The method of discrete singularities in the diffraction problem on a closed cylindrical surface (the case of H-polarization)

Y.Bakhmat

Abstract— 2-D problem of monochromatic electromagnetic waves diffraction by a perfectly conducting closed cylindrical surface (the case of H-polarization) has been considered. The mathematical model of the diffraction problem above has been built. A new approach of the transition to discrete mathematical model has been considered. The discrete mathematical model is based on the method of the discrete singularities. The numerical experiment has been made.

***Keywords* —** vector and scalar potentials, integral equation, logarithmic kernel.

I. Introduction

The purpose of the work is to construct the discrete mathematical model of the diffraction problem H-polarization electromagnetic waves by a perfectly conducting closed cylindrical surface. Vector and scalar potentials have been used for constructing the mathematical model of this problem [1]. The method of discrete singularities has been applied for constructing the discrete mathematical model of this problem. The numerical method has been made in some cases where directivity patterns modulus of the complex amplitude of the scattering field has been built.

II. Vector and scalar potentials

We introduce the Cartesian coordinate system (x_1, x_2, x_3). Let us consider a perfectly conducting cylinder which is infinite along axis x_3. The intersection has been made by plane parallel to the plane X_1OX_2. To denote L, the simple smooth contour, is the directrix of a cylindrical surface.

Vectors of the electromagnetic field have been represented as [2] :

$$E(\bar{x},t) = E(\bar{x}) \cdot e^{i\omega t}, \quad (1)$$

$$H(\bar{x},t) = H(\bar{x}) \cdot e^{i\omega t}, \quad (2)$$

where

$$H(\bar{x}) = (0, \quad 0, \quad H_{x_3}(\bar{x})), \quad H_{x_3} = u(\bar{x}), \quad (3)$$

$$E(\bar{x}) = (E_{x_1}, \quad E_{x_2}, \quad 0),$$

Y.Bakhmat, Kharkiv national university named by V.N.Karazin, Ukraine(e-mail: bakhmat23@gmail.com).

$$E_{x_1} = -\frac{1}{i\omega\varepsilon}\frac{\partial H_{x_3}}{\partial x_2}, \; E_{x_2} = \frac{1}{i\omega\varepsilon}\frac{\partial H_{x_3}}{\partial x_1} \quad (4)$$

There is exterior boundary value problem. u(x) is unknown function and all components of the electromagnetic fields are expressed by this function.

We have considered the Maxwell equations in the differential form:

$$rot\,\bar{E} = -\frac{\partial}{\partial t}\bar{B}, \quad (5)$$

$$rot\,\bar{H} = -\frac{\partial}{\partial t}\bar{D}, \quad (6)$$

$$\bar{D} = \varepsilon\,\bar{E}, \quad (7)$$

$$\bar{B} = \mu\,\bar{H}, \quad (8)$$

where ε is the dielectric permittivity of the medium, μ is the permeability of the medium.

Since

$$div\,\bar{B} = 0,$$

then, as known, vector $\bar{B}$ is represented as

$$\bar{B} = rot\bar{H}, \quad (9)$$

then from (8):

$$\bar{H} = \frac{1}{\mu} rot\bar{A} \cdot \quad (10)$$

Taking into account the equality (10), substituting (8) to (5), we obtain:

$$rot\bar{E} = -i\omega\mu\bar{H}, \quad (11)$$

$$rot\bar{E} = -i\omega\, rot\bar{A}, \quad (12)$$

$$rot(\bar{E} + i\omega\,\bar{A}) = 0. \quad (13)$$

As known,

$$rot(grad\,\psi) = 0, \quad (14)$$

so from (13), taking into account (14), we obtain:

$$-grad\,\psi = \bar{E} + i\omega\,\bar{A}, \quad (15)$$

$$\bar{E} = -grad\,\psi - i\omega\,\bar{A}. \quad (16)$$

So vector $\bar{A}$ and electrodynamics scalar potential ψ are related with $\bar{E}$ and $\bar{H}$ relations (10), (16).

