Comparison of two iteration methods for solving nonlinear Fractional Partial Differential Equations

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Abstract— This paper follows up on a research paper presented at the 2014 International Conference for Pure Mathematics Applied Mathematics and Computations Methods in July Santorini Greece [1]. We paid attention to the methodology of two integral transform methods for solving nonlinear fractional partial differential equations. On one hand, the Homotopy Perturbation Sumudu Transform Method (HPSTM) is the coupling of the Sumudu transform and the HPM using He's polynomials. On the other hand, the Homotopy Decomposition Method (HDM) is the coupling of Adomian Decomposition Method and Perturbation Method. Both methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, the HDM has an advantage over the HPSTM which is that it solves the nonlinear problems using only the inverse operator which is basically the fractional integral. There is no need to use any other inverse transform to find the components of the series solutions like in the case of HPSTM. As a consequence the calculations involved in HDM are very simple and straightforward.

Keywords— Homotopy decomposition method, Integral transforms, nonlinear fractional differential equation, Sumudu transform.

I. INTRODUCTION

Fractional Calculus has been used to model some physical and engineering processes, which are found to be best described by fractional differential equations. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields an excellent literature of this can be found in [2-11]. However, analytical solutions of these equations are quickly difficult to find.

One can find in the literature a wide class of methods dealing with approximate solutions to problems described by nonlinear fractional differential equations, asymptotic and perturbation methods for instance. Perturbation methods carry among others the inconvenient that approximate solutions engage series of small parameters which cause difficulties since most nonlinear problems have no small parameters at all. Even though a suitable choice of small parameters occasionally leads to ideal solutions, in most cases unsuitable choices lead to serious effects in the solutions. Therefore, an analytical method which does not require a small parameter in the equation modeling the phenomenon is welcome. To deal with the pitfall presented by perturbation methods for solving nonlinear equations, we present a literature review in some new asymptotic methods aiming for the search of solitary solutions of nonlinear differential equations, nonlinear differential-difference equations, and nonlinear fractional differential equations; see in [12]. The homotopy perturbation method (HPM) was first initiated by He [13]. The HPM was also studied by many authors to present approximate and exact solution of linear and nonlinear equations arising in various scientific and technological fields [14–24]. The Adomian decomposition method (ADM) [25] and variational iteration method (VIM) [26] have also been applied to study the various physical problems. The Homotopy decomposition method (HDM) was recently proposed by [27-28] to solve the groundwater flow equation and the modified fractional KDV equation [27-28]. The Homotopy decomposition method is actually the combination of perturbation method and Adomian decomposition method. Singh et al. [29] studied solutions of linear and nonlinear partial differential equations by using the homotopy perturbation Sumudu transform method (HPSTM). The HPSTM is a combination of Sumudu transform, HPM, and He's polynomials.

II. SUMUDU TRANSFORM

The Sumudu transform, is an integral transform similar to the Laplace transform, introduced in the early 1990s by Gamage K. Watugala [30] to solve differential equations and control engineering problems. It is equivalent to the Laplace-Carson transform with the substitution $p = 1/u$. Sumudu is a Sinhala word, meaning "smooth". The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by: (2.1).

$$S\big(f(t)\big) = F_s(u) = \int_0^\infty \frac{1}{u} exp\left[-\frac{t}{u}\right] f(t)dt$$

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A. *Properties of Sumudu Transform [31-34]*

- The transform of a Heaviside unit step function is a Heaviside unit step function in the transformed domain.
- The transform of a Heaviside unit ramp function is a Heaviside unit ramp function in the transformed domain.
- The transform of a monomial t^n is the called monomial $S(t^n) = n!\,u^n$.
- If $f(t)$ is a monotonically increasing function, so is $F(u)$ and the converse is true for decreasing functions.
- The Sumudu transform can be defined for functions which are discontinuous at the origin. In that case the two branches of the function should be transformed separately.
- If $f(t)$ is $C^{\,n}$ continuous at the origin, so is the transformation $F(u)$.
- The limit of $f(t)$ as t tends to zero is equal to the limit of $F(u)$ as u tends to zero provided both limits exist.
- The limit of $f(t)$ as t tends to infinity is equal to the limit of $F(u)$ as u tends to infinity provided both limits exist.
- Scaling of the function by a factor $c > 0$ to form the function $f(ct)$ gives a transform $F(cu)$ which is the result of scaling by the same factor.

