The mathematical foundations of the cell method

Elena Ferretti

***Abstract*—**Of special importance for the philosophy of the Cell Method (CM) is the classification diagram of the physical variables. Originally, the classification diagram was obtained on the basis of physical considerations on the associations between physical variables and geometry. We will show in this paper that we may obtain the same associations on the basis of mathematical considerations, thus deepening the mathematical foundations of the CM. This will allow us to recognize in the classification diagram of the Cell Method a structure of bialgebra, where the operators are generated by the outer product of the geometric algebra and the exterior product of the dual algebra of the enclosed exterior algebra. In doing so, the classification itself of the physical variables will take on a deeper meaning, by allowing us to associate the configuration variables with the geometric interpretation for the elements of a vector space and the source variables with the geometric interpretation for the elements of the dual vector space in the bialgebra. We will also discuss a new four-dimensional space/time cell-complex for studying time dependent phenomena with the CM.

***Keywords*—**Bialgebra, Cell Method, Inner Orientation, Outer Orientation.

I. Introduction

The Cell Method (CM) [1] is a numerical method that allows us to achieve a numerical modeling of Physics without starting from the differential equations.

It must be said that, from the birth of the differential calculus forth, more than three centuries ago, we are accustomed to provide each experimental law with a differential formulation. There is no doubt that the infinitesimal analysis played in the past and will play in future a major role in the mathematical treatment of Physics, but we must be aware that its introduction hides some important features of the phenomenon being described, such as the geometrical and topological features. Moreover the limit process introduces some limitations, by requiring regularity conditions on the field variables. These regularity conditions, in particular the conditions of differentiability, are the price we have to pay for using a formalism that is very advanced and easy to manipulate.

With the advent of computers, the differential equations were discretized by means of one of various discretisation methods (FEM, BEM, FVM, FDM, etc. [2]–[33]), since the numerical solution, which is no longer an exact solution, cannot be achieved in the most general case if a system of algebraic physical laws is not provided.

The very need to discretise the differential equations in order to achieve a numerical solution gives rise to the question of whether or not it is possible to formulate the physical laws in an algebraic manner directly, through a direct algebraic formulation. As we have already shown in [34]–[51], the answer to this question is affirmative and providing a direct algebraic formulation is exactly what the CM does. The aim of this paper is to study the philosophy of the CM from the mathematical point of view, in order to investigate the mathematical meanings of the choices that are at the basis of the CM.

II. Main Features of the Cell Method

The starting point of the algebraic formulation of the CM is that just few physical variables arise directly as functions of points and instants. Most of them are obtained from variables referred to extended space elements and time intervals, by performing densities and rates.

We will denote as global variables those variables that are neither densities nor rates of other variables. Specifically:

- a global variable in space is a variable that is not the line, surface or volume density of another variable;
- a global variable in time is a variable that is not the rate of another variable.

By using global variables, it is possible to obtain an algebraic formulation directly, without requiring to the global variables of being differentiable functions. Moreover, the algebraic formulation preserves the length and time scales of the global physical variables [37]–[39], [41], [47], [52] since it avoids the limit process.

By performing the limit process of the mean densities and rates of the global variables, we obtain the traditional field functions of the differential formulation. Due to their point-wise and/or instant-wise nature, the field variables are local variables.

Since performing densities and rates of the domain variables implies the assumption of continuity and differentiability of the global variables, the range of applicability of the differential formulation is restricted to regions without material discontinuities and concentrated sources, whereas the range of applicability of the algebraic formulation is not restricted to regions of regularity.

E. Ferretti is with the Department of Civil, Chemical, Environmental, and Materials Engineering - DICAM, Alma Mater Studiorum - University of Bologna, Bologna, I-40136 (phone: +39-51-2093493; fax: +39-51-2093495; e-mail: elena.ferretti2@unibo.it).

A. Association between Global Physical Variables and Space Elements

Besides than using the global rather than local nature of global variables, we can classify them on the basis of the role they play in a theory. In this second case, we can distinguish between:

- configuration variables, describing the field configuration;
- source variables, describing the field sources.

The equations used to relate the configuration variables of the same physical theory to each other and the source variables of the same physical theory to each other are known as topological equations, whereas those relating configuration to source variables, of the same physical theory, are constitutive equations. Moreover, the product between a configuration variable and a source variable gives rise to an energetic variable.

Since each physical phenomenon occurs in space, and space has a multi-dimensional geometrical structure, the global physical variables have a multi-dimensional geometrical content. As a consequence, all the global physical variables are associated with one of the four space elements, that is (Fig. 1):

- the point ($\mathbf{P}$),
- the line ($\mathbf{L}$),
- the surface ($\mathbf{S}$),
- the volume ($\mathbf{V}$).

