Comparison of selected stochastic mortality models

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Abstract— Insurance companies are affected by many different kinds of risks. In the case of life insurance there are two main risks: the investment risk and the demographic risk. The latter can be split into insurance risk due to the random deviation of the number of deaths from its expected value, and longevity risk deriving from the improvement in mortality rates.

Numbers of stochastic models have been developed to analyse the mortality improvement. We compare three stochastic models explaining improvements in mortality in the Czech Republic. This paper focuses on Lee-Carter and Cairns-Blake-Dowd models. We use data on males deaths and exposures for the Czech Republic from the Human Mortality Database. We write the code associated with models in R.

We show graphical comparison of the model fits. We find that an extension of the Cairns-Blake-Dowd model that incorporates a cohort effect fits the Czech Republic males data best.

Keywords— Cairns-Blake Dowd model, cohort effect, force of mortality, Lee-Carter model.

JEL Classification: C22, J11

I. Introduction

The mortality of the population in developed countries has improved rapidly over the last thirty years (see e.g. [3], [9]) and this has important financial implications for the insurance industry, since several important classes of liability are sensitive to the direction of future mortality trends. This uncertainty about the future development of mortality gives rise to longevity risk. Longevity risk (we refer to [11]) plays a central role in the insurance company management since only careful assumptions about future evolution of mortality phenomenon allow the company to correctly face its future obligations. Longevity risk represents a sub-modul of the underwriting risk module in the Solvency II framework. The most popular and widely used model for projecting longevity is the well-known Lee-Carter model.

This paper follows on article Gogola [6]. The paper deals with Lee-Carter and Cairns-Blake-Dowd models.

Most stochastic mortality models are constructed in a similar manner. Specifically, when they are fitted to historical data, one or more time-varying parameters (κ_t) are identified.

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By extrapolating these parameters to the future, we can obtain a forecast of future death probabilities and consequently other demographic quantities such as life expectancies (beneficial examples in [4]). They are important for quantifying longevity in pension risks and for constructing benchmarks for longevity-linked capital markets.

II. Data and notation

We use data on male deaths and exposure to risk between 1960 and 2011 from the Human Mortality Database (www.mortality.org) [7]. We consider the restricted age range from 60 to 90, the range of interest to providers of pensions.

Let calendar year t runs from exact time t to exact time t+1 and let $d_{x,t}$ be the number of deaths aged x last birthday in the calendar year t. We suppose that the data on deaths are arranged in a matrix $\boldsymbol{D} = \left(d_{x,t}\right)$. In a similar way, the data on exposure are arranged in a matrix $\boldsymbol{E} = \left(e_{x,t}\right)$ where $e_{x,t}$ is a measure of the average population size aged x last birthday in calendar year t, the so-called central exposed to risk. We suppose that $\left(d_{x,t}\right)$ and $\left(e_{x,t}\right)$ are each $n_a \times n_y$ matrices, so that we have n_a ages and n_y years.

We denote the *force of mortality* (or *hazard rate*) at exact time t for lives with exact age x by $\mu_{x,t}$. The force of mortality can be thought as an instantaneous death rate, the probability that a life subject to a force of mortality $\mu_{x,t}$ dies in the interval of time $(t, t+\mathrm{d}t)$ is approximately $\mu_{x,t} \cdot \mathrm{d}t$ where $\mathrm{d}t$ is small.

The force of mortality $\mu_{x,t}$ for human populations varies slowly in both x and t and a standard assumption is that $\mu_{x,t}$ is constant over each year of age, i.e., from exact age x to exact age x+1, and over each calendar year, i.e., from exact time t to exact time t+1. Thus,

$$\mu_{x+u,t+v} = \mu_{x,t} \text{ for } 0 \le u < 1, 0 \le v < 1, \qquad (1)$$

and so $\mu_{x,t}$ approximate the mid-year force of mortality $\mu_{x+0.5,t+0.5}$.

We suppose that $d_{x,t}$ is a realization of a Poisson variable $D_{x,t}$:

$$D_{x,t} \sim Po\left(e_{x,t} \cdot \mu_{x,t}\right), \quad (2)$$

The expected values are the product of exposures $e_{x,t}$ and the force of mortality $\mu_{x,t}$.

Assumption (2) leads us to the estimates of $\mu_{x,t}$ as

$$\widehat{\mu}_{x,t} = \frac{d_{x,t}}{e_{x,t}}, \quad (3)$$

or in a matrix form $\widehat{\boldsymbol{\mu}} = \dfrac{\boldsymbol{D}}{\boldsymbol{E}}$, that means element-wise division in **R**.

