Symmetric Association Schemes and Generalized Krein Parameters

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***Abstract*—This paper presents a generalization of the Krein parameters of a symmetric association scheme and via this generalization some bounds on the classical Krein parameters are deduced.**

***Keywords*-Association scheme, matrix analysis, strongly regular graph.**

I. Introduction

THIS paper is an extension of the work published in the conference proceedings [8]. Here, we present a generalization of the Krein parameters of an association scheme and obtain some results on them. This work is also an extension of the work developed for strongly regular graphs in [6], [7], since these graphs are symmetric association schemes with two classes. This paper is organized as follows. In Section II we will present the basic definitions and properties of symmetric association schemes which are necessary for our work. All the concepts presented are described in detail, for instance, in [1]. In Section III we generalize the Krein parameters of an association scheme and establish some bounds for these generalizations. Finally, in Section IV, we present some conclusions and examples which confirm our results.

II. Preliminaries on Association schemes

Along this section we will present all the definitions and results which are necessary and relevant to the development of our work in the further sections. For extensive reading the author may consult [1].

A *symmetric association scheme*, Ω, with d *associate classes* on a finite set X is a partition of $X \times X$ into sets $R_0, R_1, \ldots, R_d$, which are relations on X satisfying the following axyoms: (i) $R_0 = \{(x,x) : x \in X\}$; (ii) if $(x,y) \in R_i$, then $(y,x) \in R_i$, for all x, y in X and i in $\{0,1,\ldots,d\}$; (iii) for all i, j, l in $\{0,1,\ldots,d\}$ there is an integer p^l_{ij} such that, for all (x,y) in R_l

$$|\{z \in X : (x,z) \in R_i \text{ and } (z,y) \in R_j\}| = p^l_{ij}.$$

The numbers p^l_{ij} are called the *intersection numbers* of Ω. It is usual to observe the intersection numbers as the entries of the so called *intersection matrices* $L_0, L_1, \ldots, L_d$, with $(L_i)_{lj} = p^l_{ij}$, where $L_0 = I_n$.

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This definition is due to Bose and Shimamoto, [2], and by axiom (ii) the relations R_i are all symmetric. A more general definition of non necessarily symmetric association schemes can be seen in [4]. Along this text we will only consider symmetric association schemes.

One can describe the associate classes $R_0, R_1, \ldots, R_d$ of a symmetric association scheme, Ω, by their adjacency matrices $A_0, A_1, \ldots, A_d$, where each A_i is a matrix of order n defined by $(A_i)_{xy} = 1$, if $(x,y) \in R_i$, and $(A_i)_{xy} = 0$, otherwise. We also have the corresponding axyoms for these matrices: (a) $A_0 = I_n$; (b) $\sum_{i=0}^{d} A_i = J_n$; (c) $A_i = A_i^\top$, $\forall i \in \{0,1,\ldots,d\}$; (d) $A_iA_j = \sum_{l=0}^{d} p^l_{ij} A_l$, $\forall i, j \in \{0,1,\ldots,d\}$. Regard that I_n and J_n stand for the identity matrix and the all ones matrix of order n, respectively, and $A^\top$ denotes the transpose of A. Note that equality (b) implies that the matrices A_i, $i \in 0,1,\ldots,d$, are linearly independent. It is also well known (see [1, Lemma 1.3]) that the symmetry of the scheme asserts that $p^l_{ij} = p^l_{ji}$ and thus $A_iA_j = A_jA_i$, for all $i, j \in \{0,1,\ldots,d\}$.

We can acknowledge $A_1, A_2, \ldots, A_d$ as adjacency matrices of undirected simple graphs $G_1, G_2, \ldots, G_d$, with common vertex set V. Each graph G_i is regular with valency n_i. The matrices $A_0, A_1, \ldots, A_d$ of a symmetric association scheme generate a commutative algebra, $\mathcal{A}$, with dimension $d+1$, of symmetric matrices with constant diagonal. This algebra is called the *Bose-Mesner algebra* of the scheme because it was firstly studied by these two mathematicians in [3]. Note that $\mathcal{A}$ is an algebra with respect to the usual matrix product as well as to the *Hadamard* (or *Schur*) *product*, defined for two matrices A, B of order n as the componentwise product: $(A \circ B)_{ij} = A_{ij}B_{ij}$. The algebra $\mathcal{A}$ is commutative and associative relatively to this product with unit J_n.