In the CM, the global variables in space are associated with the related space elements $\mathbf{P}$, $\mathbf{L}$, $\mathbf{S}$, and $\mathbf{V}$ of three-dimensional cell-complexes. This allows us to describe global variables in space directly.

Also the global variables in time are associated with the elements of a cell-complex, which has dimension 1 and generalizes the time axis. The two time elements, that is:

- the time instant ($\mathbf{I}$),
- the time interval ($\mathbf{T}$),

are represented by the nodes and lines of this one-dimensional cell-complex, respectively.

The CM makes use of the notions provided by algebraic topology [53], [54] which denotes the nodes (points) of the

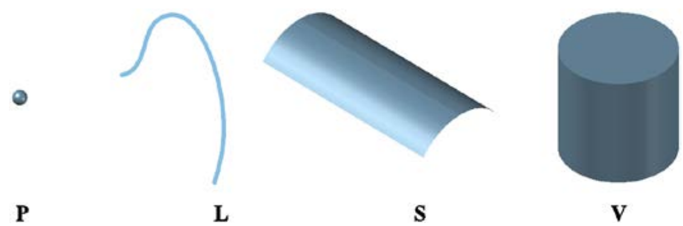

Fig. 1 notations for the four space elements in three-dimensional space

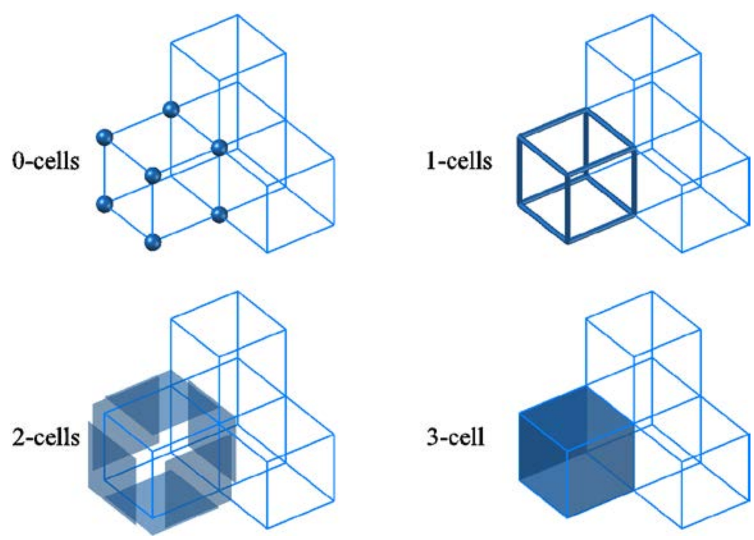

Fig. 2 The four space elements in algebraic topology

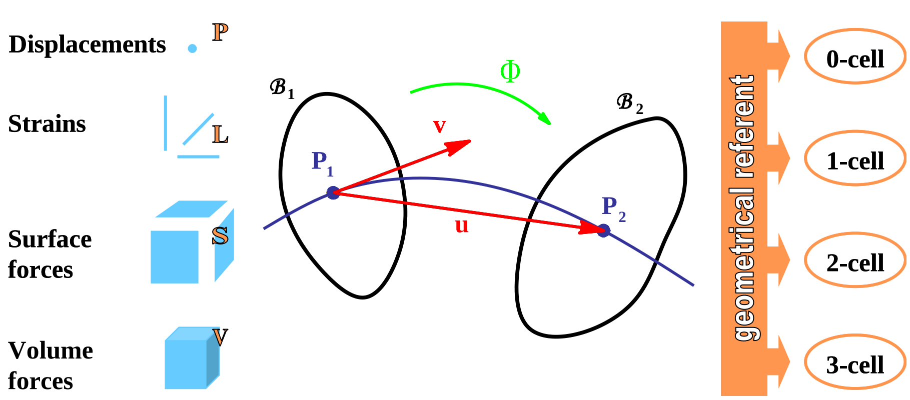

Fig. 3 Association between space elements and variables in continuum mechanics

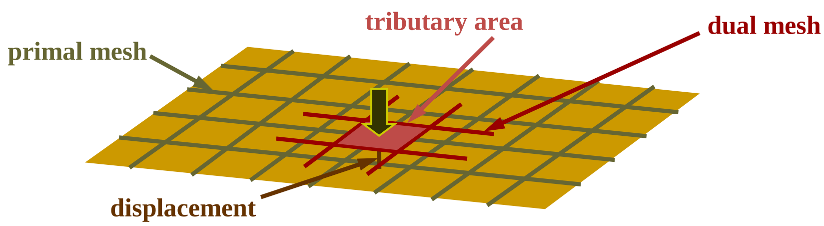

Fig. 4 Association between variables and space elements of a two-dimensional cell-complex

