# Inclusion Properties for a Certain Class of Analytic Function Related to Linear Operator

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***Abstract*—** In this paper, we introduce a new class of analytic functions defined by a new convolution operator $L_a^t(\alpha,\beta)f(z)$. The new class of analytic functions $\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$ in $U^*=\{z:0<|z|<1\}$ is defined by means of a hypergeometric function with an integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination. The author also introduces and investigates various properties of certain classes of Meromorphically univalent functions.

***Keywords*—**Analytic function, Convex function, Starlike function, Prestarlike function, Meromorphic function, Hurwitz Zeta function, Linear operator, Hadamard product.

## I. Introduction

A meromorphic function is a single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities).

A simpler definition states that a meromorphic function $f(z)$ is a function of the form

$$f(z)=\frac{g(z)}{h(z)}$$

where $g(z)$ and $h(z)$ are entire functions with $h(z)\neq 0$ (see [1], p. 64). A meromorphic function therefore may only have finite-order, isolated poles and zeros and no essential singularities in its domain.

An equivalent definition of a meromorphic function is a complex analytic map to the Riemann sphere. For example the Gamma function is meromorphic in the whole complex plane $\mathbb{C}$ (see [2], [3] and [4]).

Let *A* be the class of analytic functions $h(z)$ with $h(0)=1$, which are convex and univalent in the open unit disk $U=U^*\cup\{0\}$ and for which

$$\Re\{h(z)\}>0 \qquad (z\in U).$$

For functions $f(z)$ and $g(z)$ analytic in *U*, we state that $f(z)$ is subordinate to $g(z)$ and write

$$f\prec g \quad \text{in} \quad U \quad \text{or} \quad f(z)\prec g(z) \qquad (z\in U)$$

if there exists an analytic function $w(z)$ in *U* such that

$$|w(z)|\leq|z| \quad and \quad f(z)=g(w(z)), \qquad (z\in U).$$

Furthermore, if the function g is univalent in *U*, then $f(z)\prec g(z) \;\Leftrightarrow\; f(0)=g(0)$ and $f(U)=g(U)$, $(z\in U)$.

In the present paper, we initiate the study of functions which are meromorphic in the punctured disk

$$U^*=\{z:0<|z|<1\}=U\setminus\{0\}$$

with a Laurent expansion about the origin, see [5]. Also, we shall use the operator $L_a^t(\alpha,\beta)f(z)$ to introduce some new classes of meromorphic functions. We also, introduce and investigate various inclusion relationships and convolution properties of a certain class of meromorphic functions, which are defined in this paper by means of a linear operator.

## II. Preliminaries

Let $\Sigma$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the punctured unit disk

$$U^*=\{z:z\in\mathbb{C} \quad \text{and} \quad 0<|z|<1\}=U\setminus\{0\},$$

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\Sigma S^*(\beta)$ and $\Sigma K(\beta)$ $(\beta\geq 0)$ the subclasses of $\Sigma$ consisting of all meromorphic functions which are, respectively, starlike of order $\beta$ and convex of order $\beta$ in $U^*$ (see also the recent works [6] and [7]).

For functions $f_j(z)$ $(j=1,2)$ defined by

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}z^n, \qquad (j=1,2),$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1*f_2)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,1}a_{n,2}z^n.$$

Let us consider the function $\tilde{\phi}(\alpha,\beta;z)$ defined by

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$$\tilde{\phi}(\alpha,\beta;z)=\frac{1}{z}+\sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}a_n z^n, \tag{2}$$
$$\left(\beta\in\mathbb{C},Z_0^-;\alpha\in\mathbb{C}\right),$$

where

$$Z_0^-=\{0,-1,-2,\cdots\}=Z^-\bigcup\{0\}$$

Here, and in the remainder of this paper, $(\lambda)_\kappa$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}$$
$$=\begin{cases}1 & (\kappa=0;\lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)(\lambda+n-1) & (\kappa=n\in\mathbb{N};\lambda\in\mathbb{C})\end{cases}$$

it being understood *conventionally* that $(0)_0:=1$ and assumed tacitly that the $\Gamma$-quotient exists (see, for details, [8, p. 21 et seq.]), $\mathbb{N}$ being the set of positive integers.