We also consider the mortality rate $q_{x,t}$. This is the probability that an individual aged exactly *x* at exact time *t* will die between *t* and *t* +1.

We have the following relation between the force of mortality and the mortality rate:

$$q_{x,t} = 1 - \exp\left(\int_0^1 -\mu_{x+s,t+s}\, \mathrm{d}s\right) \approx 1 - e^{-\mu_{x,t}}. \quad (4)$$

III. The Mortality Models

We use the following conventions for our models:

- The $\alpha_x, \beta_x^{(1)}$ coefficients will reflect age-related effects.
- The $\kappa_t^{(1)}$, $\kappa_t^{(2)}$ coefficients will reflect time-related effects.
- The γ_c coefficients will reflect cohort effects, where $c = t - x$.

Our models are fitting to historical data.

a) Lee-Carter model

The Lee-Carter model was introduced by Ronald D. Lee and Lawrence Carter in 1992 with the article [10]. The model grew out of their work in the late 1980s and early 1990s attempting to use inverse projection to infer rates in historical demography. The model has been used by the United States Social Security Administration, the US Census Bureau and the United Nations. It has become the most widely used mortality forecasting technique in the world today.

Lee and Carter proposed the following model for the force of mortality:

$$\log \mu_{x,t} = \alpha_x + \beta_x^{(1)} \cdot \kappa_t^{(1)}, \quad (5)$$

with constraints

$$\sum_{x=1}^{n_a} \beta_x^{(1)} = 1, \quad (6)$$

$$\sum_{t=1}^{n_y} \kappa_t^{(1)} = 0. \quad (7)$$

The second constraint implies that, for each *x*, the estimate for α_x will be equal (at least approximately) to the mean over *t* of the $\log \mu_{x,t}$.

By the equation (5) the log of the force mortality is expressed as the sum of an age-specific component α_x that is independent of time and another component that is the product of a time-varying parameter $\kappa_t^{(1)}$ reflecting the general level of mortality and an age-specific component $\beta_x^{(1)}$ that represents how rapidly or slowly mortality at each age varies when the general level of mortality changes.

Interpretation of the parameters in Lee-Carter model is quite simple: $exp(\alpha_x)$ is the general shape of the mortality schedule and the actual forces of mortality change according to overall mortality index $\kappa_t^{(1)}$ modulated by an age response $\beta_x^{(1)}$ (the shape of the $\beta_x^{(1)}$ profile tells which rates decline rapidly and which slowly over time in response of change in $\kappa_t^{(1)}$).

b) Cairns-Blake-Dowd model (C-B-D model)

The original C-B-D model was published in Cairns et. al. [2].

The model fits mortality rates $q_{x,t}$:

$$\operatorname{logit} q_{x,t} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}), \quad (8)$$

where $\operatorname{logit} x = \log\left(\dfrac{x}{1-x}\right)$, $x \in (0,1)$,

and $\bar{x}$ is the mean age in the sample range ($\bar{x}$ = 75).

This model has no constraints.

We calculate the likelihood for all models based on the $\mu_{x,t}$. For a given model we use ϕ to represent the full set of a parameters and the notation for $\mu_{x,t}$ is extended to $\mu_{x,t}(\phi)$, to indicate its dependence on these parameters.

For both models the log-likelihood is:

$$l(\phi; \boldsymbol{D}, \boldsymbol{E}) = \sum_{x,t}\left(d_{x,t} \cdot \log[e_{x,t} \cdot \mu_{x,t}(\phi)] - e_{x,t} \cdot \mu_{x,t}(\phi) - \log(d_{x,t}!)\right), \quad (9)$$

and estimation is by maximum likelihood (MLE).

The Lee-Carter model deals with the force of mortality $\mu_{x,t}$, whereas the C-B-D model with the mortality rate $q_{x,t}$. To ensure a valid comparison between the different models, our analysis of the models for $q_{x,t}$ involves an additional step. For a given set of parameters we calculate the $q_{x,t}$ then we transform these into force of mortality using the identity $\mu_{x,t} = -\log(1 - q_{x,t})$ which comes from (4). ($\log x$ means natural logarithm of *x* throughout the article)

We can calculate the likelihood for all models consistently based on the $\mu_{x,t}$.