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III. Basic Definitions of Fractional Calculus

Definition 1 A real function $f(x), x > 0$, is said to be in the space C_μ, μ∈ℝ if there exists a real number p > μ, such that f(x) = x^ph(x), where $h(x) \epsilon\, C\,[0,\infty)$, and it is said to be in space C_μ^m if $f^{(m)} \epsilon\, C_\mu$, m∈ℕ

Definition 2 The Riemann-Liouville fractional integral operator of order α≥0, of a function $f\epsilon C_\mu$, μ≥-1, is defined as

(3.1)

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \ \ \alpha > 0, x > 0$$

$$J^0 f(x) = f(x).$$

Properties of the operator can be found in [1-4] we mention only the following:

For $f\epsilon\, C_\mu$, $\mu \geq -1, \alpha, \ \beta \geq 0$ and $\gamma > -1$:

(3.2)

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \ \ J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} .$$

Lemma 1 If $m-1 < \alpha \leq m, m \in \mathbb{N}$ *and* $f \in C_\mu^m, \mu \geq -1,$ *then*

$$D^\alpha J^\alpha f(x) = f(x) \ and, \quad J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \ x > 0 \qquad (3.3)$$

Definition 3: Partial Derivatives of Fractional order

Assume now that $f(\boldsymbol{x})$ is a function of n variables x_i $i = 1, \dots \dots, n$ also of class C on $D \in \mathbb{R}_n$. As an extension of definition 2 we define partial derivative of order α for $f(x)$ respect to x_i the function

(3.4)

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)}\int_a^{x_i} (x_i - t)^{m-\alpha-1}\, \partial_{x_i}^m f(x_j)|_{x_j=t} dt$$

If it exists, where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

Definition 4: The Sumudu transform of the Caputo fractional derivative is defined as follows [30-33]:

(3.5)

$$S[D_t^\alpha f(t)] = u^{-\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0^+), (m-1 < \alpha \leq m)$$

IV. Solution by (HPSTM) and (HDM)

IV.I Basic Idea of HPSTM

We illustrate the basic idea of this method, by considering a general fractional nonlinear non-homogeneous partial differential equation with the initial condition of the form:

(4.1)

$$D_t^\alpha U(x,t) = L\big(U(x,t)\big) + N\big(U(x,t)\big) + f(x,t); \alpha > 0$$

subjected to the initial conditions

$$D_0^k U(x,0) = g_k, \qquad (k = 0, \dots \dots n-1), D_0^n U(x,0) = 0 \ and\ n = [\alpha]$$

where, D_t^α denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator.

Applying the Sumudu Transform on Both sides of equation (4.1), we obtain:

$$S[D_t^{\alpha} U(x,t]) = S[L(U(x,t))] + S[N(U(x,t))] + S[f(x,t)] \quad (4.2)$$

Using the property of the Sumudu transform, we have

$$S[U(x,t)] = u^{\alpha} S[L(U(x,t))] + u^{\alpha} S[N(U(x,t))] + u^{\alpha} S[f(x,t)] + g(x,t) \quad (4.3)$$

Now applying the Sumudu inverse on both sides of (4.3) we obtain:

$$U(x,t) = S^{-1}\left[u^{\alpha} S[L(U(x,t))] + u^{\alpha} S[N(U(x,t))]\right] + G(x,t) \quad (4.4)$$

$G(x,t)$ represents the term arising from the known function $f(x,t)$ and the initial conditions.

Now we apply the HPM:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \quad (4.5)$$

The nonlinear tern can be decomposed

$$NU(x,t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(U) \quad (4.6)$$

using the He's polynomial$\mathcal{H}_n(U)$ [23] given as: (4.7)

$$\mathcal{H}_n(U_0, \cdots\cdots\cdots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^{\infty} p^j U_j(x,t)\right)\right], n = 0,1,2 \cdots\cdots\cdots$$

Substituting (4.5) and (4.6) $\sum_{n=0}^{\infty} p^n U_n(x,t) = G(x,t) + p\left[S^{-1}\left[u^{\alpha} S[L(\sum_{n=0}^{\infty} p^n U_n(x,t))] + u^{\alpha} S[N(\sum_{n=0}^{\infty} p^n U_n(x,t))]\right]\right]$ (4.8)

which is the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficients of like powers of p, the following approximations are obtained.

