On the solvability of the L^p and BMO Dirichlet problem for elliptic operators

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***Abstract*—We study the BMO and the L^p solvability of the Dirichlet problem for a second order divergence form elliptic operator with bounded measurable coefficients in a Lipschitz domain. We obtain a relation between the *BMO-constant* of the operator (see Definition 6) and the solvability exponents p.**

Keywords and Phrases - Bounded mean oscillation, Dirichlet problem, Elliptic measure.

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I. Introduction

Let $\Omega \subset \mathbb{R}^n$ denote a Lipschitz domain. For $K \geqslant 1$ we consider the class $\mathcal{E}(K)$ of measurable (not necessarily symmetric) matrix fields $A(x) \in L^\infty(\Omega)$ such that

$$\frac{|\xi|^2}{K} \leqslant \langle A(x)\xi, \xi\rangle \leqslant K\,|\xi|^2 \tag{1}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$.

We examine the classical Dirichlet boundary value problem:

$$\begin{cases} Lu = 0 \quad \text{in } \Omega \\ u_{|\partial\Omega} = f \in C(\partial\Omega) \end{cases} \tag{2}$$

where

$$L = \operatorname{div}(A(x)\nabla\,) \tag{3}$$

is an elliptic operator whose coefficient matrix $A(x)$ belongs to $\mathcal{E}(K)$. (See [1],[2] and [10] for some applications).

For $1 < p < \infty$, the problem (2) is called L^p- *solvable* and the operator (3) is said L^p*-resolutive*, if there exists a constant $C_p > 0$ for which the following holds: For any $f \in C(\partial\Omega)$ the unique solution $u \in W^{1,2}_{loc}(\Omega) \cap C(\bar\Omega)$ to (2) satisfies the uniform estimate

$$\|Nu\|_{L^p(\partial\Omega)} \leqslant C\,\|f\|_{L^p(\partial\Omega)}\,,$$

where Nu is the nontangential maximal function,

$$Nu(x) = \sup_{y \in \mathcal{G}(x)} |u(y)|$$

(here $\mathcal{G}(x)$ is a truncated cone with vertex at x) and where C depends only on the Lipschitz character of Ω and the ellipticity of L.

In order to state a necessary and sufficient condition that problem (2) is L^p- solvable we shall recall a key notion of the theory, namely the "elliptic measure". To this effect we assume that Ω contains the origin of $\mathbb{R}^n$ and we consider the linear functional

$$f \in C(\partial\Omega) \longrightarrow u(0)$$

where $u \in W^{1,2}_{loc}(\Omega) \cap C(\overline{\Omega})$ is the unique solution of problem (2). Then, there is a unique Borel regular probability measure ω_L on $\partial\Omega$ such that

$$u(0) = \int_{\partial\Omega} f(\sigma)\, d\omega_L(\sigma),$$

Such ω_L is called "elliptic measure" associated with L (see [14]).

Definition I.1. *We say that the measure ω supported on $\partial\Omega$ belongs to the Gehring class B_q, $1 < q < \infty$, if ω is absolutely continuous with respect to the surface measure σ on $\partial\Omega$, and the Radon-Nikodym derivative $w = \frac{d\omega}{d\sigma}$ verifies the "reverse Hölder inequality"*

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta w^q d\sigma\right)^{\frac{1}{q}} \leqslant \frac{B}{\sigma(\Delta)}\int_\Delta w\, d\sigma \tag{4}$$

with a certain constant $B \geqslant 1$ and for all surface balls $\Delta \subset \partial\Omega$.

Theorem I.1. [14] *The following conditions are equivalent ($\frac{1}{p} + \frac{1}{q} = 1$): i) problem (2) is L^p- solvable; ii) the elliptic measure ω_L of the operator L belongs to the Gehring class B_q.*

We refer the reader to the papers [5], [6], [3], [8] and to [12] for more details.

To define the BMO-solvability for L as in [7], we shall now introduce some notations.