For a model with $q_{x,t}$ we use notation $q_{x,t} = q_{x,t}(\phi)$ and we define

$$\mu_{x,t}(\phi) = -\log\left(1 - q_{x,t}(\phi)\right), \tag{10}$$

For practice the fitting of a model is usually only the first step and the main purpose is the forecasting of mortality. A number of forecasting models have been proposed in the past. We are influenced by papers [5] and [8]. For forecasting-time series we use an ARIMA approach. We apply the **R** package *Forecast* - methods and tools for displaying and analysing univariate time series forecasts including automatic ARIMA modelling. The estimated parameters ($\kappa_t^{(1)}$, $\kappa_t^{(2)}$) create a bivariate vector time-series and it is modelled by a multivariate approach.

The estimated age parameters, $\alpha_x, \beta_x^{(1)}$, are assumed invariant over time. This last assumption is certainly an approximation but the method has been very thoroughly tested in Booth at al. [1] and found to work.

IV. Lee-Carter and C-B-D models

We perform main part of our results by a graphical output. In the Figure 1. we have plotted the maximum likelihood estimates for the parameters of the Lee-Carter model, using Czech Republic males data, aged 60 - 90 from the period year 1960 - 2000.

Parameters α_x show their linear nature with respect to age, as has been proposed by Gompertz. α_x is a measure of average log(mortality) by age.

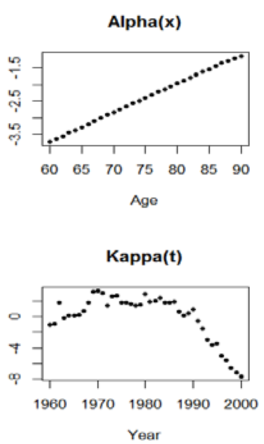

Fig. 1 Parameter Estimates for Lee-Carter model, Degree of freedom: 101 = 2 · $n_a + n_y - 2$. Source: Own Processing

In the Figure 3. we have plotted cross-section of previous 3D graphs for selected ages. Figure 3. shows observed force of mortality and the model fits (on logarithmic scale). The observed mortality data are shown as open circles, whereas model fits are shown as a dashed line.

From the middle of 80´s mortality has been declining for all ages. But the declines have been volatile. This suggests that future mortality will be uncertain and this needs to be taken into account when forecasting.

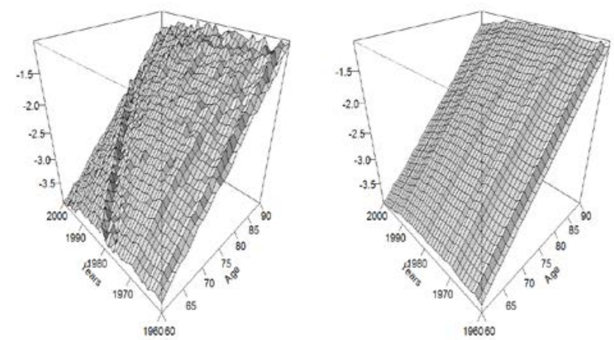

Fig. 2 3D plots of force of mortality: observed (left) and fitted by Lee-Carter model (right). Source: Own Processing

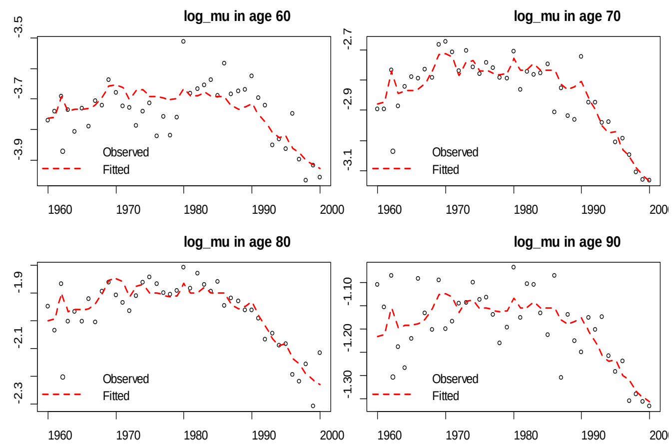

Fig. 3 The $\log \mu$ for selected ages. Source: Own Processing

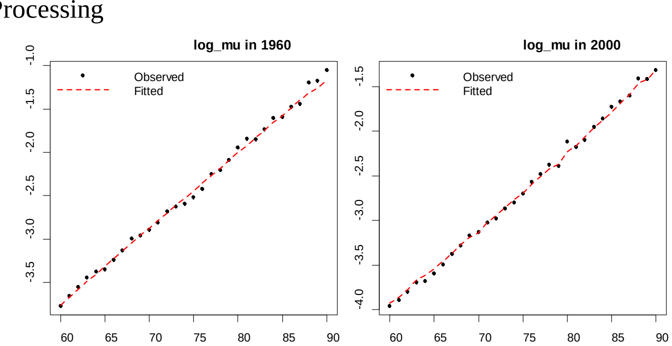

Fig. 4 The $\log \mu$ in selected years. Source: Own Processing

Figure 4. shows validity of the Gompertz law. Gompertz (1823) observed that the force of mortality is approximately linear in age (on log scale) over most of adult life. If we look at the vertical scale of both graphs we can see improvement of mortality for each ages from1960 to 2000.