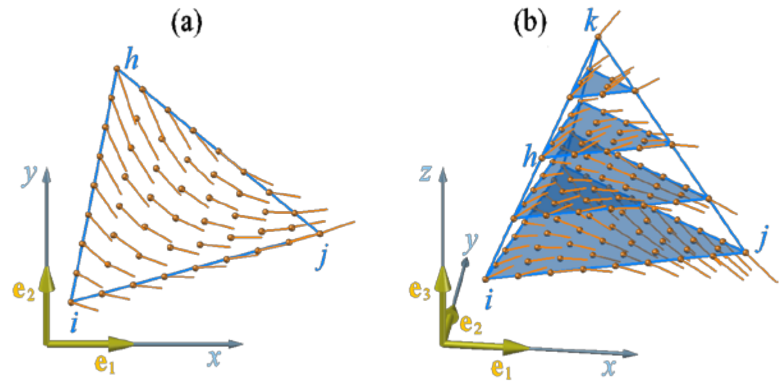

Fig. 5 Approximation of a vector field by an affine vector field defined by the vectors at the vertices of a triangle, in two-dimensional space (a), and of a tetrahedron, in three-dimensional space (b)

cell-complexes as 0-cells, the edges (lines) as 1-cells, the surfaces as 2-cells, and the volumes as 3-cells (Fig. 2). Specifically, as far as Continuum Mechanics is concerned (Fig. 3), volume forces, which are source variables, are associated with 3-cells, since their geometrical referents are volumes. Analogously, surface forces, which are source variables, have 2-cells as their geometrical referents (the surfaces), strains, which are configuration variables, have 1-cells as their geometrical referents (the lines), and displacements, which are configuration variables, have 0-cells as their geometrical referents (the points).

The CM uses two cell-complexes: the primal cell-complex and the dual cell-complex, in relation of duality with the primal cell-complex. In order to explain the reason for this choice, let us start from an example on a two-dimensional domain (Fig. 4): once a mesh has been introduced, the primal mesh, it is natural to associate the primal nodes with the displacements of the primal nodes and the total load over an area surrounding the primal nodes (tributary area), which is an area defining a dual cell-complex. Thus, the displacements, which are configuration variables, are computed on elements of the primal mesh, whereas the loads, which are source variables, are computed on elements of the dual cell-complex.

This result is general, independently of the kind of configuration or source variable, the shape of the domain, and the physical theory involved. Actually, for each physical theory and for each set of primal nodes, the source variables are always associated with the elements of the dual cell-complex and the configuration variables are always associated with the elements of the primal cell-complex.

This is a remarkable finding, which allows us to gain a new understating of the structural similarities between physical theories, commonly called "analogies" [55]. Actually, just the existence of an underlying structure, common to different physical theories, is the main responsible for the analogies, since the homologous global variables of two physical theories are those associated with the same space element. This allows us to explain the analogies in the light of the association between the global variables and the four space elements. In other words, the analogies between physical theories arise from the geometrical structure of the global variables and not from the similarity of the equations relating variables to each other in different physical theories.

In algebraic topology, it is usual to consider cell-complexes made of simplexes, that is, triangles in plane domains and tetrahedra in space. Also the Cell Method uses simplexes for building the primal mesh. The reason for this choice is that any scalar or vector field in the neighborhood of each point, in a region of regularity, can be approximated by an affine field and simplexes are compatible with the affine description of the field, whereas cells with an arbitrary number of sides are not compatible. In Fig. 5, we have depicted the displacement field as an affine field for both a plane domain made of just one triangle (Fig. 5a) and a three-dimensional domain made of just one tetrahedron (Fig. 5b).

Once the primal mesh has been generated, we have several possibilities for building the dual mesh. As the nodes of the dual mesh in plane domains we can choose, for example, the circumcenters of the primal triangles (Fig. 6b). In this case, the dual of each primal side is a straight line. Specifically, the sides of the dual polygons are the axes of the primal sides.

