Variational principle in the hydrodynamic lubrication theory

L. Savin, A. Kornaev, E. Kornaeva

***Abstract*—**Most of the laws of physics are brought to a statement that some value in the process under study has to reach its minimum or maximum. In such form these laws are called variational principles and their role in physics is hard to over-estimate. In the present paper a variational principle of the modeling of the stationary flows of viscous incompressible media of complex rheology is shown and its equivalency to the classic approach of the continuum mechanics is shown. An asymptotic case, namely a problem of a Newtonian fluid flow in an infinitely long bearing, is studied to verificate this, and the results match the results of other authors.

***Keywords*—**Continuum mechanics, hydrodynamic lubrication theory, variational problem.

I. Introduction to Variational Calculus

THE idea of the action that takes the path of least resistance has arised a long time ago, and humanity as a part of nature has found proof both in itself and in the surrounding world. Aristotle spoke of this idea in his works (IV century BC), P. Fermat used the idea to describe the laws of refraction of light (1662), and P. Maupertuis was first to formalize the principle of least action in mechanics (1744) [1]. One of the first mathematical formulations of the variational problem in mechanics is the problem of J. Bernoulli - a brachistochrone curve or curve of fastest descent, first published in ‘Acta Eruditorum’ in 1696 [1]. The basics of modern variational calculus were formed by his student – L. Euler [2] and J.L. Lagrange was the one to finish the development of the least action principle in 1760.

Variational principles of the continuum mechanics allow to substitute the problem of intergration of the closed set of equations which describe the motion of the continuum with an equivalent problem, from the formulation of which it follows that the set of solutions of differential equations are the extremals of some functional [1, 3-5]. A variational principle of Lagrange [1, 6], known in continuum mechanics, is applicable to the problems of viscoplastic media flows, however, it is insensitive to complex rheology models.

The equivalence of the variational approach to the classic one can be demonstrated in case the set of generalized differential Euler – Lagrange equations [1, 7] obtained for the target functional matches the set of differential equations that describe the motion of the continuum [8].

So, if the target functional is a definite integral:

$$I = \int_{\Omega} F\left(x_i, y_j, p_{km}\right) d\Omega, \tag{1}$$

where $y_j = y_j(x_i)$ - a set of functions of independent variables x_i, $p_{km} = \frac{\partial}{\partial x_k} y_m(x_i)$ - partial derivatives of the functions,

then the extremals of this functional are the solution of the set of generalized differential Euler – Lagrange equations [1][1]:

$$\frac{\partial}{\partial x_i}\left\{F_{p_{ik}}\right\} = F_{y_k}, \tag{2}$$

where $F_{p_{ik}} = \frac{\partial}{\partial p_{ik}} F\left(x_m, y_n, p_{rs}\right)$, $F_{y_k} = \frac{\partial}{\partial y_k} F\left(x_m, y_n, p_{rs}\right)$.

The left hand part of (2) is a sum of a so-called full partial derivatives. When calculating the full partial derivative with respect to one of the variables x_i, the rest of the variables are considered fixed, however the dependence from y_k and p_{rk} is still considered [7]:

$$\frac{\partial}{\partial x_i}\left\{F_{p_{ik}}\right\} = F_{p_{ik} x_i} + F_{p_{ik} y_k} \frac{\partial y_k}{\partial x_i} + F_{p_{ik} p_{rk}} \frac{\partial p_{rk}}{\partial x_i}, \tag{3}$$

where $F_{p_{ik} x_i} = \frac{\partial F_{p_{ik}}}{\partial x_i}$, $F_{p_{ik} y_k} = \frac{\partial F_{p_{ik}}}{\partial y_k}$, $F_{p_{ik} p_{rk}} = \frac{\partial F_{p_{ik}}}{\partial p_{rk}}$.