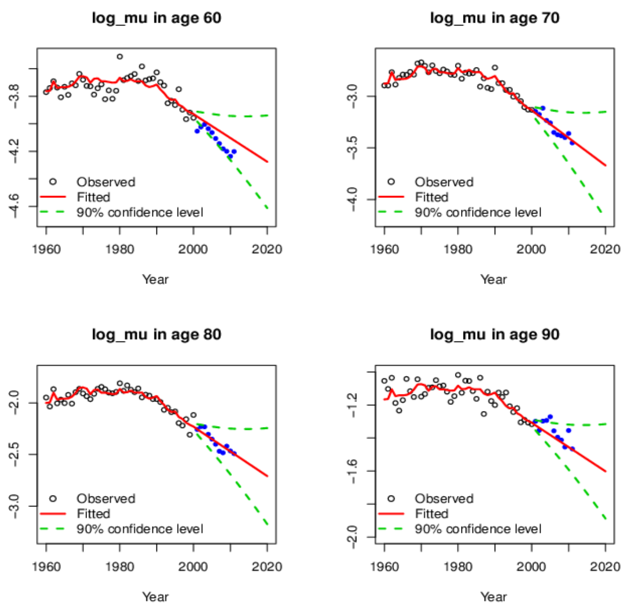

Fig. 5 Forecasted values of $\log \mu$ for selected for Lee Carter model. Source: Own Processing

We used Box-Jenkins approach to fitting an ARIMA model and used it to predict force of mortality for years 2001-2020 using data from 1960-2000. The comparison between the observed values for 2001-2011 and our predicted values provides a test of our results.

Figure 5. shows the forecasted log(mortality) at selected ages under the Lee-Carter approach. We can see that the Lee-Carter model has predictions for younger (60 years old) which is more pessimistic as the tested data from years 2001-2011 show.

The confidence intervals indicate a high level of uncertainty in the future direction of mortality.

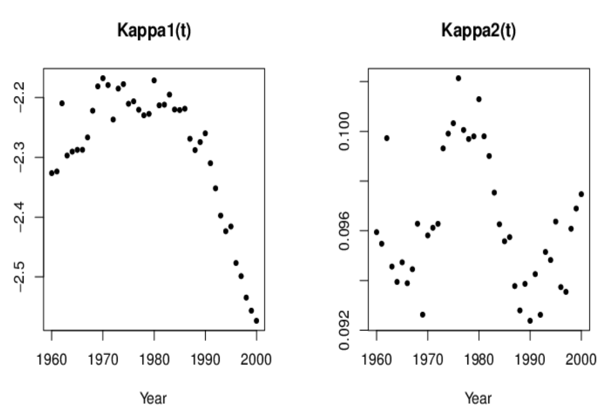

Fig. 6 Parameter Estimates for C-B-D model, DF: 82 = 2 · n_y. Source: Own Processing

In the Figure 6. we have plotted the maximum likelihood estimates for the parameters of the Cairns-Blake-Dowd model, using Czech Republic male's data, aged 60 - 90 from the period year 1960 - 2000.

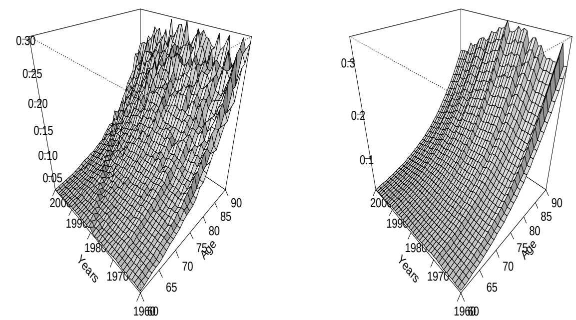

Fig. 7 3D plots of mortality rates: observed (left) and fitted by C-B-D model (right).
Source: Own Processing

Similarly as before we used Box-Jenkins approach to fitting an ARIMA model and using it forecasting mortality rates for years 2001-2020 using data from 1960-2000. The comparison between the observed values for 2001-2011 and our predicted values provides a test of our results.