An element E in $\mathcal{A}$ is an *idempotent* if $E^2 = E$. Two idempotents E and F in $\mathcal{A}$ are orthogonal if $EF = 0$. The Bose-Mesner algebra $\mathcal{A}$ has a unique basis of minimal orthogonal idempotents $\{E_0, \ldots, E_d\}$ such that $E_iE_j = \delta_{ij}E_i$, $\sum_{i=0}^{d} E_i = I_n$, where $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, otherwise, for any i, j natural numbers. Let $\mathcal{A}$ be an association scheme with d classes. If $A_j \in \mathcal{A}$, $j \in \{0,1,\ldots,d\}$ has $d+1$ distinct eigenvalues, namely $\lambda_0, \lambda_1, \ldots, \lambda_d$, the idempotents E_i can be obtained as the projectors associated to the matrix A_j through the equality:

$$E_i = \prod_{l=0, l\neq i}^{d} \frac{A_j - \lambda_l I_n}{\lambda_i - \lambda_l}. \quad (1)$$

Along this paper we will denote the rank of each E_i by μ_i, $i \in \{0,1,\ldots,d\}$.

Besides the intersection numbers already introduced in the beginning of the section each association scheme contains three more families of parameters: the eigenvalues, the dual eigenvalues and the Krein parameters. In fact, there are scalars $p_i(j)$ and $q_i(j)$ such that, for all $i \in 0, 1, \ldots, d$, we have

$$A_i = \sum_{j=0}^{d} p_i(j) E_j \text{ and} \tag{2}$$

$$E_i = \sum_{j=0}^{d} q_i(j) A_j, \tag{3}$$

where the numbers $p_i(j)$ and $q_i(j)$ are the *eigenvalues* and the *dual eigenvalues* of the scheme, respectively. We also define the *eigenmatrix*, $P = (P_{ij})$, and the *dual eigenmatrix*, $Q = (Q_{ij})$, each with dimension $(d+1)\times(d+1)$, as $P_{ij} = p_j(i)$ and $Q_{ij} = q_j(i)$, respectively. From (2) and (3) one can deduce that $PQ = I_n$. As a consequence, the dual eigenvalues are determined by the eigenvalues of $\mathcal{A}$.

Finally, the *Krein parameters* discovered by Scott, [9], of an association scheme with d classes are the numbers $q^l_{(i,j;1,1)}$, with $i, j, l \in \{0, 1, \ldots, d\}$, such that

$$E_i \circ E_j = \sum_{l=0}^{d} q^l_{(i,j;1,1)} E_l. \tag{4}$$

This notation will become clear later, in Section III, with the introduction of the generalized Krein parameters of an association scheme. These parameters can be seen as dual parameters of the intersection numbers and they are determined by the eigenvalues of the scheme. Also, the Krein parameters can be considered as the entries of the matrices $L^*_0, L^*_1, \ldots, L^*_d$, such that $(L^*_i)_{lj} = q^l_{ij}$, which are called the *dual intersection matrices* of the scheme.

Now we will emphasize some properties of the matrices P and Q that we will use in the proofs of some of the theorems that we will present in this paper.