For every $x \in \partial\Omega$ we set $B_r(x) = \{y : |y - x| \leqslant r\}$, $\Delta_r(x) = B_r(x) \cap \partial\Omega$, and we denote by $T(\Delta_r) = \Omega \cap B_r(x)$ the Carleson region above $\Delta_r(x)$.

A measure μ in Ω is Carleson if there exist $r_0 > 0$ and $C > 0$ such that for all $r \leqslant r_0$,

$$\mu(T(\Delta_r)) \leqslant C\sigma(\Delta_r).$$

For such measure μ we denote by $\|\mu\|_{Car}$ the quantity

$$\|\mu\|_{Car} = \sup_{\Delta \subset \partial\Omega} \left(\sigma(\Delta)^{-1}\mu(T(\Delta))\right)^{\frac{1}{2}}$$

We say that a function $f : \partial\Omega \to \mathbb{R}$ belongs to BMO (*Bounded Mean Oscillation*) with respect to the surface measure $d\sigma$ if

$$\sup_{I \subset \partial\Omega} \sigma(I)^{-1}\int_I |f - f_I|^2 d\sigma < \infty.$$

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Here $f_I = \sigma(I)^{-1}\int_I f d\sigma$. We denote by $\|f\|_{BMO(p)}$ the number

$$\|f\|_{BMO(p)} = \sup_{I\subset\partial\Omega}\left(\sigma(I)^{-1}\int_I |f-f_I|^p d\sigma\right)^{\frac{1}{p}}.$$

It can be shown that, for any $1 \leqslant p < \infty$, $\|f\|_{BMO(2)} < \infty$ if and only if $\|f\|_{BMO(p)} < \infty$. Moreover, $\|\cdot\|_{BMO(p)}$ and $\|\cdot\|_{BMO(2)}$ are equivalent in the sense that there is a constant $C \geqslant 1$ such that the inequality

$$C^{-1}\|f\|_{BMO(p)} \leqslant \|f\|_{BMO(2)} \leqslant C\|f\|_{BMO(p)}$$

holds for any BMO function f.

Definition I.2. *The Dirichlet problem* (2) *is called BMO-solvable for L (and the operator L is said BMO-resolutive) if the solution u for continuous boundary data f satisfies*

$$\||\nabla u|^2\delta(x)dx\|_{Car} \preceq \|f\|_{BMO(2)}.$$

Here $\delta(x)$ =dist$(x,\partial\Omega)$. Equivalently, there exists a constant C such that for all continuous f,

$$\begin{aligned}&\sup_{\Delta\subset\partial\Omega}\sigma(\Delta)^{-1}\int\int_{T(\Delta)}|\nabla u|^2\delta(x)dx\\ &\leqslant C\sup_{I\subset\partial\Omega}\sigma(I)^{-1}\int_I|f-f_I|^2d\sigma.\end{aligned}\tag{5}$$

Note that even though one defines BMO-solvability only for continuous boundary data, the solution u can be defined for any BMO function $f:\partial\Omega\to\mathbb{R}$ and moreover the estimate will hold. In addition, such a solution u will have a well-defined nontangential maximal function $Nu(x)$ for almost every point $x\in\partial\Omega$ and in the nontangential sense

$$f(x) = \lim_{y\to x,y\in\mathcal{G}(x)} u(y), \text{for a.e.} x\in\partial\Omega$$

(see [7]).

We will call BMO-constant of the operator L the quantity

$$\textit{BMO(L)} = \sup\left(\frac{\||\nabla u|^2\delta(x)dx\|_{Car}}{\|f\|_{BMO(2)}}\right)^2.\tag{6}$$

Moreover we will denote by $D_2(\sigma)$ the doubling constant of the surface measure σ on Ω, and precisely

$$D_2(\sigma) = \sup_{\Delta}\frac{\sigma(\Delta(x,2r))}{\sigma(\Delta(x,r))}.$$

An easy computation shows that, for example, if $\Omega=\mathbb{D}$, the unit disc of $\mathbb{R}^2$, then $D_2(\sigma)=3$ and in case $\Omega=\mathbb{B}$ the unit sphere of $\mathbb{R}^3$, then $D_2(\sigma)=4$.