$$p^0: U_0(x,t) = G(x,t),$$

$$p^1: \ U_1(x,t) = S^{-1}\left[u^{\alpha} S\left[L\left(U_0(x,t)\right) + H_0(U)\right]\right],$$

$$p^2: \ U_2(x,t) = S^{-1}\left[u^{\alpha} S\left[L\left(U_1(x,t)\right) + H_1(U)\right]\right],$$

(4.9)

$$p^3: \ U_3(x,t) = S^{-1}\left[u^{\alpha} S\left[L\left(U_2(x,t)\right) + H_2(U)\right]\right],$$

$$p^n: \ U_n(x,t) = S^{-1}\left[u^{\alpha} S\left[L\left(U_{n-1}(x,t)\right) + H_{n-1}(U)\right]\right],$$

Finally, we approximate the analytical solution $U(x,t)$ by the truncated series:

$$U(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} U_n(x,t) \quad (4.10)$$

The above series solution generally converges very rapidly [34].

IV.II Basic Idea of HDM [27-28]

The method first step here is to transform the fractional partial differential equation to the fractional partial integral equation by applying the inverse operator D_t^{α} of on both sides of equation (4.1) to obtain:

$$U(x,t) = \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha - j + 1)} t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [L(U(x,\tau)) + N(U(x,\tau)) + f(x,\tau)] d\tau \quad (4.11)$$

Or in general by putting

$$\sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha - j + 1)} t^{\alpha-j} = f(x,t) \ \ or \ \ f(x,t) = \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha - j + 1)} t^j$$

We obtain:

$$U(x,t) = T(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [L(U(x,\tau)) + N(U(x,\tau)) + f(x,\tau)] d\tau \quad (4.12)$$

For the homotopy decomposition method, we assume the solutions can be written as a power series in p

$$U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t) \quad (4.13)$$

$$U(x,t) = lim_{p\to 1} U(x,t,p) \quad (4.14)$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(U) \quad (4.15)$$

where $p \epsilon (0,\ 1]$ is an embedding parameter. $\mathcal{H}_n(U)$ [24] is the He's polynomials that can be generated by

(4.16)

$$\mathcal{H}_n(U_0, \cdots\cdots\cdots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^{\infty} p^j U_j(x,t)\right)\right], n = 0,1,2 \cdots\cdots\cdots$$

Now gracefully using Abel integral with the above in (4.12) we achieved (4.17)

$$\sum_{n=0}^{\infty} p^n U_n(x,t) - T(x,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[f(x,\tau) + L(\textstyle\sum_{n=0}^{\infty} p^n U_n(x,\tau)) + N(\sum_{n=0}^{\infty} p^n U_n(x,\tau))]d\tau$$

Comparing the terms of same powers of p gives solutions of various orders with the first term:

$$U_0(x,t) = T(x,t) \tag{4.18}$$

It is worth noting that, the term $T(x,t)$ is the Taylor series of the exact solution of equation (4.1) of order $n-1$.

V. Applications

In this section we solve some popular nonlinear partial differential equation with both methods.

Example 1:

Let consider the following one-dimensional fractional heat-like problem: (5.1)

$$D_t^{\alpha} u(x,t) = \frac{1}{2}x^2 u_{xx}(x,t), 0 < x < 1, 0 < \alpha \leq 1, t > 0$$

Subject to the boundary condition:

$$u(0,t) = 0, \qquad u(1,t) = \exp[t]$$

and initial condition $u(x,0) = x^2$

Example 2

Consider the following time-fractional derivative in x, y −plane as (5.2)

$$D_t^{\alpha} u(x,y,t) = \frac{1}{2}\nabla^2 u(x,y,t), 1 < \alpha \leq 2, x, y \in \mathbb{R}, t > 0$$

subjected to the initial conditions

(5.3)

$$u(x,y,0) = \sin(x+y), \qquad u_t(x,y,0) = -\cos(x+y)$$

Example 3

Consider the following nonlinear time-fractional gas dynamics equations [Kilicman]

$$D_t^{\alpha} U + \tfrac{1}{2}(U^2)_x - U(1-U) = 0, 0 < \alpha \leq 1, \tag{5.4}$$

with the initial conditions

$$U(x,0) = \exp[-x] \tag{5.5}$$

Example 4: Consider the following three-dimensional fractional heat-like equation

$$\partial_t^{\alpha} u(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right),$$

$$0 < x, y, z < 1, 0 < \alpha \leq 1 \tag{5.6}$$

Subject to the initial condition:

$$u(x,y,z,0) = 0 \tag{5.7}$$

VI. Solution via HPSTM

Example1: Apply the steps involved in HPSTM as presented in section IV.I to equation (5.1) we obtain the following:

$$p^0:\ u_0(x,t) = x^2, \tag{5.8}$$

$$p^1: u_1(x,t) = S^{-1}\left[u^{\alpha} S\left(\tfrac{1}{2}x^2(u_0)_{xx}\right)\right] = \tfrac{x^2 t^{\alpha}}{\Gamma(\alpha+1)},$$

$$p^2: u_2(x,t) = S^{-1}\left[u^{\alpha} S\left(\tfrac{1}{2}x^2(u_1)_{xx}\right)\right] = \tfrac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{5.9}$$

$$p^3: u_3(x,t) = S^{-1}\left[u^{\alpha} S\left(\tfrac{1}{2}x^2(u_2)_{xx}\right)\right] = \tfrac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n: u_n(x,t) = S^{-1}\left[u^{\alpha} S\left(\tfrac{1}{2}x^2(u_n)_{xx}\right)\right] = \tfrac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{5.10}$$

Therefore the series solution is given as:

(5.11)

$$u(x,t) = x^2\left[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]$$

This is equivalent to the exact solution in closed form:

$$u(x,t) = x^2 E_{1,\alpha}(t^{\alpha}) \tag{5.12}$$

where $E_{1,\alpha}()$ is the Mittag-Leffler function.

Example 2: Applying the steps involved in HPSTM as presented in section 4.1 to equation (5.2) we obtain:

$$p^0:\ u_0(x,y,t) = \sin(x+y) - \cos(x+y)t,$$

$$p^1: u_1(x,t) = S^{-1}\left[u^{\alpha} S\left(\tfrac{1}{2}x^2\left[(u_0)_{xx} + (u_0)_{yy}\right]\right)\right] = -\sin(x+y)\tfrac{t^2}{2} + \cos(x+y)\tfrac{t^3}{3!},$$

$$p^2: u_2(x,t) = S^{-1}\left[u^{\alpha} S\left(\frac{1}{2}x^2\left[(u_1)_{xx} + (u_1)_{yy}\right]\right)\right] = \sin(x+y)\left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)}\right] + \cos(x+y)\left[-\frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\right]$$

$$p^3: u_3(x,t) = S^{-1}\left[u^{\alpha} S\left(\tfrac{1}{2}x^2\left[(u_2)_{xx} + (u_2)_{yy}\right]\right)\right] = \sin(x+y)\left[-\tfrac{t^2}{2!} + \tfrac{t^4}{4!} + \tfrac{t^6}{6!} + \tfrac{2t^{4-\alpha}}{\Gamma(5-\alpha)} - \tfrac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \right.$$

$\frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)}\Big] + \cos(x+y)\left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{2t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right],$

Therefore the series solution is given as:

(5.13)

$$u(x,y,t) = \sin(x+y)\left[1 - \frac{3t^2}{2!} + \frac{t^4}{8} + \frac{t^6}{6!} + \frac{3t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)}\right] + \cos(x+y)\left[-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{3t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right] + \cdots$$

It is important to point out that if $\alpha = 2$ the above solution takes the form: (5.14)

$$u_{N=4}(x,y,t) = \sin(x+y)\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right] - \cos(x+y)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right]$$

which is the first four terms of the series expansion of the exact solution $u(x,y,t) = \sin(x+y-t)$

Example 3: Apply the steps involved in HPSTM as presented in section 4.1 to equation (5.4) Kilicman et al [34] obtained the following:

$p^0:\ u_0(x,t) = exp(-x),$

$p^1: u_1(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{2}x^2(u_0)_{xx}\right)\right] = \frac{exp\,(-x)t^\alpha}{\Gamma(\alpha+1)},$

$p^2: u_2(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{2}x^2(u_1)_{xx}\right)\right] = \frac{exp\,(-x)t^{2\alpha}}{\Gamma(2\alpha+1)},$

$p^3: u_3(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{2}x^2(u_2)_{xx}\right)\right] = \frac{exp\,(-x)t^{3\alpha}}{\Gamma(3\alpha+1)},$

(5.15)

$p^n: u_n(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{2}x^2(u_n)_{xx}\right)\right] = \frac{exp\,(-x)t^{n\alpha}}{\Gamma(n\alpha+1)},$