$$Q(i,j)Q(i,k) = \sum_{l=0}^{2} q^l_{(j,k;1,1)} Q(i,l); \tag{5}$$

$$|Q(i,j)| \leq \frac{\mu_j}{n}; \tag{6}$$

$$|P(i,j)| \leq n_j; \tag{7}$$

$$\sum_{i=0}^{d} n_i Q(i,j)Q(i,k) \leq \frac{\mu_j}{n}\delta(j,k). \tag{8}$$

III. Generalized Krein parameters and some bounds

In what follows we generalize the Krein parameters of an association scheme. Let $A_0, A_1, \ldots, A_d$ be the adjacency matrices of an association scheme with d classes, Ω, on a finite set of order n, $\mathcal{A}$ the underlying Bose-Mesner algebra and $\mathcal{S} = \{E_0, E_1, \ldots, E_d\}$ be the associated unique basis of minimal orthogonal idempotents. Let p be a natural number and denote by $\mathcal{M}_n(\mathbb{R})$ the set of square matrices of order n with real entries. Then, for $B \in \mathcal{M}_n(\mathbb{R})$, we denote by $B^{\circ p}$ the *Hadamard power* of order p of B, with $B^{\circ 1} = B$.

Now, we introduce the following compact notation for the Hadamard powers of the elements of $\mathcal{S}$. Let x, y, α and β be natural numbers such that $0 \leq \alpha, \beta \leq d$. Then we define $E^{\circ x}_\alpha = (E_\alpha)^{\circ x}$ and $E^{\circ x,y}_{\alpha,\beta} = (E_\alpha)^{\circ x} \circ (E_\beta)^{\circ y}$. Note that, when $\alpha = \beta$, there is a connection between the two notations: $E^{\circ x,y}_{\alpha,\alpha} = E^{\circ x+y}_\alpha$.

Since the Bose-Mesner algebra $\mathcal{A}$, that is generated by the adjacency matrices of Ω, is closed under the Hadamard product, then there exist real numbers $q^i_{(\alpha,\beta;x,y)}$ such that

$$E^{\circ x,y}_{\alpha,\beta} = \sum_{i=0}^{d} q^i_{(\alpha,\beta;x,y)} E_i. \tag{9}$$

We call the parameters $q^i_{(\alpha,\beta;x,y)}$, $i \in \{0, 1, \ldots, d\}$, the *generalized Krein parameters* of the association scheme Ω, since for $x = y = 1$ we obtain the "classical" Krein parameters already presented in (4). With this notation, the greek letters are used as idempotent indices and the latin letters are used as exponents of Hadamard powers.

Next we present a formula to compute the generalized Krein parameters by making use of just the entries of the matrices P and Q.

Theorem 1: Let Ω be a symmetric association scheme with d classes and let i, x, y, α and β be natural numbers such that $0 \leq i, \alpha, \beta \leq d$. Then the generalized Krein parameters of Ω, defined in (9), satisfy the equality

$$q^i_{(\alpha,\beta;x,y)} = \sum_{t=0}^{d} (Q(t,\alpha))^x (Q(t,\beta))^y P(i,t). \tag{10}$$

Proof: We have

$$E^{\circ x,y}_{\alpha,\beta} = E^{\circ x}_\alpha \circ E^{\circ y}_\beta = \sum_{t=0}^{d} (Q(t,\alpha))^x A_t \circ \sum_{t=0}^{d} (Q(t,\beta))^y A_t.$$

It follows that $E^{\circ x}_\alpha \circ E^{\circ y}_\beta = \sum_{t=0}^{d} (Q(t,\alpha))^x (Q(t,\beta))^y A_t$. But then, from (2)-(3) one can write

$$E^{\circ x}_\alpha \circ E^{\circ y}_\beta E_i = \sum_{t=0}^{d} (Q(t,\alpha))^x (Q(t,\beta))^y A_t E_i.$$

Then, we have

$$q^i_{(\alpha,\beta;x,y)} E_i = \sum_{t=0}^{d} (Q(t,\alpha))^x (Q(t,\beta))^y P(i,t) E_i.$$

Therefore (10) follows. ∎

From Theorem 1 we obtain the following consequence.

Corollary 1: Let Ω be a symmetric association scheme with d classes and let i, α and β be natural numbers such that $0 \leq i, \alpha, \beta \leq d$. Then, the classical Krein parameters of Ω satisfy

$$q^i_{(\alpha,\beta;1,1)} = \sum_{t=0}^{d} Q(t,\alpha) Q(t,\beta) P(i,t).$$

Now we present some bounds on the generalized Krein parameters. They can be obtained by making use of the properties (5)-(8).