In [7] the following result is obtained (see [7], Theorem 2.2).

Theorem I.2 ([7]). *Assume that L is BMO-resolutive. Then there exists $p_0>1$ such that the L^p Dirichlet problem for L is solvable for all $p_0<p<\infty$.*

Last result is obtained by the authors by proving that an operator L in our class $\mathcal{E}(K)$ is BMO-solvable if and only if the elliptic measure ω_L belongs to the Muckenhoupt class $A_\infty = \cup B_q$ with respect to the surface measure σ on the boundary of the domain of solvability Ω. And when the density of elliptic measure with respect to σ belongs to some B_{q_0}, using Theorem I.1, it turns out that the Dirichlet problem is L^{p_0} solvable where $1/q_0+1/p_0=1$. The range of solvability (p_0,∞) can be then obtained by observing that $B_{q_0}\subset B_q$ for $q<q_0$.

In this note our aim is to give an upper bound for such exponent p_0 in terms of the BMO constant of L appearing in (6). In particular our main result is the following

Theorem I.3. *Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz domain and let L be a divergence form elliptic operator with bounded coefficiens, satisfying the strong ellipticity condition. Assume the operator L be BMO-resolutive. Then L is L^p resolutive, for all $p\geqslant 1+\rho_0$ where*

$$\rho_0 = C\cdot D_2(\omega_L)^2\cdot BMO(L) + e\cdot\log D_2(\sigma).\tag{7}$$

Here $C=C(n)$, $D_2(\omega_L)$ is the doubling constant of the elliptic measure ω_L and $BMO(L)$ is the BMO constant of L defined in (6).

We note explicitely that (7) gives a control for the greatest lower bound of the solvability exponents p which is linear with respect to the BMO-costant of L.

It is worth to point out that a similar result can be easily obtained in the context of the Orlicz boundary data using Theorem 4.4 in [19] (see also [21] and [20]).

Finally, using [18] Theorem 1.3, we shall prove a simultaneous BMO-solvability result for two different operators without assumption on the distance of the operator's coefficients near the boundary (see Section IV).

II. Preliminaries

In this section we recall some results about the real variable theory of weights, which will be useful in the sequel (see [5], [6], [3], [8], [12]).

Definition II.1. *Let ν be a finite measure on $\partial\Omega$. Then ν belongs to $A_\infty(d\sigma)$ if for all $\varepsilon>0$ there exists an $\eta>0$ such that for every surface ball Δ and subset $E\subset\Delta$, whenever $\frac{\sigma(E)}{\sigma(\Delta)}<\eta$ then $\frac{\nu(E)}{\nu(\Delta)}<\varepsilon$.*

Theorem II.1. *Assume that the measure ν supported on $\partial\Omega$ belongs to A_∞. Then there exist constants $0<\beta\leqslant 1\leqslant H<\infty$ so that*

$$\frac{\nu(E)}{\nu(\Delta)}\leqslant H\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^\beta,\tag{8}$$

for any surface ball $\Delta\subset\partial\Omega$ and any measurable set $E\subset\Delta$.

It is well known that A_∞ is the union of Gehring classes B_q:

$$A_\infty = \cup_{q>1}B_q$$

Definition II.2. *For any A_∞ measure ν on $\partial\Omega$ we define*

$$\tilde{B}_1(\nu) = \inf\left\{\frac{H}{\beta} : 0<\beta\leqslant 1\leqslant H \textit{ and condition (8) holds}\right\}.\tag{9}$$

If we switch the role of the measures σ and ν on $\partial\Omega$ in (8) are preserved the properties of the weights supported (see [4])

Theorem II.2. *The measure ν supported on $\partial\Omega$ belongs to A_∞ with respect to σ if and only if there exist constants $0 < \alpha \leqslant 1 \leqslant M$ such that*

$$\frac{\sigma(F)}{\sigma(\Lambda)} \leqslant M \left(\frac{\nu(F)}{\nu(\Lambda)}\right)^\alpha, \tag{10}$$

for any surface ball $\Lambda \subset \partial\Omega$ and for any measurable set $F \subset \Lambda$.