Therefore the series solution is given as:

(5.16)

$$u(x,t) = exp(-x)\left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]$$

Example 4: Applying the steps involved in HPSTM as presented in section 4.1 to equation (5.2) we obtain:

$$p^0:\ u_0(x,t) = x^4y^4z^4$$

$$p^1: u_1(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{36}\left(x^2(u_0)_{xx} + y^2(u_0)_{yy} + z^2(u_0)_{zz}\right)\right)\right] = \frac{t^\alpha x^4y^4z^4}{\Gamma(\alpha+1)}$$

$p^2: u_2(x,t) = S^{-1}\left[u^\alpha S\frac{1}{36}\left(x^2(u_1)_{xx} + y^2(u_1)_{yy} + z^2(u_1)_{zz}\right)\right] = \frac{t^{2\alpha}x^4y^4z^4}{\Gamma(2\alpha+1)}$

$p^3: u_3(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{36}\left(x^2(u_2)_{xx} + y^2(u_2)_{yy} + z^2(u_2)_{zz}\right)\right)\right] = \frac{t^{3\alpha}x^4y^4z^4}{\Gamma(3\alpha+1)}$

(5.17)

$$p^n: u_n(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{36}\left(x^2(u_{n-1})_{xx} + y^2(u_{n-1})_{yy} + z^2(u_{n-1})_{zz}\right)\right)\right] = \frac{t^{n\alpha}x^4y^4z^4}{\Gamma(n\alpha+1)}$$

Therefore the approximate solution of equation for the first n is given below as:

(5.18)

$u_N(x,y,z,t) = \sum_{n=1}^{N}\frac{t^{n\alpha}x^4y^4z^4}{\Gamma(n\alpha+1)}.$

V.II. Solution via HDM

Example 1: Apply the steps involved in HDM as presented in section 4.2 to equation (5.1) we obtain the following

(5.19)

$$\sum_{n=0}^{\infty}p^n\mathrm{u}_n(x,t) - x^2 = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[x^2\sum_{n=0}^{\infty}p^n\frac{\partial^2 u_n(x,\tau)}{\partial x^2}\right]d\tau$$

Comparing the terms of the same powers of p we obtain:

$u_0(x,t) = x^2$

$$u_1(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{x^2\partial^2 u_0(x,\tau)}{\partial x^2}d\tau = \frac{x^2t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{x^2\partial^2 u_1(x,\tau)}{\partial x^2}d\tau = \frac{x^2t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{x^2\partial^2 u_2(x,\tau)}{\partial x^2}d\tau = \frac{x^2 t^{3\alpha}}{\Gamma(2\alpha+1)}$$

$$u_n(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{x^2\partial^2 u_{n-1}(x,\tau)}{\partial x^2}d\tau = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)}$$

The asymptotic solution is given by

$$u_N(x,t) = x^2\left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{N\alpha}}{\Gamma(N\alpha+1)} + \cdots\right]$$

$$\lim_{N\to\infty} u_N(x,t,\alpha) = u(x,t,\alpha) \qquad (5.20)$$

$$\lim_{\substack{\alpha\to 1\\ N\to\infty}} u_N(x,t,\alpha) = x^2\exp(t)$$

This is the exact solution of (5.1) when $\alpha = 1$.

Example 2

Following the discussion presented earlier, applying the initial conditions and comparing the terms of the same power of p, integrating, we obtain the following solutions:

$$u_0(x,t) = \sin(x+y) - \cos(x+y)t$$

$$u_1(x,t) = -\sin(x+y)\frac{t^2}{2} + \cos(x+y)\ \frac{t^3}{3!}$$

$$u_2(x,t) = \sin(x+y)\left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)}\right] + \cos(x+y)\left[-\frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\right]$$

(5.21)

$$u_3(x,t) = \sin(x+y)\left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)}\right] + \cos(x+y)\left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{2t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right]$$

Using the package Mathematica, in the same manner one can obtain the rest of the components. But for four terms were computed and the asymptotic solution is given by:

(5. 22)

$$u(x,y,t) = \sin(x+y)\left[1 - \frac{3t^2}{2!} + \frac{t^4}{8} + \frac{t^6}{6!} + \frac{3t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)}\right] + \cos(x+y)\left[-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{3t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right] + \cdots$$

It is important to point out that if $\alpha = 2$ the above solution takes the form:

$$u_{N=4}(x,y,t) = \sin(x+y)\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right] - \cos(x+y)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right]$$