As known,

$$rot\{rot\bar{A}\} = grad\,\{div\bar{A}\} - \nabla^2\bar{A}. \quad (17)$$

Substitute (10) and (16) to (6)

$$rot\left\{\frac{1}{\mu} rot\bar{A}\right\} = i\omega\varepsilon\{-grad\,\psi - i\omega\bar{A}\}, \quad (18)$$

Taking into account (17), we obtain:

$$grad\{div\overline{A}\} - \nabla^2 \overline{A} = i\omega\varepsilon\mu\{-grad\,\psi - i\omega\overline{A}\}. \quad (19)$$

We finally have:

$$\nabla^2 \overline{A} + k^2 \overline{A} = grad\{i\omega\varepsilon\mu\psi + div\overline{A}\},\ k = \omega\sqrt{\varepsilon\mu}\,. \quad (20)$$

As $\overline{A}$ uniquely defined, so the condition is imposed:

$$i\omega\varepsilon\mu\psi + div\overline{A} = 0\,, \quad (21)$$

Then we obtain equation:

$$\nabla^2 \overline{A} + k^2 \overline{A} = 0\,. \quad (22)$$

The solution of Gelmgolt`s equation is vector potential [3]:

$$\overline{A}(N) = \frac{\mu}{4\pi}\int_S \bar{j}(M)\frac{\exp\{-ikL_{MN}\}}{L_{MN}}ds_M\,, \quad (23)$$

where L_{MN} is the distance from the integration point M to the observation point N.

Now to imagine scalar potential in the form of an integral over the surface S. the density of the current $\bar{j}(M)$ is defined by points, that belong to the surface. Substitute (23) to (21), also assuming, that observation point N does not belong to the surface S, than have divergence operator under the integral sign. Taking into account:

$$div\{\bar{a}\cdot\varphi\} = \bar{a}\cdot grad\varphi + \varphi\cdot div\bar{a}\,, \quad (24)$$

we obtained the scalar potential:

$$\psi(N) = \frac{-i}{4\pi\omega\varepsilon}\int_S \bar{j}(M)\cdot grad\left\{\frac{\exp\{-ikL_{MN}\}}{L_{MN}}\right\}ds_M\,. \quad (25)$$

III. OUTPUT THE BOUNDARY INTEGRAL EQUATION

The tangential component of the intensity vector of the total electric field by the surface becomes equally 0. This is boundary condition:

$$\left[\overline{E}^0, \bar{n}^0\right] = -\left[\overline{E},\ \bar{n}^0\right], \quad (26)$$

$$\left[\overline{E}^0, \bar{n}^0\right] = -\lim_{N\to M}\left\{\left[\bar{n}^0, grad\,\psi(N)\right] + i\omega\left[\bar{n}^0, \overline{A}(N)\right]\right\}, \quad M \in S \quad (27)$$

We introduced curvilinear coordinate system q, τ, z for convenience, so that surface S coincides with the part of the coordinate surface $q = q_0 = const$. Point M has coordinates $x_1 = \xi,\ x_2 = \eta,\ x_3 = \zeta$.

As the directrix of the cylindrical surface in this case is ellipse, then parametric surface has the form:

$$\begin{cases} x_1 = a\cdot\cos t, \\ x_2 = a\cdot\sin t, \\ x_3 = z, \qquad t\in[0, 2\pi), \qquad z \in (-\infty;+\infty). \end{cases} \quad (28)$$

Let $\bar{t}^0$ is the basis vector of the variable τ in the point M, flux density is present in the form of

$$\bar{j}(M) = \bar{j}(t,\zeta) = j_\tau(t,\zeta)\cdot\bar{t}^0 + j_z(t,\zeta)\cdot\bar{z}^0\,, \quad (29)$$

where $j_\tau(t,\zeta),\ j_z(t,\zeta)$ are respectively transverse and longitudinal components of the vector on the point M.