Taking (3) into account, the set of generalized differential Euler – Lagrange equations (1) can be written as follows:

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[1] The A. Einstein’s summation notation and the A.I. Lourie’s exception are used.

$$F_{p_{ik}x_i} + F_{p_{ik}y_k}\frac{\partial y_k}{\partial x_i} + F_{p_{ik}p_{rk}}\frac{\partial p_{rk}}{\partial x_i} = F_{y_k}. \qquad (4)$$

Thereby, if the set of the differential equations (4) matches the set of the differential equations that describe the motion of the continuum, then the solution of these equations is equivalent to finding the extremum of the functional (1).

The present paper is dedicated to the problem of justification of the variational principle and estimation of the advantages and disadvantages of the variational approach in comparison to the classic approach to mathematical modeling of the hydrodynamic lubrication theory problems.

II. Variational Principle in Problems of Viscous Media Flow

As it is known, the classic formulation of the boundary problem of the media motion includes the formulation of the closed set of equations and boundary conditions [6, 8]. The basis of the classic formulation are the following differential equations: the equation of motion

$$\nabla \cdot T_\sigma + \rho\vec{f} = \rho\frac{d\vec{V}}{dt} \qquad (5)$$

and the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0, \qquad (6)$$

where T_σ - is the stress tensor with the components σ_{ij}; $\nabla \cdot T_\sigma$ - the scalar product of the W. Hamilton operator and stress tensor, the result is a vector which in Cartesian coordinates has components $\partial\sigma_{ij}/\partial x_i$; ρ - is the density of the media; $\vec{f}$ - mass force, e.g. gravity; $\vec{V}$ - the velocity of the media with components V_i.

The equations (5) and (6) are formulated in a so-called tensor form, which is convenient in a way that it has the same form in any set of orthogonal curvilinear coordinates.

For further exposition the standard procedure of tensor decomposition into spheric and deviatoric parts has to take place:

$$T_\sigma = S_\sigma + D_\sigma, \qquad (7)$$

where S_σ - is the spheric part of the stress tensor with null side components and diagonal components equal to average stress or hydrostatic pressure taken with an opposite sign $\sigma_0 = \sigma_{ii}/3 = -p_0$, D_σ - deviatoric stress tensor with components s_{ij}.

The physical meaning of the decomposition (7) is the following: the speric part of the stress tensor is responsible for the change in volume, and the deviatoric part is responsible for reshaping the neighborhood of a particle of the continuum.

Differential equations (5) and (6) are supplemented with cinematic, rheologic and other relations as well as with initial and boundary conditions, and form a classic mathematical formulation of the problem of the isometric motion of the continuum [6, 8].

The variational principle equivalent to the classic approach with some additional assumptions is justified below.

Suppose that the moving media completely fills the volume Ω with surface area S, which is characterized by a single outer normal $\vec{n}$. Let us write the functional that includes the sum of the power of inner forces and the power of the continuum flow through the closed area S, , developed by the hydrostatic pressure p_0 on this surface:

$$I = \int_\Omega D_\sigma \cdot \left(\nabla\otimes\vec{V}\right)d\Omega + \int_S \left(p_0\vec{V}\right)\cdot\vec{n}\,dS. \qquad (8)$$

where $\nabla\otimes\vec{V}$ - is the gradient of the velocity vector field, the result is a tensor which in the Cartesian coordinates has the components $\partial V_j/\partial x_i$.

In case when the contribution of the deviatoric stress tensor D_σ on the surface of S to the flow of full stress $\vec{\sigma}^n$ across this surface is small or absent, the second term in (8) is the power of "pumping" the media across the volume Ω.