Theorem 2: Consider a symmetric association scheme Ω with d classes. Then, for all natural numbers i, x, y, α and β such that $0 \leq i, \alpha, \beta \leq d$, we have

$$0 \leq q^i_{(\alpha,\beta;x,y)} \leq 1.$$

The following result presents another upper-bound on the generalized Krein parameters associated to only one idempotent.

Theorem 3: Let Ω be a symmetric association scheme with d classes on a finite set of order n and let i, x, y and α be natural numbers such that $0 \leq i, \alpha, \leq d$. Then the generalized Krein parameter $q^i_{(\alpha,\alpha;x,y)}$, with $x + y = m$, satisfies

$$q^i_{(\alpha,\alpha;x,y)} \leq \left(\frac{\mu_\alpha}{n}\right)^{m-1}.$$

Proof: Since $q^i_{\alpha m} = \sum_{t=0}^d (Q(t,\alpha))^m P(i,t)$ and $|Q(t,\alpha)| \leq \frac{\mu_\alpha}{n}$, and $|P(i,t)| \leq n_t$ see (i)-(iv), we conclude that

$$\begin{aligned}
q^i_{\alpha m} &= \sum_{t=0}^d (Q(t,\alpha))^m P(i,t) \\
&= \left|\sum_{t=0}^d (Q(t,\alpha))^m P(i,t)\right| \\
&\leq \sum_{t=0}^d |(Q(t,\alpha))^m|\,|P(i,t)| \\
&\leq \sum_{t=0}^d |Q(t,\alpha)|^{m-2} (Q(t,\alpha))^2 n_t \\
&\leq \sum_{t=0}^d \left(\frac{\mu_\alpha}{n}\right)^{m-2} (Q(t,\alpha))^2 n_t \\
&\leq \left(\frac{\mu_\alpha}{n}\right)^{m-2} \sum_{t=0}^d (Q(t,\alpha))^2 n_t \\
&\leq \left(\frac{\mu_\alpha}{n}\right)^{m-2} \frac{\mu_\alpha}{n} \\
&= \left(\frac{\mu_\alpha}{n}\right)^{m-1}.
\end{aligned}$$

■

Proceeding in an analogous manner as we have done in the proof of Theorem 3, we obtain the following result.

Theorem 4: Let Ω be a symmetric association scheme with d classes on a finite set of order n and let i, x, α and β be natural numbers such that $0 \leq i, \alpha, \beta \leq d$ and $\alpha < \beta$. Then, the generalized Krein parameter $q^i_{(\alpha,\beta;x,x)}$ satisfies the inequality

$$q^i_{(\alpha,\beta;x,x)} \leq \left(\frac{1}{2}\right)^x \left(\frac{2(\max\{\mu_\alpha,\mu_\beta\})^2}{n^2}\right)^{x-1}.$$

Proof: Since

$$q^i_{\alpha\beta xy} = \sum_{d=0}^2 (Q(d,\alpha))^x (Q(d,\beta))^y P(i,d)$$

and $|Q(d,\alpha)| \leq (\frac{\mu_\alpha}{n})$, $|Q(d,\beta)| \leq \frac{\mu_\beta}{n}$, $|P(i,d)| \leq n_d$ (see (7) and (8)), we conclude that