Figure 8. shows the forecasted logit(mortality) at selected ages under the Cairns-Dowd-Blake model.

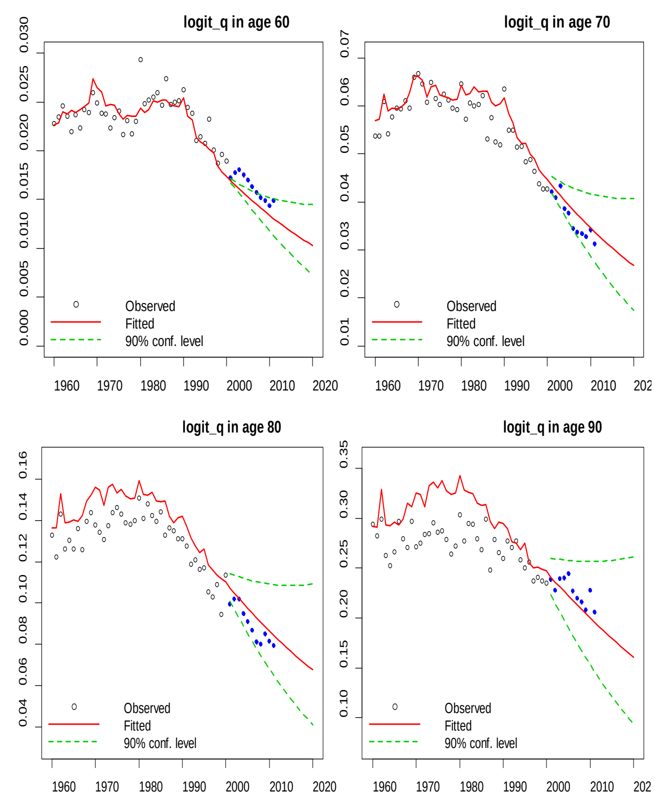

Fig. 8 Forecasted value of logit q for selected for C-B-D model. Source: Own Processing

In the Figures 9.-11. we have plotted the maximum likelihood estimates for the parameters for both models (Lee-Carter and C-B-D) in different range of years.

Figure 9. shows that the parameter $\kappa_t^{(1)}$ values from the Lee-Carter model changes if new data are considered.

This is because:

- When one more year of data is included, the maximum likelihood estimates of all model parameters, that is, α_x, β_x and κ_t for all x and t will be updated.
- Parameter constraints are involved in the estimation process. In particular, the constraint $\sum_t \kappa_t$ re-scales the series of κ_t as new data are included.

For the C-B-D model the inclusion of new data will not affect previous parameters values. We can call this property as "new data invariant".

Reasons for this special property, that adding new data will have no effect on the parameters that are already estimated, is due to no constraint in this model.

Figure 10. and 11. show the data-invariant property of MLE estimates of mortality parameters from the C-B-D model using Czech Republic males data.

$\kappa_t^{(1)}$ in C-B-D model presents the level of the logit-transformed mortality curve. A reduction in $\kappa_t^{(1)}$, that is a parallel downward shift of the logit-transformed mortality curve, represents an overall mortality improvement.

$\kappa_t^{(2)}$ presents the steepness of the logit-transformed mortality curve. An increase in $\kappa_t^{(2)}$, that is an increase in the steepness of the logit-transformed mortality curve, means that mortality (in logit scale) at younger ages improves more rapidly than at older ages.

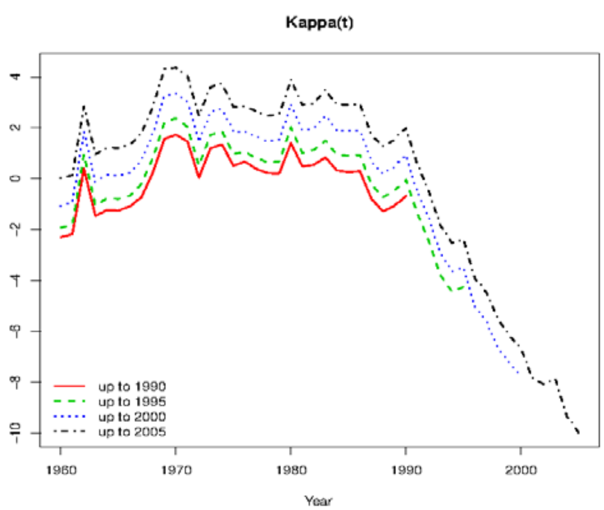

Fig. 9. Parameter $\kappa_t^{(1)}$ estimates for Lee-Carter model. Source: Own Processing