Applying the M.V. Ostrogradsky - C. Gauss formula allows to convert the functional (8) to the following form:

$$I = \int_\Omega \left[D_\sigma\cdot\left(\nabla\otimes\vec{V}\right) + \nabla\cdot\left(p_0\vec{V}\right)\right]d\Omega. \qquad (9)$$

If during the media motion the contribution of mass and inertial forces can be neglected, the equation of motion (5) becomes an equation of equilibrium:

$$\nabla\cdot T_\sigma = 0. \qquad (10)$$

If the motion of the media occurs without changes in density, then the continuity equation (7) takes a more simplified form:

$$\nabla\cdot\vec{V} = 0, \qquad (11)$$

which is called the incompressibility condition. Considering this in Cartesian coordinates the integral (9) takes the following form:

$$I = \int_\Omega \left[s_{ij}\frac{\partial V_i}{\partial xj} + \frac{\partial p_0}{\partial x_k}V_k\right]d\Omega. \qquad (12)$$

According to the notation in (1) and (4) the functional (12) includes 13 unknown functions y_j (s_{ij}, V_i, p_0) and 10 partial derivatives p_{km} ($\partial V_i/\partial x_j$, $\partial p_0/\partial x_k$). The components of the Euler – Lagrange equation for the corresponding unknown functions are presented in the table I.

I The components of the Euler – Lagrange equations

y_k	F_{y_k}	$F_{p_{ik}x_i}$	$F_{p_{ik}y_k}\dfrac{\partial y_k}{\partial x_i}$	$F_{p_{ik}p_{rk}}\dfrac{\partial p_{rk}}{\partial x_i}$
s_{ij}	$\dfrac{\partial V_i}{\partial x_j}$	0	0	0
V_k	$\dfrac{\partial p_0}{\partial x_k}$	$\dfrac{\partial s_{kj}}{\partial x_k}$	0	0
p_0	0	$\dfrac{\partial V_i}{\partial x_i}$	0	0

The set of equations (14) obtained on functions s_{ij} leads to a trivial solution. The set of equations obtained on functions V_i leads to the equations of equilibrium (10) and the set of equations obtained on function p_0 leads to an incompressibility condition (11). Thus, functions V_i and p_0, which convey the extremum to the functional (9), meet the equation of equilibrium and the incompressibility condition. So, the problem of determining the extremum of the functional (8) on the preassigned set of velocity fields $V_i = V_i(x_j)$ and the set of the pressure fields $p_0 = p_0(x_i)$ is equivalent to solving the set of differential equations (10) and (11) and can be written in a form of a variational principle:

$$\delta I = 0 , \tag{13}$$

where δI - is the variation of the functional I.

The variational principle (13) reads as follows: among the set of the velocity and pressure fields only those fields are valid which convey to the functional I a steady state value.

The variation of the unknown functions must be equal zero on the border of the area, i.e. values of the unknown functions must be set as border conditions [1]. In case the incompressibility condition (11) is met by all elements of the set of velocity fields in the whole volume of the moving media then we say that the set of cinematically admissible velocity fields (CAVF) is defined. In this case the necessity of pressure field $p_0 = p_0(x_i)$ determination in solving the variational problem is eliminated.

The main shortcoming of the variational approaches to solving the problems of the continuum mechanics is the necessity of determining the set of velocity fields or other unknown functions beforehand. This causes a severe limitation in terms of the accuracy of the result. In case a precise solution exists in the initial set, the variational principle allows to determine it, otherwise a relatively close solution is accepted as the result.

III. Comparison of the Basic Assumptions of the Variational Principle and the O. Reynolds's Equation

The most used equation in hydrodynamic lubrication theory is the Reynolds equation, obtained from the continuity equation (6) with some assumptions [9]. Analysis of these assumptions will allow to compare the variational and classic approaches in solving the problems of the hydrodynamic lubrication theory (table II).

II Assumptions of the variational and classical approaches

Factor	the Reynolds equation	the variational principle (13)
Inertial and mass forces	-	-
compressibility of the media	+	-
non-Newtonian properties of the media	-	+
three-dimensionalmotion of the media	-	+

Both Reynolds equation and the variational principle are obtained with an assumption that inertial and mass forces need not be considered for their insignificant effect. What follows is, in particular, that the velocity fields are taken steady-state. The problems of dynamics, given that, can be successfully solved with obtaining a so-called quasi-stationary solutions, i.e. it is assumed that the velocity fields change in time very slowly, and this reshaping does not significantly affect the development of the inertial forces.