Very recently, Ghanim ([4]; see also [7]) made use of the Hadamard product for functions $f(z)\in\Sigma$ in order to introduce a new linear operator $L_a^t(\alpha,\beta)f(z)$ defined on $\Sigma$ by

$$L_a^t(\alpha,\beta)f(z)=\tilde{\phi}(\alpha,\beta;z)*G_{t,a}(z) \tag{3}$$
$$=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\left(\frac{a+1}{a+n}\right)^t a_n z^n \qquad (z\in U^*)$$

Where

$$G_{t,a}(z):=(a+1)^t\left[\Phi(z,t,a)-a^t+\frac{1}{z(a+1)^t}\right]$$
$$=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{a+1}{a+n}\right)^t z^n \qquad (z\in U^*) \tag{4}$$

and the function $\Phi(z,t,a)$ is the well-known Hurwitz-Lerch zeta function defined by (see, for example, [9, p. 121 et seq.]; see also [10], [11, p. 194 et seq.], [12] and [13])

$$\Phi(z,t,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^t} \tag{5}$$
$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;t\in\mathbb{C}\ \text{ when } |z|<1;\Re(t)>1\ \text{ when } |z|=1\right)$$

Many papers considered the above operator along with the meromorphic functions and generalized hypergeometric functions, see for example [[6], [7], [14], [15], [16], and [17]].

It follows from (3) that

$$z\left(L_a^t(\alpha,\beta)f(z)\right)'$$
$$=\alpha\left(L_a^t(\alpha+1,\beta)f(z)\right)-(\alpha+1)L_a^t(\alpha,\beta)f(z). \tag{6}$$

Let $\Omega$ represent the class of analytic functions $h(z)$ with $h(z)=1$, which are convex and univalent in the open unit disk $U=U^*\bigcup\{0\}$.

***Definition 1*** A function $f(z)\in\Sigma$ is said to be in the $\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$, if it satisfies the subordination condition

$$(1+\rho)z\left(L_a^t(\alpha,\beta)f(z)\right)+\rho z^2\left(L_a^t(\alpha,\beta)f(z)\right)'\prec h(z) \tag{7}$$

Where $\rho$ is a complex number and $h(z)\in\Omega$..

Let *A* be class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{8}$$

which are analytic in *U*. A function $h(z)\in A$ is said to be in the class $S^*(\gamma)$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\gamma \qquad (z\in U)$$

For some $\gamma(\gamma<1)$. When $0<\gamma<1$, $S^*(\gamma)$ is the class of starlike functions of order $\gamma$ in *U*. A function $h(z)\in A$ is said to be prestarlike of order $\gamma$ in *U*, if

$$\frac{z}{(1-z)^{2(1-\gamma)}}*f(z)\in S^*(\gamma) \qquad (\gamma<1)$$

where the symbol * is used to refer to the familiar Hadamard product (or convolution) of two analytic functions in *U*. We denote this class by $R(\gamma)$. A function $f(z)\in A$ is in the class $R(0)$, if and only if $f(z)$ is convex univalent in *U* and

$$R\left(\frac{1}{2}\right)=S^*\left(\frac{1}{2}\right)$$

## III. Main results

In order to establish our main results, the following lemmas will be required:

***Lemma 2*** (*See* [18]) *Let* $g(z)$ *be analytic in U, and* $h(z)$ *be analytic and convex univalent in U with* $h(0)=g(0)$. *If,*

$$g(z)+\frac{1}{\mu}zg'(z)\prec h(z) \tag{9}$$

where $\Re(\mu)\geq 0$ and $\mu\neq 0$, then

$$g(z)\prec\tilde{h}(z)=\mu z^{-\mu}\int_0^z t^{\mu-1}h(t)\,dt\prec h(z)$$

and $\tilde{h}(z)$ is the best dominant of (9).

***Lemma 3*** [19] *Let* $\Re(\alpha)\geq 0$ *and* $\alpha\neq 0$. *Then,*

$$\Sigma_{\alpha,\beta}^{a,t}(\rho;h)\subset\Sigma_{\alpha,\beta}^{a,t}(\rho;\tilde{h}),$$

*where*

$$\tilde{h}(z)=\alpha z^{-\alpha}\int_0^z t^{\alpha-1}h(t)\,dt\prec h(z).$$

***Lemma 4*** [19] *Let* $f(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$, $g(z)\in\Sigma$ and

$$\Re(zg(z))>\frac{1}{2} \qquad (z\in U).$$

*Then,*

$$(f*g)(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$$

***Lemma 5*** (See [19]) *Let* $a<1$, $f(z)\in S^{*}(a)$ *and* $g(z)\in R(a)$. For any analytic function $F(z)$ in *U*, then

$$\frac{g*F}{g*f}(U)\subset\overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the convex hull of $F(U)$.