As we have seen H-polarized field, then the surface currents are only transverse

$$\bar{j}(M) = j_\tau(t)\cdot\bar{t}^0\,. \quad (29.1)$$

In this case the boundary condition has the form:

$$\frac{1}{h_\tau}\frac{\partial\psi}{\partial\tau} + i\omega A_\tau(N) = E_\tau^0, \quad q = q_0,\ \ t\in[0, 2\pi), \quad (30)$$

$$h_\tau = \sqrt{\left(\frac{\partial x_1(\tau)}{\partial\tau}\right)^2 + \left(\frac{\partial x_2(\tau)}{\partial\tau}\right)^2}\,. \quad (32)$$

Considered the scalar potential (25). As

$$\left(grad\left\{\frac{\exp(-ikL_{MN})}{L_{MN}}\right\}, \bar{j}\right) = \frac{j_\tau(t)}{l(t)}\frac{\partial}{\partial t}\left(\frac{\exp(-ikL_{MN})}{L_{MN}}\right), \quad (33)$$

$$l(t) = \sqrt{\left(\frac{\partial x_1(t)}{\partial t}\right)^2 + \left(\frac{\partial x_2(t)}{\partial t}\right)^2}\,, \quad (34)$$

$$H_0^{(2)}(x) = \frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{\exp\left\{-i\sqrt{x^2+t^2}\right\}}{\sqrt{x^2+t^2}}dt\,, \quad (35)$$

then we obtained finally

$$\psi(N) = \frac{-i}{4\omega\varepsilon}\int_0^{2\pi} j_\tau(t)\frac{\partial}{\partial t}H_0^{(2)}(k\,L_{MN})dt\,. \quad (36)$$

Then considered the vector potential and found $A_\tau(N)$. As

$$A_\tau(N) = \left(\bar{\tau}^0, \overline{A}(N)\right), \quad (37)$$

We obtained

$$A_\tau(N) = \frac{\mu}{4l(\tau)i}\int_0^{2\pi} s(t,\tau)H_0^{(2)}(k\,L_{MN})\,j_\tau(t)dt\,, \quad (38)$$

$$s(t,\tau) = \frac{\partial x_1(t)}{dt}\cdot\frac{\partial x_1(\tau)}{d\tau} + \frac{\partial x_2(t)}{dt}\cdot\frac{\partial x_2(\tau)}{d\tau}\,. \quad (40)$$

In (30) the point N can be omitted on the contour L, if we understand the integral in the principal value sense. In order to obtain an integro-differential equation, we must substitute the representation for the scalar potential (36) and the τ – component of the vector potential at the boundary condition.

Thus we have the boundary integral equation:

$$\frac{-1}{4l(\tau)\omega\varepsilon}\lim_{q\to q_0}\frac{\partial}{\partial\tau}\int_0^{2\pi}\frac{\partial}{\partial t}H_0^{(2)}(kL)\,j_\tau(t)dt + {} + \frac{\omega\mu}{4l(\tau)}\int_0^{2\pi}s(t,\tau)H_0^{(2)}(kL_0)\,j_\tau(t)dt = E_\tau^0 \quad (41)$$

$$L_0 = \sqrt{(x_1(t) - x_1(\tau))^2 + (x_2(t) - x_2(\tau))^2}\,, \quad (42)$$

$$L(q,t,\tau) = \sqrt{(x_1(q_0,t) - x_1(q,\tau))^2 + (x_2(q_0,t) - x_2(q,\tau))^2}\,, \quad (43)$$

$$k = \omega\sqrt{\varepsilon\mu}\,, \quad (44)$$

where ($x_1(q_0,t)$ and $x_2(q_0,t)$) are coordinate points, which belong to the contour, ($x_1(q, \tau)$ and $x_2(q_,\ \tau)$) are coordinate points, which belong to the contour of the normal to the surface. Under the limit $q \to q_0$, we understand that research point is raised above the contour and then that point falls along the normal to the contour.