The Reynolds equation is obtained assuming the compressibility of the media, which makes it almost irreplaceable in solving the problems of gas flows. In these problems the application of the variational principle is vastly limited.

The variational principle (13) is valid for viscous media which can be described by almost any rheological model, unlike the Reynolds equation in which the dynamic viscosity coefficient is taken as a constant before the sign of integration. Moreover, the variational principle allows to study the three-dimensional media flows with additional effects, e.g. Taylor vortex flow [8].

It would be fair to note that scientists have put effort to take into account the factors in the Reynolds equation that are not determined in this particular case (table II), e.g. some papers consider inertial forces when deriving the Reynolds equation [10] . However, the resulting generalized equations are too complex to apply.

Next, a mathematical model of a fluid-friction bearing in a simple formulation of a two-dimensional flow. The results will be compared to the known precise results and the results of the Reynolds equation solution.

IV. Mathematical Model of the Fluid-Film Bearing

Two-dimensional isothermal flux of the Newtonian fluid in the gap between the spinning at a constant peripheral speed pin and static bearing. Motion is taken stationary and volume and mass forces are weak. On the surfaces of the pin and the bearing the no-slip condition is applied.

The medium motion is better to study not in the set of

Cartesian coordinates x_i but in the bipolar coordinates β_i [11], isolines of which coincide with the edges of the area. Isolines of the β_1 coordinate are the eccentrically located circles, to two of which, i.e. $\beta_1 = \beta_1^+$ and $\beta_1 = \beta_1^-$, can be associated with two circles with radius r as the pin and radius R as the bearing (Fig. 1). Then, e^{β_1} is the ratio of the polar radius of a point C. The coordinate β_2 in the Cartesian set of coordinates x_i is the angle between the beams from the pole A $(-a, 0)$ and B $(a, 0)$. The limits $\beta_2 = \beta_2^- = -\pi$ and $\beta_2 = \beta_2^+ = \pi$ determine the uniqueness condition of the flux area in the set of coordinates β_i. In the bipolar coordinates the flux area is the rectangle.

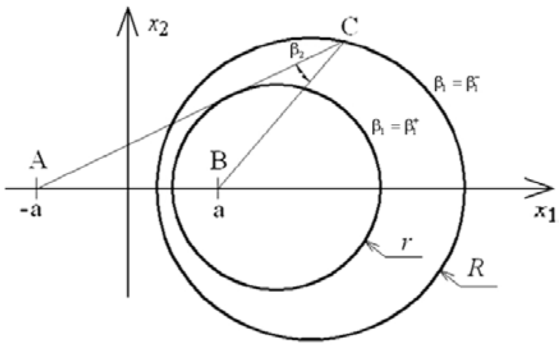

Fig. 1 Cartesian and bipolar set of coordinates correspondence

Lamés coefficients for the coordinate transformation are [11]:

$$H_1 = H_2 = H = \frac{a}{ch(\beta_1) - \cos(\beta_2)}, \quad H_3 = 1. \tag{14}$$

Cinematically admissible velocity field can be easily determined with a flux function Ψ [8] as follows:

$$V_1 = \frac{1}{H}\frac{\partial\Psi}{\partial\beta_2}, \quad V_2 = -\frac{1}{H}\frac{\partial\Psi}{\partial\beta_1}. \tag{15}$$

The velocity field (15) draws the incompressibility condition to identity, which shortens the number of equations to solve.