***Theorem 6*** *Let* $f(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$. *Then the function* $F(z)$ *defined by*

$$F(z)=\frac{\mu-1}{z^{\mu}}\int_0^z t^{\mu-1}f(t)\,dt \qquad (\Re(\mu)>1) \tag{10}$$

*is in the class* $\Sigma_{\alpha,\beta}^{a,t}(\rho;\tilde{h})$, *where*

$$\tilde{h}(z)=(\mu-1)z^{1-\mu}\int_0^z t^{\mu-2}h(t)\,dt\prec h(z)$$

***Proof:*** For $f(z)\in\Sigma$ and $\Re(\mu)>1$, we find from (10) that $F(z)\in\Sigma$ and

$$(\mu-1)f(z)=\mu F(z)+zF'(z) \tag{11}$$

$f(z)\in\Sigma$.

Define *G (z)* by

$$zG(z)=(1+\rho)z\left(L_a^t(\alpha,\beta)F(z)\right)+\rho z\left(L_a^t(\alpha,\beta)F(z)\right)'. \tag{12}$$

By differentiating both sides of (12) with respect to *z*, we get:

$$zG'(z)-G(z)=(1+\rho)z\left(L_a^t(\alpha,\beta)(zF'(z))\right)+\rho z^2\left(L_a^t(\alpha,\beta)(zF'(z))\right)'. \tag{13}$$

Furthermore, it follows from (11), (12) and (13) that:

$$\begin{aligned}&(1+\rho)z\left(L_a^t(\alpha,\beta)f(z)\right)+\rho z^2\left(L_a^t(\alpha,\beta)f(z)\right)'\\&=(1+\rho)z\left(L_a^t(\alpha,\beta)\left(\frac{\mu F(z)+zF'(z)}{\mu-1}\right)\right)\\&+\rho z^2\left(L_a^t(\alpha,\beta)\left(\frac{\mu F(z)+zF'(z)}{\mu-1}\right)\right)'\\&=\frac{\mu}{\mu-1}G(z)+\frac{1}{\mu-1}\left(zG'(z)-G(z)\right)\\&=G(z)+\frac{zG'(z)}{\mu-1}.\end{aligned} \tag{14}$$

Let $f(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$. Then, by (14)

$$G(z)+\frac{zG'(z)}{\mu-1}\prec h(z) \qquad (\Re(\mu)>1),$$

by using Lemma 2, we get

$$G(z)\prec\tilde{h}(z)=(\mu-1)z^{1-\mu}\int_0^z t^{\mu-2}h(t)\,dt\prec h(z)$$

Hence, by Lemma 3, we arrive at:

$$F(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;\tilde{h})\subset\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$$

**Theorem 7** *Let* $f(z)\in\Sigma$ *and* $F(z)$ *be defined as in Theorem 6. If*

$$(1+\gamma)z\left(L_a^t(\alpha,\beta)F(z)\right)+\gamma z\left(L_a^t(\alpha,\beta)F(z)\right)\prec h(z) \qquad (\gamma>0), \tag{15}$$

*then* $F(z)\in\Sigma_{\alpha,\beta}^{a,t}(0;\tilde{h})$, *where* $\Re(\mu)>1$ *and*

$$\tilde{h}(z)=\frac{(\mu-1)}{\gamma}z^{\frac{1-\mu}{\gamma}}\int_0^z t^{\frac{\mu-1}{\gamma}-1}h(t)\,dt\prec h(z).$$