It is therefore natural to associate to weight ν a constant defined as

$$\tilde{A}_\infty(\nu) = \inf\left\{\frac{M}{\alpha} : 0 < \alpha \leqslant 1 \leqslant M \text{ and (10) holds}\right\}. \tag{11}$$

We emphasize explicitly that a measure ν belongs to A_∞ if and only if $\tilde{A}_\infty(\nu) < \infty$ or, equivalently, $\tilde{B}_1(\nu) < \infty$. That is why we will call (9) and (11) A_∞- constants of ν. For example, in dimension $n = 2$, if ω is defined by $\frac{d\omega}{d\sigma} = \sigma^\alpha$ with $\alpha \in (-1, 0]$, then $\omega \in A_\infty$ and $\tilde{B}_1(\omega) = \frac{1}{\alpha+1}$.

III. Proof of Theorem I.3

A main tool in our proof will be the following result:

Theorem III.1 ([7]). *Let L be BMO resolutive. Then the ellipic measure ω_L belongs to A_∞. In particular, for any $\varepsilon > 0$, assuming $\eta = e^{-\frac{Cd^2\bar{C}}{\varepsilon}}$, for all surface ball $\Delta \subset \partial\Omega$ and for all measurable subset $E \subset \Delta$,*

$$\frac{\sigma(E)}{\sigma(\Delta)} < \eta \Longrightarrow \frac{\omega_L(E)}{\omega_L(\Delta)} < \varepsilon \tag{12}$$

Here $C = C(n)$, $d = D_2(\omega_L)$ is the doubling constant of ω_L and $\bar{C} = BMO(L)$ is the BMO-constant of L.

Proof. The thesis can be obtained by following line by line the proof of Theorem 2.1 in [7]. For the convenience of the reader we give here the details. Let Δ be a surface ball on the boundary of Ω and assume f be a positive and continuous function supported on Δ. By the assumptions and also by using [13], one can see that if $Lu = 0$ and $u = f$ on the boundary then, for some constant $C_0 = C_0(n, \Omega)$,

$$\omega_L(\Delta)^{-1}\int_\Delta f d\omega_L \leqslant C_0\bar{C}\|f\|_{BMO}. \tag{13}$$

Suppose that $\sigma(\Delta) = r$ and let $\varepsilon > 0$. Let $E \subset \Delta$ be an open set. We shall find $\eta > 0$ such that $\sigma(E)/\sigma(\Delta) < \eta$ implies $\omega_L(E)/\omega_L(\Delta) < \varepsilon$. To this aim let $\chi(E)$ be the characteristic function of E. Define the BMO function

$$f = max\{0, 1 + \delta \log M(\chi(E))\},$$

where $M(\chi(E))$ denotes the Hardy-Littlewood maximal function of $\chi(E)$ with respect to surface measure on the boundary of Ω, i.e.

$$M(\chi_E)(x) = \sup_{\Delta\ni x} \fint_\Delta \chi_E d\sigma$$

and where δ is to be determined. The function f verifies the following properties:

i) $f \geqslant 0$,
ii) $\|f\|_{BMO} \leqslant \delta$,
iii) $f = 1$ on E.

Observe that if $x \notin 2\Delta$, then

$$M(\chi_E)(x) < \frac{\sigma(E)}{\sigma(\Delta)} < \eta.$$

Then, for any δ, if one choose η sufficiently small, then $1 + \delta \log M(\chi_E)(x) \leqslant 0$ so that $f = 0$ outside 2Δ. Assume

$$\eta = e^{-\frac{1}{\delta}}. \tag{14}$$

Using a standard mollification process, one can find a family of continuous functions, f_t, $t > 0$ verifying:

- $f_t \to f$ in L^p, as $t \to 0$
- $\forall t$, $\exists C_1 : \|f_t\|_{BMO} \leqslant C_1\|f\|_{BMO}$,
- $\mathrm{supp} f_t \subseteq 3\Delta$.