The flow function Ψ can be shown as a set as follows:

$$\Psi(\tilde{\beta}_i) = c\alpha_k \beta_1^{k\tilde{\hbar}}, \tag{16}$$

where $\tilde{\beta}_i$ are dimensionless coordinates $\tilde{\beta}_i = \frac{\beta_i - \beta_i^-}{\beta_i^+ - \beta_i^-}$, c a constant coefficient that provides the non-slip condition met, α_k are unknown coefficients of the set ($k = 1,2...$), $\tilde{\hbar}$ is Lamés dimensionless coefficient (14) on the surface of the pin $\beta_1 = \beta_1^+$.

Then, CAVF (14) will be as follows:

$$V_1 = \frac{1}{H}\frac{ck\alpha_k}{(\beta_2^+ - \beta_2^-)}\frac{\partial\tilde{\hbar}}{\partial\tilde{\beta}_2}\beta_1^{k\tilde{\hbar}}\ln(\beta_1), \quad V_2 = -\frac{1}{H}\frac{ck\alpha_k}{h(\beta_1^+ - \beta_1^-)}\tilde{\hbar}\beta_1^{k\tilde{\hbar}-1}. \tag{17}$$

Then, the components of the strain rate tensor can be determined with the J. Stokes's formula [6, 8]:

$$T_\xi = \frac{1}{2}\left(\nabla \otimes \vec{V} + \vec{V} \otimes \nabla\right). \tag{18}$$

The connection between the stress and strain states for viscous incompressible medium is determined by the generalized R. Hooke's law:

$$D_\sigma = 2\mu D_\xi, \tag{19}$$

where μ is the coefficient of the dynamic viscosity, D_ξ is the deviator of the strain rate tensor T_ξ (for the incompressible medium $T_\xi = D_\xi$).

The components of the pressure gradient in the functional (9) can be obtained from the equation of equilibrium considering (7):

$$\nabla \cdot S_\sigma = -\nabla \cdot D_\sigma, \tag{13}$$

Substitution of (17-20) in the functional (9) allows it to be determined by the set of CAVF (17). Representation of the CAVF (10) in the form of a set makes the W. Ritz's method easy to use [6].

V. RESULTS OF THE IMITATIONAL MODEL OF THE FLUID-FILM BEARING TESTING

The mathematical model was developed as a program by means of a free software «GNU Octave» [12], using MATLAB compatible high-level programming language. The search for the minimum of the functional (9) was implemented with an inbuilt combined algorithm of finding the extremum of the function of several variables.

After a series of computational experiments with different values of the eccentricity e_p of the pin in the bearing the following results were achieved.

In the Fig. 2 contour graphs of the flux function fields are shown, isolines of which represent the trajectory of the flux, and the vector velocity fields (17). Due to the small value of the average gap $h_0 = R - r$, are in the Fig. 2 was stretched in the direction of the coordinate β_1 so that the results are clear to see. As one can notice in the Fig. 2, the increase in eccentricity results in partial locking and backflow. Similar results were obtained by other authors theoretically [13, 14] and experimentally [15].