$$\begin{aligned}
q^i_{\alpha\beta xx} &= \sum_{t=0}^d (Q(t,\alpha))^x (Q(t,\beta))^x P(i,t) \\
&= \left|\sum_{t=0}^d ((Q(t,\alpha))(Q(t,\beta)))^x P(i,t)\right| \\
&\leq \sum_{t=0}^d \left(\frac{1}{2}\right)^x \left((Q(t,\alpha))^2 + (Q(t,\beta))^2\right)^x |P(i,t)| \\
&\leq \sum_{t=0}^d \left(\frac{1}{2}\right)^x \left((Q(t,\alpha))^2 + (Q(t,\beta))^2\right)^{x-1} \\
&\times \left((Q(t,\alpha))^2 + (Q(t,\beta))^2\right) n_t \\
&\leq \sum_{t=0}^d \left(\frac{1}{2}\right)^x \left(\left(\frac{\mu_\alpha}{n}\right)^2 + \left(\frac{\mu_\beta}{n}\right)^2\right)^{x-1} \\
&\times \left((Q(t,\alpha))^2 + (Q(t,\beta))^2\right) n_t \\
&\leq \sum_{t=0}^d \left(\frac{1}{2}\right)^x \left(2\frac{(\max\{\mu_\alpha,\mu_\beta\})^2}{n^2}\right)^{x-1} \\
&\times \left((Q(t,\alpha))^2 + (Q(t,\beta))^2\right) n_t \\
&\leq \left(\frac{1}{2}\right)^x \left(2\frac{(\max\{\mu_\alpha,\mu_\beta\})^2}{n^2}\right)^{x-1} \\
&\times \sum_{t=0}^d \left((Q(t,\alpha))^2 + (Q(t,\beta))^2\right) n_t \\
&\leq \left(\frac{1}{2}\right)^x \left(2\frac{(\max\{\mu_\alpha,\mu_\beta\})^2}{n^2}\right)^{x-1} \left(\frac{\mu_\alpha}{n} + \frac{\mu_\beta}{n}\right) \\
&\leq \left(\frac{1}{2}\right)^x \left(2\frac{(\max\{\mu_\alpha,\mu_\beta\})^2}{n^2}\right)^{x-1} \left(\frac{\mu_\alpha}{n} + \frac{\mu_\beta}{n}\right) \\
&\leq \left(\frac{1}{2}\right)^x \left(2\frac{(\max\{\mu_\alpha,\mu_\beta\})^2}{n^2}\right)^{x-1} \left(\frac{n-1}{n}\right) \\
&\leq \left(\frac{1}{2}\right)^x \left(2\frac{(\max\{\mu_\alpha,\mu_\beta\})^2}{n^2}\right)^{x-1}.
\end{aligned}$$

■

From Theorems 3 and 4 we conclude the following corollary that states the above bounds for the classical Krein parameters of association schemes.

Corollary 2: Let Ω be a symmetric association scheme with d classes on a finite set of order n and let i, α and β be natural numbers such that $0 \leq i, \alpha, \beta \leq d$ and $\alpha < \beta$.Then:

(i) $q^i_{(\alpha,\alpha;1,1)} \leq \frac{\mu_\alpha}{n}$;
(ii) $q^i_{(\alpha,\beta;1,1)} \leq \frac{1}{2}$.

IV. CONCLUSIONS

From the analysis of the generalized Krein parameters we have deduced new upper-bounds over the generalized Krein parameters of a symmetric association scheme. Finally, from the results of Corollary 2, we present upper-bounds on the classical Krein parameters of a symmetric association scheme that we will show that cannot be improved by presenting some examples.

Example 1: In this example we consider association schemes with two classes which are equivalent to strongly regular graphs.

(a) Let us consider the family of strongly regular graphs known as the conference graphs. A member of this family of order n satisfies $\mu_0 = 1, \mu_1 = \frac{n-1}{2}$ and $\mu_2 = \frac{n-1}{2}$. Also, we have: $q^0_{(1,1;1,1)} = 1/2 - 1/2n = \mu_1/n$. Therefore, the upper-bound presented in (i) of Corollary 2 is attained.

(b) Now, we consider the family of strongly regular graphs known as the cocktail party graphs. For a member of this family of order $2l$ we have:

$$q^1_{(1,2;1,1)} = \frac{l-1}{2l},$$

and therefore the upper-bound presented in (ii) of Corollary 2 is asymptotically attained.

Example 2: In this example we present a family of association schemes with three classes constructed from symmetric designs. This family has an infinite number of elements and it is presented and studied in [10], where the following definition can be seen.