Now, since $f \geqslant 1$ on E, by (13) we have

$$\begin{aligned}
\frac{\omega_L(E)}{\omega_L(3\Delta)} &\leqslant \frac{1}{\omega_L(3\Delta)}\int_E f d\omega_L \\
&\leqslant \frac{1}{\omega_L(3\Delta)}\lim_{t\to 0^+}\int_{3\Delta} f_t d\omega_L \\
&\leqslant \bar{C}C_1 \limsup_{t\to 0^+}\|f_t\|_{BMO} \leqslant \bar{C}C_1C_2\|f\|_{BMO}.
\end{aligned}$$

Now we choose

$$\delta = \frac{\varepsilon}{2\bar{C}C_1C_2},$$

so that by last inequality we find

$$\frac{\omega_L(E)}{\omega_L(\Delta)} < \frac{\varepsilon}{2d^2}$$

where $d = D_2(\omega_L)$ is the doubling constant of ω_L. At this point the thesis easily follows.

□

Proof of Theorem I.3. Using Theorem III.1, the statement of Theorem I.3 follows by using a well known argument (see for example [9]). For the convenience of the reader we give here some details.

Assume that L is BMO resolutive. Then, by Theorem III.1, for any $\varepsilon > 0$, assuming $\eta = e^{-\frac{Cd^2\bar{C}}{\varepsilon}}$, for all surface ball $\Delta \subset \partial\Omega$ and for all measurable subset $E \subset \Delta$,

$$\frac{\sigma(E)}{\sigma(\Delta)} < \eta \Longrightarrow \frac{\omega_L(E)}{\omega_L(\Delta)} < \varepsilon \tag{15}$$

Moreover, since $\omega_L << \sigma$ we consider the Radon-Nikodym derivative $w = d\omega_L/d\sigma$. To obtain the thesis of Theorem I.3, we shall prove that for $\varepsilon \in (0, 1)$ one can determine $h = h(\varepsilon, d, \bar{C}, n, \Omega)$ such that $\omega_L \in B_{1+h}(d\sigma)$. To this aim we will use a classical argument due to Coiffman and Feffermann [4] and Muckenhoupt [15]. Let us fix $0 < \varepsilon < 1$ and consider the $\eta \in (0, 1)$ associated to ε according to (15). Let $\Delta \subset \partial\Omega$ be a surface ball. We take an increasing sequence $\lambda_0 < \lambda_1 < ... < \lambda_k < ...$ with $\lambda_0 = \fint_\Delta w d\sigma$ and, for any $k \in \mathbb{N}$, $\lambda_k =$

$\lambda_0 \left(\frac{S}{\eta}\right)^k$, where $S = D_2(\sigma)$ is the doubling constant of the surface measure σ on Ω.

Now we make the Calderòn-Zygmund decomposition of Δ for the function w and the value λ_0, that is we consider a family $\Delta_{0,j}$ of disjoint surface ball satisfying

$$\begin{aligned} &\lambda_0 < \fint_{\Delta_{0,j}} w d\sigma \leqslant S\lambda_0 \\ &w(x) \leqslant \lambda_0 \qquad a.e. x \notin \cup_{j\in\mathbb{N}} \Delta_{0,j} =: D_0. \end{aligned} \tag{16}$$

Then, we make the Calderòn-Zygmund decomposition of any $\Delta_{0,j}$ for the function w and the value λ_1. In this way we obtain a family $\Delta_{1,j}$ of disjoint surface ball satisfying

$$\begin{aligned} &\lambda_1 < \fint_{\Delta_{1,j}} w d\sigma \leqslant S\lambda_1 \\ &w(x) \leqslant \lambda_1 \qquad a.e. x \notin \cup_{j\in\mathbb{N}} \Delta_{1,j} =: D_1, \end{aligned} \tag{17}$$

and so on. In this way we obtain a family $\Delta_{k,j}$ of surface balls such that

$$\begin{aligned} &\forall k, \ \{\Delta_{k,j}\}_{j\in\mathbb{N}} \text{ is a disjoint family} \\ &\lambda_k < \fint_{\Delta_{k,j}} w d\sigma \leqslant S\lambda_k \\ &w(x) \leqslant \lambda_k \qquad a.e. x \notin \cup_{j\in\mathbb{N}} \Delta_{k,j} =: D_k, \end{aligned} \tag{18}$$

Moreover, since each $\Delta_{k+1,j}$ is contained in $\Delta_{k,i}$ for some i, than $D_{k+1} \subset D_k$.