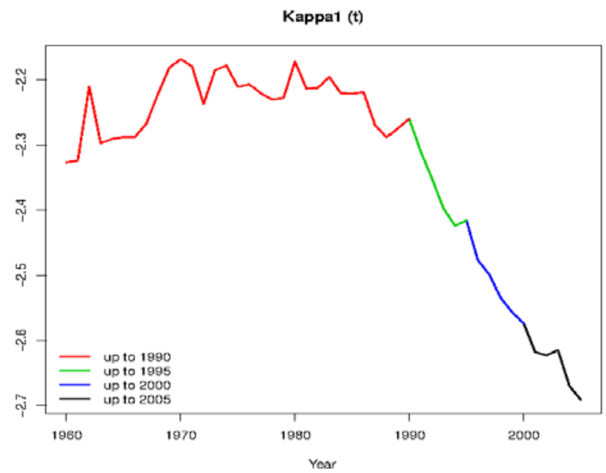

Fig. 10. Parameter $\kappa_t^{(1)}$ estimates for Cairns-Blake-Dowd model. Source: Own Processing

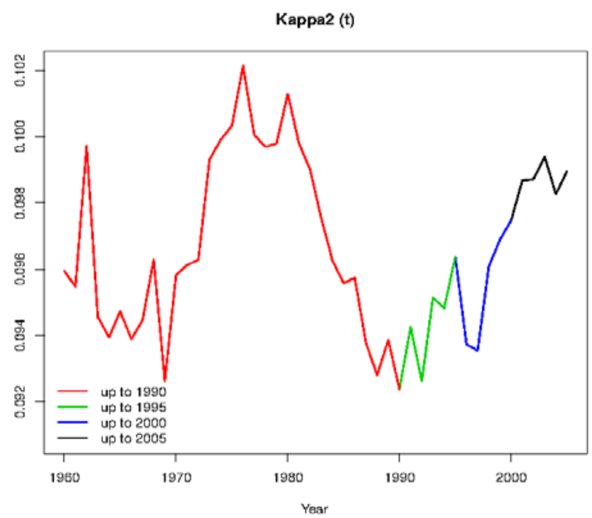

Fig. 11. Parameter $\kappa_t^{(2)}$ estimates for Cairns-Blake-Dowd model. Source: Own Processing

If we look better on 3D plots (Figures 2. and 7.) we can see a "kink" which draws from the bottom (age around 60 in 1980) upwards. This "kink" is connected with cohort born around 1920 (those born after World War I).

We can see that nor Lee-Carter model not C-B-D model can describe this "kink". Therefore we propose to apply an extension of the C-B-D model that incorporates a cohort effect.

V. Cairns-Blake-Dowd Model with Cohort Effect

Cairns et. al. (2009) in [2] introduced the C-B-D model with cohort effect:

$$\log \mu_{x,t} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_c , \qquad (11)$$

with constraints

$$\sum_{c=1}^{n_c} \gamma_c = 0\,, \qquad (12)$$

$$\sum_{c=1}^{n_c} c \cdot \gamma_c = 0\,, \qquad (13)$$

where n_c is the number of cohorts ($n_c = n_a + n_y - 1$)

In the Figure 12. we have plotted the maximum likelihood estimates for the parameters of the Cairns-Blake-Dowd model with cohort effect, using Czech Republic males data, aged 60 - 90 from the period year 1960 - 2000.

We used Box-Jenkins approach again to fitting an ARIMA model and using it forecasting mortality rates for years 2001-2020 using data from 1960-2000. The comparison between the observed values for 2001-2011 and our predicted values provides a test of our results.

3D plots (Figure 13.) show that this model gives an improved fit.

Figure 14. shows the forecasted logit(mortality) at selected ages under the Cairns-Dowd-Blake model with cohort effect. One notable feature is that the model has wider confidence interval for older males, but it seems to be reasonable.