Which are the first four terms of the series expansion of the exact solution $u(x,y,t) = \sin(x+y-t)$

Example 3:

p^0: $u_0(x,t) = exp(-x)$,

p^1: $u_1(x,t) = \frac{exp\,(-x)}{\Gamma(\alpha+1)}t^\alpha$,

p^2: $u_2(x,t) = \frac{exp\,(-x)}{\Gamma(2\alpha+1)}t^{2\alpha}$,

p^3: $u_3(x,t) = \frac{exp\,(-x)}{\Gamma(3\alpha+1)}t^{3\alpha}$,

(5.23)

⋮

p^n: $u_n(x,t) = \frac{exp\,(-x)}{\Gamma(n\alpha+1)}t^{n\alpha}$,

Therefore the series solution is given as:

(5.24)

$$u(x,t) = exp(-x)\left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]$$

Example 4: Following carefully the steps involved in the HDM, we arrive at the following equations

$$\sum_{n=0}^\infty p^n u_n(x,y,z,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Big(x^4y^4z^4 + \frac{1}{36}\big(x^2(\sum_{n=0}^\infty p^n u_n(x,y,z,t))_{xx} + y^2(\sum_{n=0}^\infty p^n u_n(x,y,z,t))_{yy} + z^2(\sum_{n=0}^\infty p^n u_n(x,y,z,t))_{zz}\big)\Big)d\tau$$

(5.25)

Now comparing the terms of the same power of p yields:

p^0: $u_0(x,y,z,t)$ (5.26)

$$p^1:\ u_1(x,y,z,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x^4y^4z^4 d\tau$$

⋮

$$p^n: u_n(x,y,z,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\frac{1}{36}\left(x^2(u_{n-1})_{xx} + y^2(u_{n-1})_{yy} + z^2(u_{n-1})_{zz}\right)\right)d\tau, u_n(x,y,z,0) = 0, n \geq 2$$

Thus the following components are obtained as results of the above integrals

$$u_0(x,y,z,t) = 0$$

$$u_1(x,y,z,t) = \frac{t^{\alpha}x^4y^4z^4}{\Gamma(\alpha+1)}$$

$$u_2(x,y,z,t) = \frac{t^{2\alpha}x^4y^4z^4}{\Gamma(2\alpha+1)}$$

$$u_3(x,y,z,t) = \frac{t^{3\alpha}x^4y^4z^4}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$u_n(x,y,z,t) = \frac{t^{n\alpha}x^4y^4z^4}{\Gamma(n\alpha+1)} \tag{5.27}$$

Therefore the approximate solution of equation for the first n is given below as:

$$u_N(x,y,z,t) = \sum_{n=1}^{N}\frac{t^{n\alpha}x^4y^4z^4}{\Gamma(n\alpha+1)}$$

Now when $N \to \infty$ we obtained the follow solution

$$\tag{5.28}$$

$$u(x,y,z,t) = \sum_{n=0}^{\infty}\frac{t^{n\alpha}x^4y^4z^4}{\Gamma(n\alpha+1)} - x^4y^4z^4 = x^4y^4z^4(E_{\alpha}(t^{\alpha}) - 1)$$

Where $E_{\alpha}(t^{\alpha})$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x,y,z,t) = x^4y^4z^4(\exp(t) - 1) \tag{5.29}$$

This is the exact solution for this case.

VI. Comparison of Methods

This section is devoted to the comparison between the two integral transform methods.

The two methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, it can be noted that the HDM has an advantage over the HPSTM which is that it solves the nonlinear problems using only the inverse operator which is simply the fractional integral. There is no need to use any other inverse transform to find the components of the series solutions as in the case of HPSTM. In addition the calculations involved in HDM are very simple and straightforward. In conclusion, the HDM and the HPSTM may be considered as a nice refinement in existing numerical techniques and might find wide applications.