Let $\mathcal{P}$ be a set of points and $\mathcal{B}$ be a set of blocks, where a *block* is a subset of $\mathcal{P}$. Then, the ordered pair $(\mathcal{P}, \mathcal{B})$ is a *symmetric design* with parameters (n, k, c), with $c < k$, if it satisfies the following properties:

(i) $\mathcal{B}$ is a subset of the power set of $\mathcal{P}$;
(ii) $|\mathcal{P}| = |\mathcal{B}| = n$;
(iii) $\forall b \in \mathcal{B}$, $|b| = k$;
(iv) $\forall p \in \mathcal{P}$, $|\{b \in \mathcal{B} : p \in b\}| = k$;
(v) $\forall p_1, p_2 \in \mathcal{P}$, $p_1 \neq p_2$, $|\{b \in \mathcal{B} : p_1, p_2 \in b\}| = c$;
(vi) $\forall b_1, b_2 \in \mathcal{B}$, $b_1 \neq b_2$, $|\{p \in \mathcal{P} : p \in b_1 \wedge p \in b_2\}| = c$.

Given a symmetric design with parameters (n, k, c), we build a three class association scheme, as in [10], in the following manner. Let $X = \mathcal{P} \cup \mathcal{B}$. We define the following relations in $X \times X$:

$$R_0 = \{(x,x) : x \in X\};$$
$$R_1 = \{(x,y) \in \mathcal{P} \times \mathcal{B} : x \in y\} \cup \{(y,x) \in \mathcal{B} \times \mathcal{P} : x \in y\};$$
$$R_2 = \{(x,y) \in \mathcal{P} \times \mathcal{P} : x \neq y\} \cup \{(x,y) \in \mathcal{B} \times \mathcal{B} : x \neq y\};$$
$$R_3 = \{(x,y) \in \mathcal{P} \times \mathcal{B} : x \notin y\} \cup \{(y,x) \in \mathcal{B} \times \mathcal{P} : x \notin y\}\ .$$

Through the axioms $(i) - (vi)$ of a symmetric design it is proved that R_0, R_1, R_2, R_3 constitute an association scheme with three classes over X. From the relations above we compute the intersection matrices of the association scheme, given by $L_0 = I_4$,

$$L_1 = \begin{pmatrix} 0 & k & 0 & 0 \\ 1 & 0 & k-1 & 0 \\ 0 & c & 0 & k-c \\ 0 & 0 & k & 0 \end{pmatrix},$$
$$L_2 = \begin{pmatrix} 0 & 0 & n-1 & 0 \\ 0 & k-1 & 0 & n-k \\ 1 & 0 & n-2 & 0 \\ 0 & k & 0 & n-k-1 \end{pmatrix},$$
$$L_3 = \begin{pmatrix} 0 & 0 & 0 & n-k \\ 0 & 0 & n-k & 0 \\ 0 & k-c & 0 & n-2k+c \\ 1 & 0 & n-k-1 & 0 \end{pmatrix}.$$

Now, using axioms $(a) - (d)$ of the matrices of the Bose-Mesner algebra, $\mathcal{A} = \{A_0, A_1, A_2, A_3\}$, we obtain:

- $A_0 \times A_i = A_i \times A_0 = A_i$, for $i \in \{0, 1, 2, 3\}$;
- $A_1 \times A_1 = kA_0 + cA_2$;
- $A_1 \times A_2 = A_2 \times A_1 = (k-1)A_1 + kA_3$;
- $A_1 \times A_3 = A_3 \times A_1 = (k-c)A_2$;
- $A_2 \times A_2 = (n-1)A_0 + (n-2)A_2$;
- $A_2 \times A_3 = A_3 \times A_2 = (n-k)A_1 + (n-k-1)A_3$;
- $A_3 \times A_3 = (n-k)A_0 + (n-2k+c)A_2$.

Now we can calculate the powers of A_1 to obtain the following polynomial:

$$p_{A_1}(\lambda) = \lambda^4 + (-k^2 - k + c)\lambda^2 + k^2(k-c), \qquad (11)$$

such that $p_{A_1}(A_1) = \mathcal{O}_n$, where $\mathcal{O}_n$ denotes the n dimensional null matrix. Then A_1 has four distinct eigenvalues and therefore the least natural number such that the set $\{I_n, A_1, A_1^2, \ldots, A_1^k\}$ is linear dependent is 4. Then, we conclude that the polynomial (11) is the minimal polynomial of A_1.