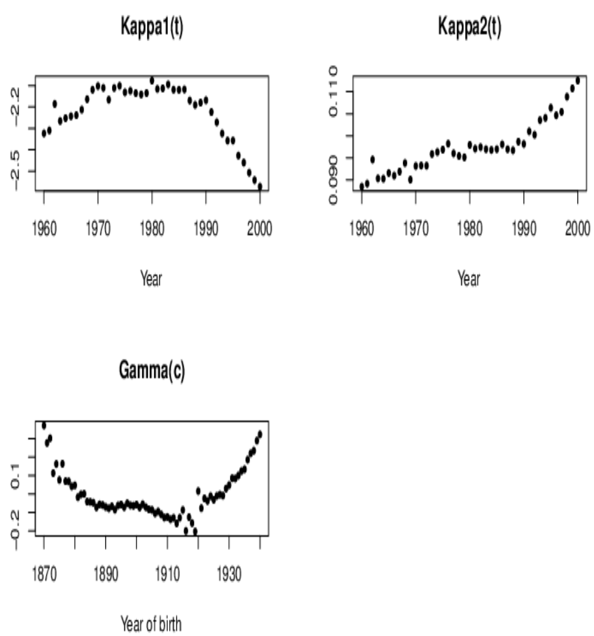

Fig. 12 Parameter Estimates for C-B-D model with cohort effect, DF: 151 = 2 · n_a + n_c −2. Source: Own Processing

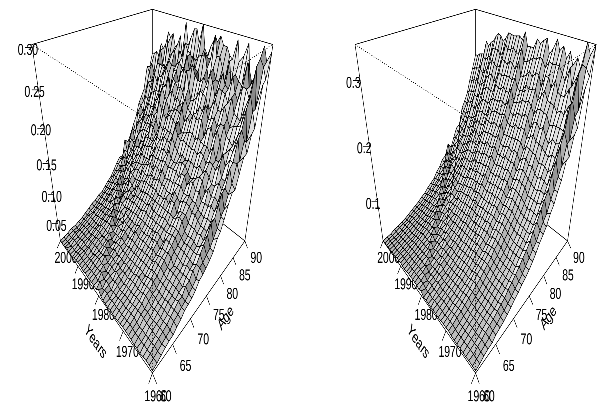

Fig. 13 3D plots of mortality rates: observed (left) and fitted by C-B-D model with cohort effect (right). Source: Own Processing

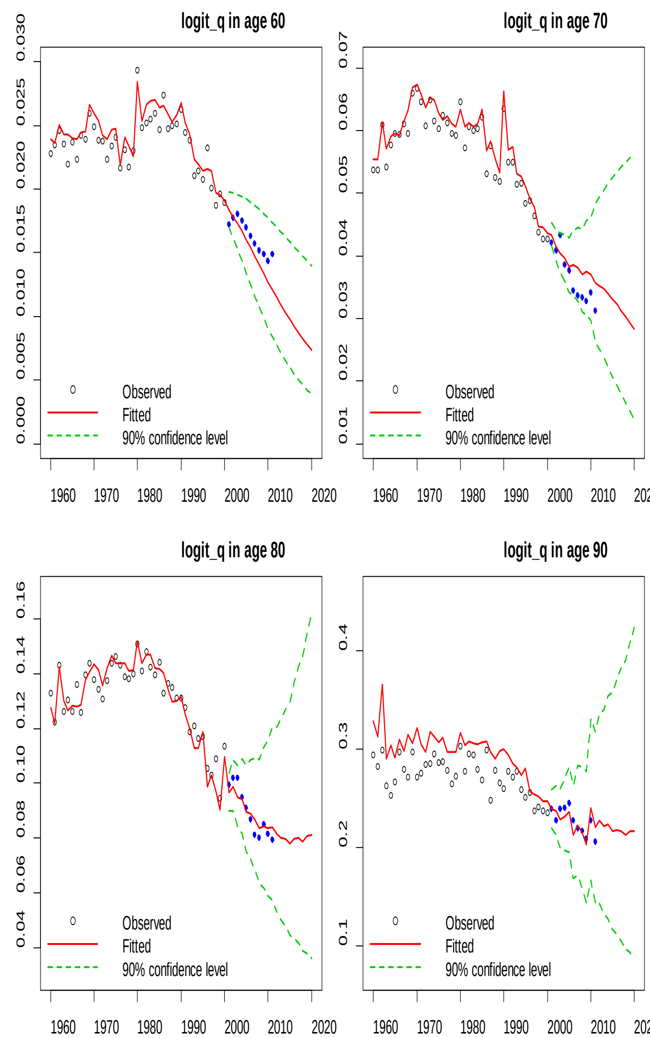

Fig. 14 Forecasted value of logit q for selected ages for C-B-D model with cohort effect. Source: Own Processing

VI. Conclusion

National governments and the WHO announce life expectancies of different populations every year. To financial institutions, life expectancy is not an adequate measure of risk, because all it does not give any idea about how mortality rates at different ages vary over time. On the other hand, indicators of longevity risk cannot be too complicated. An indicator that is composed by a huge array of numbers is difficult to interpret and will lose the purpose as a “summary” of a mortality pattern.