Table 1: Numerical results of equation (5.2) via mathematica

			HPSTM and HDM	HPSTM and HDM	HPSTM and HDM		
t	x	y	$\alpha = 1.25$	$\alpha = 1.75$	$\alpha = 2.0$	Exact	Errors
0.25	0.5	0.5	0.643624	0.666050	0.681639	0.681639	0
	0.5	1.0	0.901511	0.929440	0.948985	0.948985	0
	1.0	0.5	0.901511	0.929440	0.948985	0.948985	0
	1.0	1.0	0.938676	0.965271	0.983986	0.983986	0
0.5	0.5	0.5	0.366563	0.440270	0.479425	0.479426	0.000006
	0.5	1.0	0.691370	0.788806	0.841471	0.841471	0
	1.0	0.5	0.691370	0.788806	0.841471	0.841471	0
	1.0	1.0	0.846906	0.944215	0.997495	0.997495	0
0.75	0.5	0.5	0.067024	0.192505	0.247402	0.247404	0.000002

	1.	1.	0.4215	0.6004	0.6816	0.6816	0.0000
	0	0	20	31	36	39	03
	0.	0.	0.4215	0.6004	0.6816	0.6816	0.0000
	5	5	20	31	36	39	03
	1.	1.	0.6728	0.8613	0.9489	0.9489	0.0000
	0	0	13	1	82	85	03
1.0	0.	0.	-	-	-	0.0000	0.0000
	5	5	0.2082	0.0543	0.0000	00	19
	1.	1.	46	57	019	0.4794	0.0000
	0	0	0.1417	0.3857	0.4794	26	25
	0.	0.	97	25	01	0.4794	0.0000
	5	5	0.1417	0.3857	0.4794	26	25
	1.	1.	97	25	01	0.8414	0
	0	0	0.4571	0.7313	0.8414	48	
			23	69	48		

The approximate solution of equation (5.2) obtained by the present methods is close at hand to the exact solution. It is to be noted that only the fourth-order term of the HDM and HPSTM were used to evaluate the approximate solutions for Figures 1. It is evident that the efficiency of the present method can be noticeably improved by computing additional terms of $u(x,t)$ when the HDM is used.

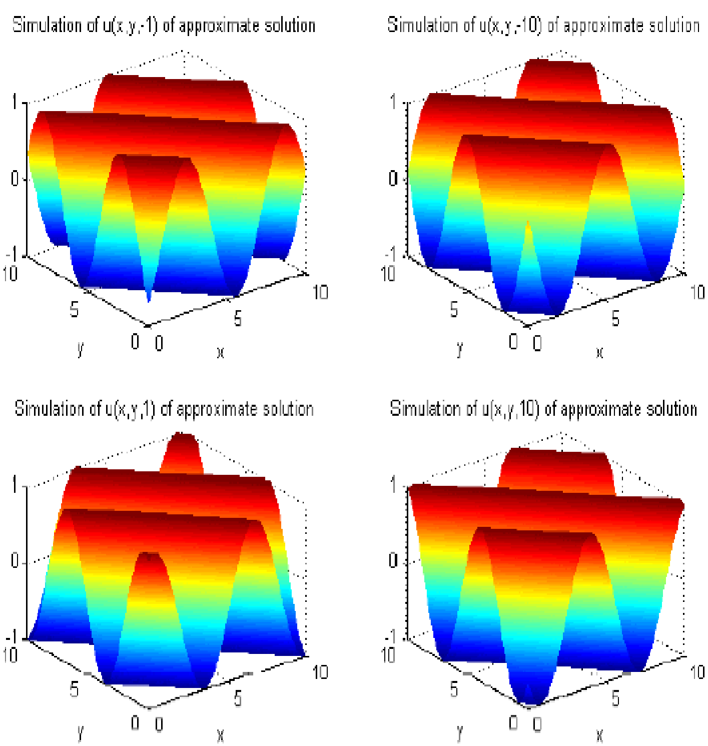

Figure 1: Numerical simulation of the approximated solution of equation (5.2).

VII. Conclusion

We studied two integral transform methods for solving fractional nonlinear partial differential equation. The first method namely homotopy perturbation Sumudu transform method is the coupling of the Sumudu transform and the HPM using He's polynomials. The second method namely Homotopy decomposition method is the combination of Adomian decomposition method, pertubation technique and an astute use of Abel's integral. Both methods fared well. We presented numerical simulations on different equations. We also gave a graphical representation of a solution. The two methods are efficient on different kinds of linear and nonlinear fractional differential equations arising in fields of science and engineering. The HDM nonetheless has a complexity advantage over the HPSTM. It needs only one inverse operator, which is simply the fractional integral to solve those nonlinear problems. No extra inverse transform is needed to find the components of the series solutions as in the case of HPSTM. Thereof calculations involved in HDM are slimpler and straightforward. In comparison the HDM is more user friendly than the HSPTM.

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