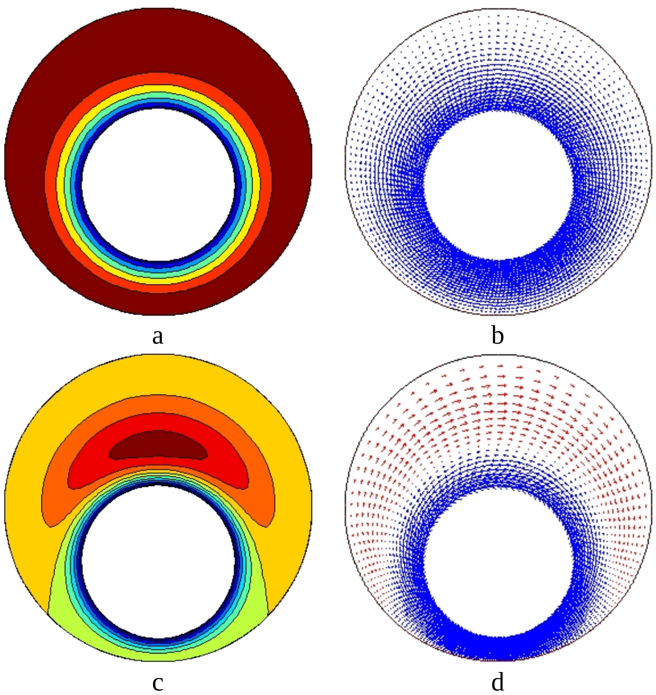

Fig. 2 flux and velocity functions fields; (a) and (b) with relative eccentricity $e_p/h_0 = 0.3$, (c) and (d) - $e_p/h_0 = 0.7$

As a test the problem of the flow of Newtonian fluid with viscosity $\mu_0 = 0.032$ Pa·s, $\rho_f = 886$ kg/m^3 density in the flow area between the pin with $r = 2.5 \cdot 10^{-2}$ m radius, and bearing with $R = 2.51 \cdot 10^{-2}$ m radius was considered. Pressure on the surface of the pin in the point of minimal gap is $P_0 = 2.5 \cdot 10^6$ Pa and the peripheral speed of the pin is $V^0 = 3$ m/s.

The results of friction coefficient calculation as a relation of the resulting friction force T to the resulting lifting force P showed the presence of a minimum (Fig. 3) which could be the expression of the Magnus's effect [8].

Quantitative comparison of the results of calculation of hydrodynamic forces in the developed program with the results of the calculation using the method of N.E. Zhukovsky and S.A. Chaplygin [16] showed a divergence of not more than 2%, which most likely occurred due to the rounding error. The comparison of the results by N.E. Zhukovsky and S.A. Chaplygin and the results of the solution based of the Reynolds equation showed a discrepancy of more than 50% with the relative eccentricities greater than 0.3 which is caused likely by theReynolds's assumptions of smallness of a normal component of the media velocity.

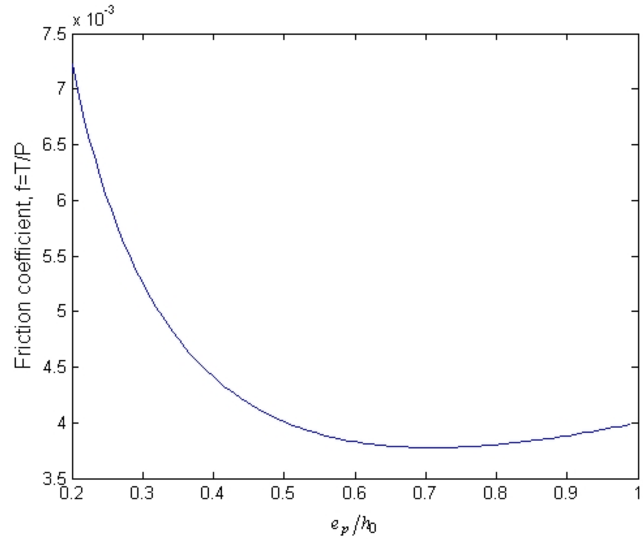

Fig. 3 dependence of the friction coefficient value on the relative eccentricity

VI. Conclusion

In this paper a theoreticaly justified and proved by the calculation results, a possibility of applying the variational principle in solving the problems of hydrodynamic lubrication theory. The paper examined the possibility of application of the isoperimetric form of the variational problem of the viscous incompressible fluid flux in the gap in the fluid-film bearing, the use of the modified target functional was proposed and justified. The application of variational principles allows the substitution of the differential equations integration with the search for the extremum of the function of several variables as well as to obtain the solution in analytic form. One of the most important advantages of the approach is the possibility to study the medium with complex rheology. As for the drawbacks of the method, the relative complexity of CAVF formation procedure, inability to solve the time-dependent and complex flux area form problems can be highlighted.

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