If we compare only first two models i.e. Lee-Carter model and Cairns-Blake-Dowd model. We propose using the C-B-D model mortality parameters ($\kappa_t^{(1)}$, $\kappa_t^{(2)}$) as a longevity risk indicator. It is a “simple“ summary of a mortality pattern. The indicator contains only two set of numbers, $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$, each of which is readily interpretable and they together tell how mortality rates at different ages change with time.

The main reason why we propose is that it has the new-data invariant property. This property is important; because, as a proper indicator, we cannot allow new data to alter the index values of previous years.

We could see that there is a significant cohort effect in mortality improvements and we found that the Cairns-Dowd – Blake model with cohort effect fits our data best.

These models will only be reliable if past trends continue. The medical advances can invalidate projection by changing the trend.

We have attempted to explain mortality improvements for males aged 60-90 in the Czech Republic using three stochastic mortality models. It is not the aim of this paper to provide an exhaustive comparison of all the mortality models in existence. There is the obvious question of whether results based on the whole population are applicable to annuitants and pensions.

Afterwards we can turn to the industry requirement to forecast future mortality.

But forecasting of mortality should be approached with both caution and humility. Any prediction is unlikely to be correct.

There is a need for awareness of model risk when assessing longevity-related liabilities, especially for annuities and pensions. The fact that parameters can be estimated does not imply that they can sensibly be forecast.

Such forecasting should enable actuaries to examine the financial consequences with different models and hence to come to an informed assessment of the impact of longevity risk on the portfolios in their care.

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Model fitting was done in **R**, which was also used for all graphs.

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References

[1] H. Booth, L. Tickle, L. Smith, “Evaluation of the variants of the Lee-Carter method of forecasting mortality: a multi-country comparison,” *New Zealand Population Review*, 31, p. 13-34, 2005.

[2] A. J. G. Cairns, D. Blake, K. Dowd, G.D. Coughlan, et. al., “A quantitative comparison of stochastic mortality models using data from England and Wales and the United States,” *North American Actuarial Journal*, 13, No. 1, p. 1-35, 2009.

[3] E. L. Croitoru, F. Alexandru, S. A. Toader, “Sustainability of the Public Pensions System in Terms of Demographic Factors. Case Study: Romania,“ *International Journal of Economics and Statistics*, ISSN: 2309-0685, Volume 2, p. 179-185, 2014.

[4] I. D. Currie, M. Durban, P. H. C. Eilers, “Smoothing and forecasting mortality rates,” *Statistical Modelling*, **4**, p. 279-298, 2004.

[5] C. P. Da Veiga, C. R. P. Da Veiga, A. Catapan, U. Tortato, W. V. Da Silva, “Demand Forecasting in Food Retail: A Comparison between the Holt-Winters and ARIMA Models,” *WSEAS Transactions on Business and Economics*, ISSN / E-ISSN: 1109-9526 / 2224-2899, Volume 11, Art. #56, p. 608-614, 2014.

[6] J. Gogola, “A comparison of Lee-Carter and Cairns-Blake-Dowd stochastic mortality models,” *Proceedings of the 16th International Conference on Mathematical Methods, Computational Techniques and Intelligent Systems* (MAMECTIS´14), WSEAS Press, Lisbon, ISBN: 978-960-474-396-4, p. 130-134, 2014.

[7] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www. mortaliy.org

[8] Jun M. Liu, “Nonlinear Forecasting using Nonparametric Transfer Function Models,” *WSEAS Transactions on BUSINESS and ECONOMICS*, Issue 5, Volume 6, May 2009, ISSN 1109-9526.

[9] C. Lazar, M. Lazar, “The Actual and Perspective Situation of Romania's Population,” *WSEAS TRANSACTIONS on BUSINESS and ECONOMICS*, Issue 1, Volume 4, January 2007, ISSN 1109-9526.

[10] R. D. Lee, L. Carter, “Modelling and forecasting the time series of U.S. mortality,” *Journal of the American Statistical Association*, 87, p. 659-671, 1992.

[11] V. Pacáková, P. Jindrová, “Quantification of Selected Factors of Longevity,” *Proceedings of the 2014 International Conference on Mathematical Methods in Applied Sciences* (MMAS’14), Saint Petersburg State Polytechnic University, ISBN 978-1-61804-251-4, p. 170 – 174, 2014.

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