We have

$$\begin{aligned} S\lambda_k &\geqslant \frac{1}{\sigma(\Delta_{k,i})} \int_{\Delta_{k,i}\cap D_{k+1}} w d\sigma \\ &= \frac{1}{\sigma(\Delta_{k,i})} \sum_{\Delta_{k+1,j}\subset\Delta_{k,i}} \int_{\Delta_{k+1,j}} w d\sigma \\ &> \lambda_{k+1} \frac{\sigma(\Delta_{k,i}\cap D_{k+1})}{\sigma(\Delta_{k,i})}. \end{aligned} \tag{19}$$

Thus,

$$\frac{\sigma(\Delta_{k,i}\cap D_{k+1})}{\sigma(\Delta_{k,i})} < \frac{S\lambda_k}{\lambda_{k+1}} = \eta$$

and hence

$$\frac{\omega_L(\Delta_{k,i}\cap D_{k+1})}{\omega_L(\Delta_{k,i})} < \varepsilon.$$

Summing over i,

$$\omega_L(D_{k+1}) < \varepsilon \omega_L(D_k),$$

which leads to

$$\omega_L(D_k) < \varepsilon^k \omega_L(D_0).$$

Of course we also have

$$\sigma(D_{k+1}) \leqslant \eta \sigma(D_k)$$

and

$$\sigma(D_k) \leqslant \eta^k \sigma(D_0)$$

which implies that

$$\sigma(\cap_{k=0}^{\infty} D_k) = \lim_{k\to\infty} \sigma(D_k) = 0.$$

Then, for any $h > 0$,

$$\begin{aligned} \int_\Delta w^{1+h} d\sigma &= \int_{\Delta\setminus D_0} w^{1+h} d\sigma + \sum_{k=0}^{\infty} \int_{D_k\setminus D_{k+1}} w^{1+h} d\sigma \\ &\leqslant \lambda_0^h \omega_L(\Delta\setminus D_0) + \sum_{k=0}^{\infty} \lambda_{k+1}^h \omega_L(D_k\setminus D_{k+1}) \\ &\leqslant \lambda_0^h \left\{ \omega_L(\Delta\setminus D_0) + \sum_{k=0}^{\infty} (S\eta^{-1})^{(k+1)h} \varepsilon^k \omega_L(D_0) \right\} \\ &\leqslant \lambda_0^h \left\{ \omega_L(\Delta\setminus D_0) + (S\eta^{-1})^h \sum_{k=0}^{\infty} ((S\eta^{-1})^h \varepsilon)^k \omega_L(D_0) \right\}. \end{aligned} \tag{20}$$

Now, if we take $h > 0$ small enough in order to have $(S\eta^{-1})^h \varepsilon < 1$, i.e.

$$h < \frac{\varepsilon \log(\varepsilon^{-1})}{Cd^2\bar{C} + \varepsilon \log S} \tag{21}$$

the series in the right hand side of (20) will have a finite sum and we shall get

$$\begin{aligned} \int_\Delta w^{1+h} d\sigma &\leqslant C\lambda_0^h (\omega_L(\Delta\setminus D_0) + \omega_L(D_0)) \\ &= C\left(\fint_\Delta w d\sigma\right)^h \omega_L(\Delta) \end{aligned}$$

that is $\omega_L \in B_{1+h}(d\sigma)$. At this point, using Theorem I.1 the thesis easily follows. □