After we have allocated hypersingular and logarithmic features, finally we have obtained the integral equation:

$$\frac{i}{2\pi}\int_0^{2\pi}\frac{j_\tau(t)\,dt}{\sin^2\left(\frac{t-\tau}{2}\right)}+\frac{i\,k^2\,l(\tau)}{\pi}\int_0^{2\pi}\ln\left|\sin\left(\frac{t-\tau}{2}\right)\right| j_\tau(t)dt+ \tag{45}$$
$$+\int_0^{2\pi}Q(t,\tau)\,j_\tau(t)\,dt=f(\tau),$$

$$Q(t,\tau)=-\frac{\partial^2}{\partial\tau\partial t}H_0^{(1)}(k\,L_0)-\frac{i}{2\pi}\frac{1}{\sin^2\left(\frac{t-\tau}{2}\right)}-$$
$$-\frac{2ik^2}{2\pi}\left(\left(x_1{}'(\tau)\right)^2+\left(x_2{}'(\tau)\right)^2\right)\cdot\ln\left|\sin\left(\frac{t-\tau}{2}\right)\right|+ \tag{46}$$
$$+k^2\left(x_1{}'(\tau)\cdot x_1{}'(t)+x_2{}'(\tau)\cdot x_2{}'(t)\right)\cdot H_0^{(2)}(k\,L_0)+$$
$$+k^2\left(\left(x_1{}'(\tau)\right)^2+\left(x_2{}'(\tau)\right)^2\right)\frac{2i}{\pi}\ln\left|\sin\left(\frac{t-\tau}{2}\right)\right|,$$

$$f(\tau)=4l(\tau)\,\omega\,\varepsilon\,E_\tau^0. \tag{47}$$

IV. The Discrete Mathematical Model of the Problem

For the construction of the discrete model we have formulated the problem for the approximate solution in the form of the interpolating polynomials. We have replaced all smooth function in (47) by corresponding trigonometric interpolation polynomials [4]:

$$\frac{i}{2\pi}\int_0^{2\pi}\frac{\left(P_n^{(1)}j_\tau\right)(t)\,dt}{\sin^2\left(\frac{t-\tau}{2}\right)}+\frac{i\,k^2\,l(\tau)}{\pi}\int_0^{2\pi}\ln\left|\sin\left(\frac{t-\tau}{2}\right)\right|\cdot\left(P_n^{(1)}j_\tau\right)(t)dt+$$
$$+\int_0^{2\pi}\left(P_n^{(2)}P_n^{(1)}Q\right)(t,\tau)\cdot\left(P_n^{(1)}j_\tau\right)(t)\,dt=\left(P_n^{(2)}f\right)(\tau), \tag{48}$$

$$\left(P_n^{(i)}g\right)(\varphi)=\frac{1}{2n+1}\sum_{k=0}^{2n}g\left(\varphi_k^{(i,n)}\right)\frac{\sin\frac{2n+1}{2}\left(\varphi-\varphi_k^{(i,n)}\right)}{\sin\frac{1}{2}\left(\varphi-\varphi_k^{(i,n)}\right)}, \tag{49}$$

$$\varphi_k^{(1,n)}=\varphi_k^n=\frac{2\pi k}{2n+1},\quad k=0,1,...,2n, \tag{50}$$

$$\varphi_j^{(2,n)}=\varphi_{0j}^n=\frac{2j+1}{2n+1}\pi,\quad j=0,1,...,2n. \tag{51}$$

We have used interpolation quadrature formula with special set of points as nod. We have obtained the system of linear algebraic equations where unknown vectors are current density values ($j_\tau(t)$) in special set of points:

$$\frac{4\pi}{2n+1}\sum_{k=0}^{2n}j_\tau\left(\varphi_k^n\right)\left(\frac{\sin^2\frac{n}{2}\left(\varphi_{0j}^n-\varphi_k^n\right)}{\sin^2\frac{1}{2}\left(\varphi_{0j}^n-\varphi_k^n\right)}-\frac{n\cdot\sin\left(n+\frac{1}{2}\right)\left(\varphi_{0j}^n-\varphi_k^n\right)}{\sin\frac{1}{2}\left(\varphi_{0j}^n-\varphi_k^n\right)}\right)-$$
$$-\frac{ik^2l(\tau)}{2\pi}\frac{\pi}{2n+1}\sum_{k=0}^{2n}j_\tau\left(\varphi_k^n\right)\cdot\left(\ln 2+\sum_{p=1}^{n}\frac{\cos p\left(\varphi_{0j}^n-\varphi_k^n\right)}{p}\right)+$$
$$+\frac{2\pi}{2n+1}\sum_{k=0}^{2n}j_\tau\left(\varphi_k^n\right)\cdot Q\left(\varphi_k^n,\varphi_{0j}^n\right)=f\left(\varphi_{0j}^n\right),\quad j=0,1,\ldots,2n. \tag{52}$$