***Proof:*** Let us define,

$$G(z)=z\left(L_a^t(\alpha,\beta)F(z)\right) \tag{16}$$

Then the analytic function $G(z)$ in the unit disk U, $G(0)=1$ and

$$zG'(z)=G(z)+z^2\left(L_a^t(\alpha,\beta)F(z)\right)'. \tag{17}$$

Making use of (11), (15), (16) and (17), we deduce that:

$$\begin{aligned}&(1+\gamma)z\left(L_a^t(\alpha,\beta)F(z)\right)+\gamma z\left(L_a^t(\alpha,\beta)F(z)\right)\\&=(1+\gamma)z\left(L_a^t(\alpha,\beta)F(z)\right)\\&+\frac{\gamma}{\mu-1}\left(\mu zL_a^t(\alpha,\beta)F(z)\right)+z^2\left(L_a^t(\alpha,\beta)F(z)\right)'\\&=G(z)+\frac{zG'(z)}{\mu-1}\prec h(z)\end{aligned}$$

for $\Re(\mu)>1$ and $\gamma>0$.

Thus, an application of Lemma 2 evidently completes the proof of Theorem 7.

***Theorem 8*** *Let* $F(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$. *If the function* $f(z)$ *is defined by*

$$F(z)=\frac{\mu-1}{z^{\mu}}\int_0^z t^{\mu-1}f(t)\,dt \qquad (\mu>1) \tag{18}$$

then,

$$\sigma f(\sigma z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$$

where

$$\sigma=\sigma(\mu)=\frac{\sqrt{\mu^2-2(\mu-1)}-1}{(\mu-1)}\in(0,1) \tag{19}$$

The bound $\sigma$ is sharp when

$$h(z)=\delta+(1-\delta)\frac{1+z}{1-z} \qquad (\delta\neq 1). \tag{20}$$

***Proof:*** For $F(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$, we can verify that:

$$F(z)=F(z)*\frac{z}{1-z}$$

and

$$zF'(z)=F(z)*\left(\frac{1}{z(1-z)^2}-\frac{2}{z(1-z)}\right).$$

Hence, by (18), we have:

$$f(z)=\frac{\mu F(z)+zF'(z)}{\mu-1}=(F*g)(z) \tag{21}$$

$(z\in U^{*},\ \mu>1)$, where g (z) =

$$\frac{1}{\mu-1}\left(\frac{1}{(1-z)^2}-(\mu-2)\frac{1}{(1-z)}\right)\in\Sigma. \tag{22}$$

We then show that:

$$\Re\{zg(z)\}>\frac{1}{2} \qquad (|z|<\sigma), \tag{23}$$

where $\sigma=\sigma(\mu)$ is given by (19). Setting

$$\frac{1}{1-z}=\mathrm{R}e^{i\theta} \qquad (R>0,\ |z|=r<1)$$

we find that:

$$\cos\theta=\frac{1+R^2(1-r^2)}{2R} \quad and \quad R\geq\frac{1}{1+r} \tag{24}$$

For $\mu>1$ it follows from (22) and (24) that:

$$2R\{zg(z)\}$$
$$=\frac{2}{\mu-1}\left[(\mu-2)R\cos\theta+R^2\left(2\cos^2\theta-1\right)\right]$$
$$=\frac{1}{\mu-1}\left[(\mu-2)\left(1+R^2(1-r^2)\right)+\left(1+R^2(1-r^2)\right)^2-2R^2\right]$$
$$=\frac{R^2}{\mu-1}\left[(1-r^2)^2+\mu(1-r^2)-2\right]+1$$
$$=\frac{R^2}{\mu-1}\left[(1-\mu)r^2+\mu-2r-1\right]+1.$$

This evidently gives (23), which is equivalent to

$$\Re\{z\sigma g(\sigma z)\}>\frac{1}{2} \qquad (z\in U^*) \tag{25}$$

Let $F(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$. Then, by using (21) and (25), an application of Lemma 4 yields:

$$\sigma f(\sigma z)=F(z)*\sigma g(\sigma z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho;h)$$

For $h(z)$ given by (20). We consider the function $F(z)\in\Sigma$ defined by:

$$(1+\lambda)z\left(L_a^t(\alpha,\beta)F(z)\right)$$
$$+\lambda z^2\left(L_a^t(\alpha,\beta)F(z)\right)'=\delta+(1-\delta)\frac{1+z}{1-z}. \tag{26}$$

$(\delta\neq1)$. Then, by (26), (12) and (14) (used in the proof of Theorem 1), we find that:

$$(1+\rho)z\left(L_a^t(\alpha,\beta)f(z)\right)+\rho z^2\left(L_a^t(\alpha,\beta)f(z)\right)'$$
$$=\delta+(1-\delta)\frac{1+z}{1-z}+\frac{z}{\mu-1}\left(\delta+(1-\delta)\frac{1+z}{1-z}\right)'$$
$$=\delta+\frac{(1-\delta)\left(\mu+2z-1+(1-\mu)z^2\right)}{(\mu-1)(1-z)^2}=\delta$$

$(\sigma=-z)$.