IV. Simultaneous BMO-solvability for two different operators in the plane

Let $\mathbb{D}$ be the unit disc in $\mathbb{R}^2$. We denote by $\mathcal{E}_1(K)$ the subclass of $\mathcal{E}(K)$ of matrix functions satisfying the condition

$$\det \mathcal{A}(x) = 1 \quad a.e. \ x \in \mathbb{D}.$$

The restriction to coefficient matrices $\mathcal{A} \in \mathcal{E}_1(K)$ poses any loss of generality. For this we recall [11] that, if $u \in W^{1,2}_{loc}$ solves $\operatorname{div}(A(x)\nabla u) = 0$ for some $A \in \mathcal{E}(K)$ then there is a correction $\mathcal{A} \in \mathcal{E}_1(K)$ such that $\operatorname{div}(\mathcal{A}\nabla u) = 0$.

Actually, *all* matrices in $\mathcal{E}_1(K)$ generate *pull-back* of Laplacian via K- quasiconformal mappings. More precisely, let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F = (\alpha, \beta)$ be K- *quasiconformal*; that is, F is a homeomorphism of class $W^{1,2}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$|DF(x)|^2 \leqslant \left(K + \frac{1}{K}\right) J_F(x) \qquad \text{a.e..} \tag{22}$$

Here $|DF(x)|$ stands for the Hilbert-Schmidt norm of the differential matrix $DF(x)$ and $J_F(x)$ for the Jacobian determinant of F. Then, with $\mathbb{R}^2_+$ denoting the half-plane $x_2 > 0$, we have $F(\mathbb{R}^2_+) = \mathbb{R}^2_+$ and $F(\mathbb{R}) = \mathbb{R}$. Moreover, if u satisfies $\Delta u = 0$, then $v = u \circ F$ is a solution to $Lv = \operatorname{div}(\mathcal{A}\nabla v) = 0$ where $\mathcal{A} = \mathcal{A}(x_1, x_2)$ is given by

$$\mathcal{A} = \frac{1}{J_F} \begin{pmatrix} \beta_{x_1}^2 + \beta_{x_2}^2 & -\alpha_{x_1}\beta_{x_1} - \alpha_{x_2}\beta_{x_2} \\ -\alpha_{x_1}\beta_{x_1} - \alpha_{x_2}\beta_{x_2} & \alpha_{x_1}^2 + \alpha_{x_2}^2 \end{pmatrix} \tag{23}$$

and verifies (1). Hence

$$L := \Delta_F$$

is the *pull-back* under F of the Laplacian. It is well known that $\mathcal{A}$ belongs to $\mathcal{E}_1(K)$, see e.g. [11] and [16]. In [4] the authors construct examples of such operators having the elliptic measure completely singular with respect to arc length, also with coefficients continuous in the closed unit disc.

We prove the following theorem of simultaneous BMO-solvability.

Theorem IV.1. *Let $F : \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal mapping. Then, the operator*

$$L_0 = \Delta_F \tag{24}$$

is BMO-resolutive if and only if

$$L_1 = \Delta_{F^{-1}} \tag{25}$$

is BMO-resolutive.

Proof. The result is obtained by using Theorem 1.3 in [18], and Theorem 2.1 in [7]. Infact, as in [18] one can prove the following equalities between the A_∞-constants of ω_{L_0} and ω_{L_1} (see Section II)

$$\tilde{A}_\infty(\omega_{L_0}) = \tilde{B}_1(\omega_{L_1}), \tag{26}$$

$$\tilde{A}_\infty(\omega_{L_1}) = \tilde{B}_1(\omega_{L_0}) \tag{27}$$

so that, combining (26), (27) and Theorem 2.1 in [7], the thesis follows.

□

We point out that under the definitions (24) and (25) it is not really meaningful to speak of the distance between L_0 and L_1. Indeed the domains of operators L_0 and L_1 are $\mathbb{D}$ and $F(\mathbb{D})$ respectively. On the other hand, even after composition with most natural map F, the coefficient matrix $\mathcal{A}_1 \circ F$ is not close to $\mathcal{A}_o$ in the sense of any natural distance between coefficients (see for example [18], Example 4.1).

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