Applying formula (1) to the matrix A_1, considering the eigenvalues of the polynomial (11), $\lambda_0 = k$, $\lambda_1 = -k$, $\lambda_2 = \sqrt{k-c}$ and $\lambda_3 = -\sqrt{k-c}$, and taking into account the equality

$$(n-1)c = k(k-1), \qquad (12)$$

satisfied by these symmetric designs with parameters (n, k, c), see [5], we obtain the elements of the unique basis of minimal orthogonal idempotents of $\mathcal{A}$:

$$E_0 = \frac{A_0 + A_1 + A_2 + A_3}{2n} = \frac{J_n}{2n};$$
$$E_1 = \frac{A_0 - A_1 + A_2 - A_3}{2n};$$
$$E_2 = \frac{(n-1)\sqrt{k-c}A_0 + (n-k)A_1 - \sqrt{k-c}A_2 - kA_3}{2n\sqrt{k-c}};$$
$$E_3 = \frac{(n-1)\sqrt{k-c}A_0 - (n-k)A_1 - \sqrt{k-c}A_2 + kA_3}{2n\sqrt{k-c}}.$$

Now we apply equalities (2) and (3) to compute the matrices

P and Q, respectively:

$$P = \begin{pmatrix} 1 & k & n-1 & n-k \\ 1 & -k & n-1 & k-n \\ 1 & \sqrt{k-c} & -1 & -\sqrt{k-c} \\ 1 & -\sqrt{k-c} & -1 & \sqrt{k-c} \end{pmatrix},$$

$$Q = \frac{1}{2n} \begin{pmatrix} 1 & 1 & n-1 & n-1 \\ 1 & -1 & -\frac{k-n}{\sqrt{k-c}} & \frac{k-n}{\sqrt{k-c}} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\frac{k}{\sqrt{k-c}} & \frac{k}{\sqrt{k-c}} \end{pmatrix}.$$

Finally, we obtain the dual intersection matrices of this association scheme by applying formula (10) from Proposition 1 and taking into account equality (12): $L_0^* = I_4/2n$,

$$L_1^* = \frac{1}{2n} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$L_2^* = \frac{1}{2n} \begin{pmatrix} 0 & 0 & n-1 & 0 \\ 0 & 0 & 0 & n-1 \\ 1 & 0 & \frac{n-2}{2} + \frac{n-2k}{2\sqrt{k-c}} & \frac{n-2}{2} - \frac{n-2k}{2\sqrt{k-c}} \\ 0 & 1 & \frac{n-2}{2} - \frac{n-2k}{2\sqrt{k-c}} & \frac{n-2}{2} + \frac{n-2k}{2\sqrt{k-c}} \end{pmatrix},$$

$$L_3^* = \frac{1}{2n} \begin{pmatrix} 0 & 0 & 0 & n-1 \\ 0 & 0 & n-1 & 0 \\ 0 & 1 & \frac{n-2}{2} - \frac{n-2k}{2\sqrt{k-c}} & \frac{n-2}{2} + \frac{n-2k}{2\sqrt{k-c}} \\ 1 & 0 & \frac{n-2}{2} + \frac{n-2k}{2\sqrt{k-c}} & \frac{n-2}{2} - \frac{n-2k}{2\sqrt{k-c}} \end{pmatrix}.$$

From the dual intersection matrices presented above, it is possible to extract some evidence of the optimality of the upper bound $1/2$, for the Krein parameters q_{ij}^l, with $i \neq j$, presented in Corollary 2, (ii). In fact, we can observe that

$$q_{23}^0 = (L_2^*)_{03} = \frac{n-1}{2n}$$

and this value converges to $1/2$, when n tends to infinity.

Regardless the examples presented above, we can construct other examples of symmetric association schemes with a number of classes greater than three for which the Krein parameters converge for the values of the upper-bounds presented in Theorems 3 and 4, by calling upon the the Kronecker product of symmetric association schemes.

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