V. The Directivity Pattern of Scattered Field

Consider the case, when the directrix of the cylinder is ellipse, whose center is at the origin. Then the parametric representation of the contour directrix can be written as follows:

$$\begin{cases}x_1=a\cdot\cos t,\\ x_2=b\cdot\sin t,\end{cases}\qquad t\in[0,\ 2\pi). \tag{53}$$

And let

$$\begin{cases}y_1=r\cdot\cos t,\\ y_2=r\cdot\sin t,\end{cases}\qquad t\in[0,\ 2\pi). \tag{54}$$

As $\bar{y}\in C\overline{\Omega}$, we have assumed, that $r>>R$.

Asymptotic behavior of the Hankel functions at infinity [5] is

$$H_\nu^{(2)}(z)\approx\sqrt{\frac{2}{\pi\,z}}e^{-i\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)},\qquad z\to+\infty. \tag{55}$$

The directivity pattern of scattered field is determined by the formula [6]

$$D_H(t_0)=\lim_{r\to+\infty}\frac{U(r,t_0)}{\sqrt{\frac{2}{\pi\,r}}e^{i\left(kr-\frac{\pi}{4}\right)}}, \tag{56}$$

$$U(r,t_0)=\frac{k}{4i}\int_0^{2\pi}\left\{x_2{}'(t)\frac{x_1(t)-r\cos t_0}{L_{MN}}-x_1{}'(t)\frac{x_2(t)-r\sin t_0}{L_{MN}}\right\}\cdot$$
$$\cdot\sqrt{\frac{2}{\pi\,k\,L_{MN}}}e^{-i\left(k\,L_{MN}-\frac{\pi}{2}-\frac{\pi}{4}\right)}j_\tau(t)\,dt. \tag{57}$$

Thus, finding limit (56), we have got directivity diagram of the complex amplitude of the scattered field:

$$D_H(t_0)=\frac{\sqrt{k}}{2\sqrt{2\pi}}\int_0^{2\pi}\left(-x_2{}'(t)\cos t_0+x_2{}'(t)\sin t_0\right)\cdot \tag{58}$$
$$\cdot e^{ik\left(x_1(t)\cos t_0+x_2(t)\sin t_0\right)}\cdot j_\tau(t)dt.$$

To denote

$$g(t,t_0)=\left(-x_2{}'(t)\cos t_0+x_2{}'(t)\sin t_0\right)\cdot e^{ik\left(x_1(t)\cos t_0+x_2(t)\sin t_0\right)} \tag{59}$$

Replacing $g(t,t_0)$ and $j_\tau(t)$ the corresponding interpolation functions by trigonometric polynomials, using appropriate quadrature formula, we have finally obtained

$$D_H(t_0)=\frac{2\pi}{2n+1}\sum_{k=0}^{2n}g\left(\varphi_k^n,\varphi_0\right)\cdot j_\tau\left(\varphi_k^n\right). \tag{60}$$

VI. Conclusion

Thus, mathematical and discrete mathematical models of the diffraction problem H-polarization electromagnetic waves by the perfectly conducting closed cylindrical surface had been built. The numerical experiment had been made. The directivity pattern of the scattered field module complex amplitude had been built. The method of discrete singularities is based on the special quadrature formulas of interpolation that guarantees high accuracy and high rate convergence of the algorithm.