Therefore, we conclude that the bound $\sigma=\sigma(\mu)$ cannot be increased for each $\mu$ $(\mu>1)$.

## IV. INCLUSION RELATIONS

***Theorem 9*** *Let* $0\leq\rho_1<\rho_2$. *Then*

$$\Sigma_{\alpha,\beta}^{a,t}(\rho_2;h)\subset\Sigma_{\alpha,\beta}^{a,t}(\rho_1;h).$$

***Proof:*** Let $0\leq\rho_1<\rho_2$ and suppose that:

$$g(z)=z\left(L_a^t(\alpha,\beta)f(z)\right) \tag{27}$$

For $f(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho_2;h)$. Then the function $g(z)$ is analytic in $\mathrm{U}^*$ with $g(0)=1$. Differentiating both sides of (27) with respect to $z$ and using (6), we have:

$$(1+\rho_2)z\left(L_a^t(\alpha,\beta)f(z)\right)+\rho_2z^2\left(L_a^t(\alpha,\beta)f(z)\right)'$$
$$=g(z)+\rho_2zg'(z)\prec h(z) \tag{28}$$

Hence an application of Lemma 2 with $m=\frac{1}{\rho_2}>0$ yields:

$$g(z)\prec h(z) \tag{29}$$

Noting that $0<\frac{\rho_1}{\rho_2}<1$ and that $h(z)$ is convex univalent in $U$, it follows from (27), (28) and (29) that:

$$(1+\rho_1)z\left(L_a^t(\alpha,\beta)f(z)\right)+\rho_1z^2\left(L_a^t(\alpha,\beta)f(z)\right)'$$
$$=\frac{\rho_1}{\rho_2}\left[(1+\rho_2)z\left(L_a^t(\alpha,\beta)f(z)\right)\right.$$
$$\left.+\rho_2z^2\left(L_a^t(\alpha,\beta)f(z)\right)'\right]+\left(1-\frac{\rho_1}{\rho_2}\right)g(z)\prec h(z).$$

Thus, $f(z)\in\Sigma_{\alpha,\beta}^{a,t}(\rho_1;h)$ and the proof of Theorem 9 is complete.

***Theorem 10*** Let,

$$\Re\left\{z\tilde{\phi}(\alpha_1,\alpha_2;z)\right\}>\frac{1}{2} \tag{30}$$

$\left(z\in U^*;\ \alpha_2\notin\{0,-1,-2,\cdots\}\right)$,

where $\tilde{\phi}(\alpha_1,\alpha_2;z)$ is defined as in (2). Then,

$$\Sigma_{\alpha_2,\beta}^{a,t}(\rho;h)\subset\Sigma_{\alpha_1,\beta}^{a,t}(\rho;h).$$

***Proof:***

For $f(z)\in\Sigma$, we can verify that:

$$z\left(L_a^t(\alpha_1,\beta)f(z)\right)$$
$$=\left(z\tilde{\phi}(\alpha_1,\alpha_2;z)*\left(z\left(L_a^t(\alpha_2,\beta)f(z)\right)\right)\right) \tag{31}$$

and

$$z^2\left(L_a^t(\alpha_1,\beta)f(z)\right)'$$
$$=\left(z\tilde{\phi}(\alpha_1,\alpha_2;z)*\left(z^2\left(L_a^t(\alpha_2,\beta)f(z)\right)'\right)\right) \tag{32}$$

Let $f(z)\in\Sigma_{\alpha_2,\beta}^{a,t}(\rho;h)$. Then from (31) and (32), we deduce that:

$$(1+\rho)z\left(L_a^t(\alpha_1,\beta)f(z)\right)+\rho z^2\left(L_a^t(\alpha_1,\beta)f(z)\right)'$$
$$=\left(z\tilde{\phi}(\alpha_1,\alpha_2;z)\right)*\Psi(z) \tag{33}$$

and

$$\Psi(z) = (1+\rho) z \left( L_a^t(\alpha_1,\beta) f(z) \right) + \rho z^2 \left( L_a^t(\alpha_1,\beta) f(z) \right)' \prec h(z) \qquad (34)$$

