatures" in *Journal of Thermal Stresses*. 26 (11-12), 2003. pp.1221-1228.

[5] Khomasuridze N.G. "The symmetry principle in continuum mechanics" in *Journal of Applied Mathematics and Mechanics*. 71 (1), 2007. pp. 20-29.

[6] Arslan H., Sture S. "Finite element simulation of localization in granular materials by micropolar continuum approach" in *Computers and Geotechnics.* 35 (4), 2008. pp. 548-562.

[7] Jeong J., Neff P. "Existence, uniqueness and stability in linear cosserat elasticity for weakest curvature conditions" in *Mathematics and Mechanics of Solids.* 15 (1), 2010. pp. 78-95.

[8] Ieşan D. "Deformation of porous Cosserat elastic bars" in *International Journal of Solids and Structures.* 48 (3-4), 2011. pp. 573-583.

[9] Badanin A., Bugrov A., Krotov A. The determination of the first critical load on particulate medium of sandy loam foundation // Magazine of Civil Engineering. 9. 2012. Pp 29-34. (in Russian).

[10] Harris D. "Double-slip and Spin: A generalisation of the plastic potential model in the mechanics of granular materials" in *International Journal of Engineering Science*. Volume 47, Issues 11–12, 2009. pp.208-1215.

[11] Heinrich M. Jaeger, Sidney R. Nagel. "Granular solids, liquids, and gases*" Reviews of modern physics*. vol.68, №4, 1996. pp 1259-1273.

[12] Prosvetov, V.I., Sumets, P.P., Verveyko, N.D. "Modeling of flow of medium with homogeneous microstructure" in *International Journal of Mathematical Models and Methods in Applied Sciences*. V.5 (3),2011. pp. 508-516

[13] Kurbatskiy E.N., Golosova O.A." Features of the propagation of stress waves in natural and artificial granular media" in *Structural mechanics and calculation of structures.* №2, 2011. pp. 45-50. (in rus).

[14] Grekova E.F., Herman G.C." Wave propagation in rocks modelled as reduced Cosserat continuum" in *proceedings of 66th EAGE conference*, Paris, 2004.

[15] Grekova E.F., Kulesh M.A., Herman G.C. "Waves in linear elastic media with microrotations, part 2: Isotropic reduced Cosserat model" in *Bulletin of the Seismological Society of America*. 99 (2 B), 2009. pp. 1423-1428.

[16] Harris D. *Douple-slip and Spin: Dilatant Shear in a Reduced Cosserat Model. Modern Trends in Geomechanics*. Springer, 2006.

[17] Grekova E.F." Linear reduced cosserat medium with spherical tensor of inertia, where rotations are not observed in experiment" in *Mechanics of solids*. vol 47, issue 5, 2012. pp538-543.

[18] Grekova E.F."Nonlinear isotropic elastic reduced Cosserat continuum as a possible model for geomedium and geomaterials. Spherical prestressed state in the semilinear material" in *Journal of seismology*. vol 16, issue 4, 2012. pp695-707.

[19] Lalin V., Zdanchuk E. "On the Cauchy problem for nonlinear reduced Cosserat continuum" in *Proceedings of the XXXIX Summer School – Conference ADVANCED PROBLEMS IN MECHANICS*, Saint Petersburg. 2011.

[20] Lalin V., Zdanchuk E. "A model of continuous granular medium. Waves in the reduced Cosserat continuum" in *Magazine of Civil Engineering*. 5(31). 2012. Pp65-71. (in Russian).

[21] Lalin V., Zdanchuk E. "Nonlinear thermodynamic model for granular medium" *Recent Advances in Mechanical Engineering and Mechanics.Proceedings of the 2014 International Conference on Theoretical Mechanics and Applied Mechanics* , Venice, Italy. 2014. Pp32-35.

[22] Maalouf, C., Tran Le, A.D.,Lachi, M., Wurtz, E., Mai, T.H. "Effect of moisture transfer on thermal inertia in simple layer walls: Case of a vegetal fibre material" in *International Journal of Mathematical Models and Methods in Applied Sciences*. V.5 (1),2011. pp. 33-47

[23] Makukula, Z.G., Sibanda, P.,Motsa, S.S. "On new solutions for heat transfer in a visco-elastic fluid between parallel plates" in *International Journal of Mathematical Models and Methods in Applied Sciences*. V.4 (4),2010. pp. 221-230

[24] Bachok, N., Ishak, A., Nazar, R. "Flow and heat transfer over an unsteady stretching sheet in a micropolar fluid with prescribed surface heat flux" in *International Journal of Mathematical Models and Methods in Applied Sciences*. V.4 (3),2010. pp. 167-176

[25] Lurie A.I. *Nonlinear theory of elasticity*. Amsterdam. 1990.

[26] Lalin V. "Non-linear dynamics equations of the moment elastic medium" in *Scientific and Technical Sheets SPbGPU*. 2007. (in Russian).

[27] Zhilin P.A. "A new approach to the analysis of free rotations of rigid bodies" in *ZAMM-Z. angew. Math.Mech*, 76, №4, 1996. pp187-204.