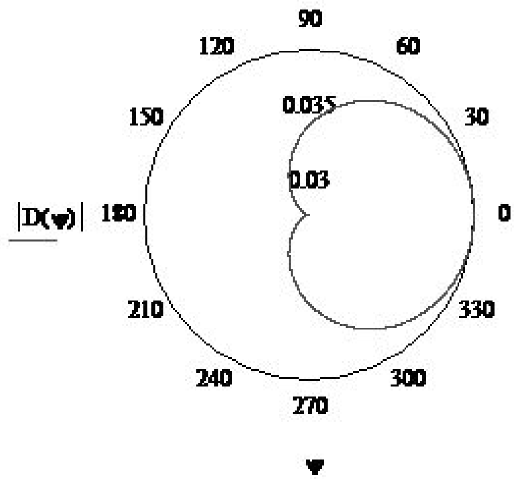

Fig.1. The directivity diagram of the complex amplitude of the scattered field, directrix is circle, R=1, $\alpha = \dfrac{\pi}{2}$.

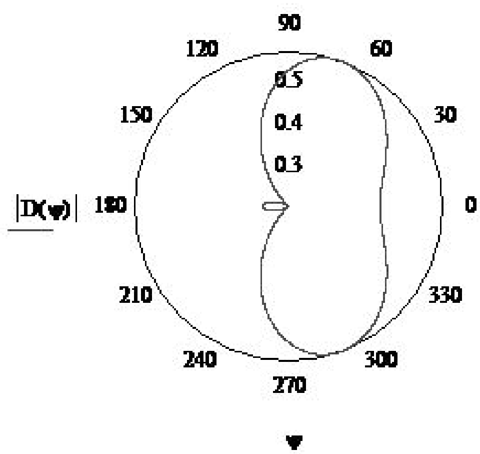

Fig.2. The directivity diagram of the complex amplitude of the scattered field, directrix is ellipse, $a = \dfrac{10}{9},\ b = \sqrt{a^2 - 1}$, $\alpha = \dfrac{\pi}{2}$.

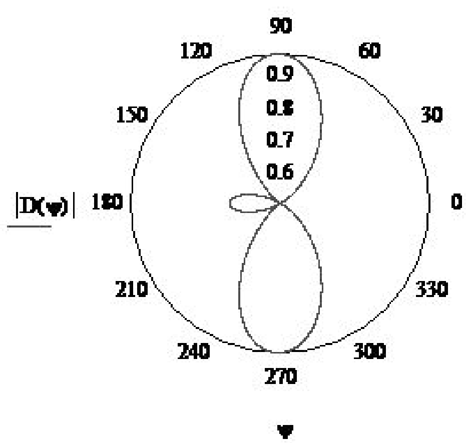

Fig.3. The directivity diagram of the complex amplitude of the scattered field, directrix is ellipse, $a = \dfrac{15}{9},\ b = \sqrt{a^2 - 1}$, $\alpha = \dfrac{\pi}{2}$.

REFERENCES

[1] Захаров Е.В., Пименов Ю.В. Численный анализ дифракции электромагнитных волн на идеально проводящих экранах. --- М.: Радио и связь, 1984. -- 71с.

[2] Гандель Ю.В. Введение в методы вычисления сингулярных и гиперсингулярных интегралов. Учебное пособие.---Харьков: ХНУ, 2002. -- 92 с.

[3] Пименов Ю.В. Техническая электродинамика: Учебное пособие для вузов. --- М.: Радио и связь, 2000. -- 536с.

[4] Гандель Ю.В., Еременко С.В., Полянская Т.М. Обоснование численного метода дискретных особенностей решения двумерных задач дифракции электромагнитных волн: учебное пособие. Часть 2 --- Харьков: ХГУ, 1992. -- 145 с..

[5] Справочник по специальным функциям под редакциею М.Абрамовица и И.Стиган.---М.: Наука, 1979. -- 832с.

[6] Панасюк В.В., Саврук М.П., Назарчук З.Т. Метод сингулярных интегральных уравнений в двумерных задачах дифракции. --- Киев: Наук.думка, 1984. -- 344с.

[7] D. Colton, R.Kress. Integral Equation Methods in Scattering Theory. New York et al., John Wiley & Sons, 1983.