In view of (30), the function $z\tilde{\phi}(\alpha_1,\alpha_2;z)$ has the Herglotz representation:

$$z\tilde{\phi}(\alpha_1,\alpha_2;z) = \int_{|x|=1} \frac{dm(x)}{1-xz} \qquad (z\in U) \qquad (35)$$

where $m(x)$ is a probability measure defined on the unit circle $|x|=1$ and

$$\int_{|x|=1} dm(x) = 1$$

Since $h(z)$ is convex univalent in *U*, it follows from (33), (34) and (35) that:

$$(1+\rho) z \left( L_a^t(\alpha,\beta) f(z) \right) + \rho z^2 \left( L_a^t(\alpha,\beta) f(z) \right)'$$
$$\int_{|x|=1} \Psi(xz)\, dm(x) \prec h(z)$$

This shows that $f(z) \in \Sigma_{\alpha_1,\beta}^{a,t}(\rho;h)$ and the theorem is proved.

***Theorem 11*** Let $0<\alpha_1<\alpha_2$. Then

$$\Sigma_{\alpha_2,\beta}^{a,t}(\rho;h) \subset \Sigma_{\alpha_1,\beta}^{a,t}(\rho;h)$$

***Proof:*** Define,

$$g(z) = z + \sum_{n=1}^{\infty} \left| \frac{(\alpha_1)_{n+1}}{(\alpha_2)_{n+1}} \right| z^{n+1}$$

$\left( z\in U,\ 0<\alpha_1<\alpha_2 \right)$.

Then,

$$z^2\tilde{\phi}(\alpha_1,\alpha_2;z) = g(z) \in A \qquad (36)$$

where $\tilde{\phi}(\alpha_1,\alpha_2;z)$ is defined as in (2), and

$$\frac{z}{(1-z)^{\alpha_2}} * g(z) = \frac{z}{(1-z)^{\alpha_1}} \qquad (37)$$

By (37), we see that

$$\frac{z}{(1-z)^{\alpha_2}} * g(z) \in S^*\left(1-\frac{\alpha_1}{2}\right) \subset S^*\left(1-\frac{\alpha_2}{2}\right)$$

for $0<\alpha_1<\alpha_2$, with implies that:

$$g(z) \in R\left(1-\frac{\alpha_2}{2}\right) \qquad (38)$$

Let $f(z) \in \Sigma_{\alpha_2,\beta}^{a,t}(\rho;h)$. Then we deduce from (33), (34) and (36) that:

$$(1+\rho) z \left( L_a^t(\alpha,\beta) f(z) \right) + \rho z^2 \left( L_a^t(\alpha,\beta) f(z) \right)'$$
$$= \frac{g(z)}{z} * \Psi(z) = \frac{g(z) * (z\Psi(z))}{g(z) * z} \qquad (39)$$

where

$$\Psi(z) = (1+\rho) z \left( L_a^t(\alpha,\beta) f(z) \right) + \rho z^2 \left( L_a^t(\alpha,\beta) f(z) \right)' \prec h(z) \qquad (40)$$

Since $z$ belongs to $S^*\left(1-\frac{\alpha_2}{2}\right)$ and $h(z)$ is convex univalent in *U*, it follows from (38), (39), (40) and Lemma 5 that:

$$(1+\rho) z \left( L_a^t(\alpha_1,\beta) f(z) \right) + \rho z^2 \left( L_a^t(\alpha_1,\beta) f(z) \right)' \prec h(z)$$

Thus, $f(z) \in \Sigma_{\alpha_1,\beta}^{a,t}(\rho;h)$ and the proof is completed.

## V. Conclusion

In our present investigation, we have successfully applied a remarkably general family of linear operators which are associated with the hyper geometric functions. By means of this general linear operator, we have introduced and investigated various properties of some new subclasses of meromorphically univalent functions in the punctured unit disk $U^*$. We have also considered several closely related consequences of the main results (Theorems 6, 7, 8, 9, 10 and 11) presented in this paper.

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