Another possibility for building the dual mesh is to connect the barycenters of the primal triangles to the mid-points of the primal sides (Fig. 6a). In this second case, the dual of each

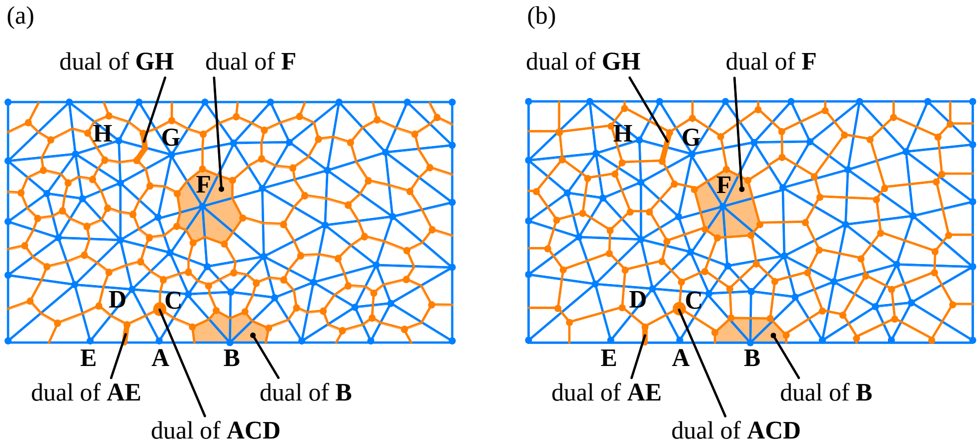

Fig. 6 The barycentric dual mesh (a) and the circumcentric dual mesh of Voronoi (b) in two-dimensional domains

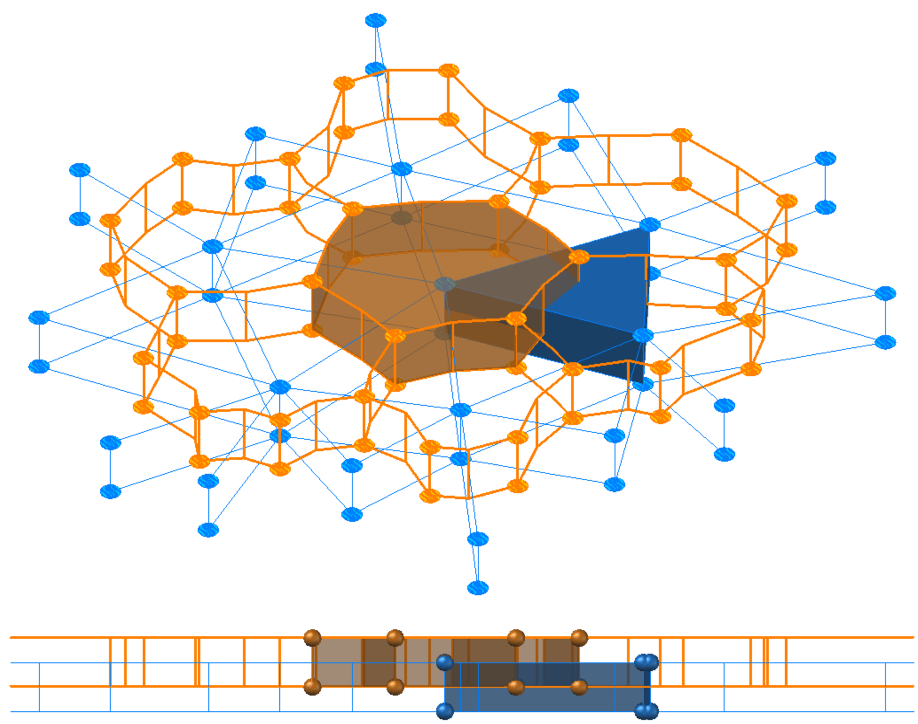

Fig. 7 Staggering in 2D cell-complexes with barycentric dual cells

primal side is no longer a straight line.

The choice for the dual mesh is arbitrary, but the circumcentric dual mesh has a disadvantage. In fact, whereas the barycentric dual mesh does not involve any restriction on the shape of the primal mesh, the circumcentric dual mesh requires that all the angles of the primal triangles are lower than 90°, in order to avoid that some triangle has a circumcenter that lies outside the triangle itself, since this generates numerical errors. This is the reason why we will prefer to use the barycentric dual mesh.

The primal mesh is provided with a thickness also in plane domains, as we can see in Fig. 7, where the thickness is a unit thickness. Therefore, the two meshes are shifted for half the thickness and the dual nodes are not in the same planes as the primal nodes.

B. The Classification Diagram of the Cell Method

In the classification diagram of the Cell Method [1], the global variables are stored in two columns (Fig. 8), the column of the configuration variables, with their topological equations, and the column of the source variables, with their topological equations.

The configuration variables are arranged from top to bottom in their column, in order of increasing multiplicity of the associated space element, thus realizing a downward cochain [56]. Conversely, the source variables are arranged from bottom to top in their column, in order of increasing multiplicity of the associated space element, thus realizing an upward cochain.

With this choice, each primal space element is at the same level of the corresponding dual space element. Actually, the sum of the dimensions of a space element and its dual element

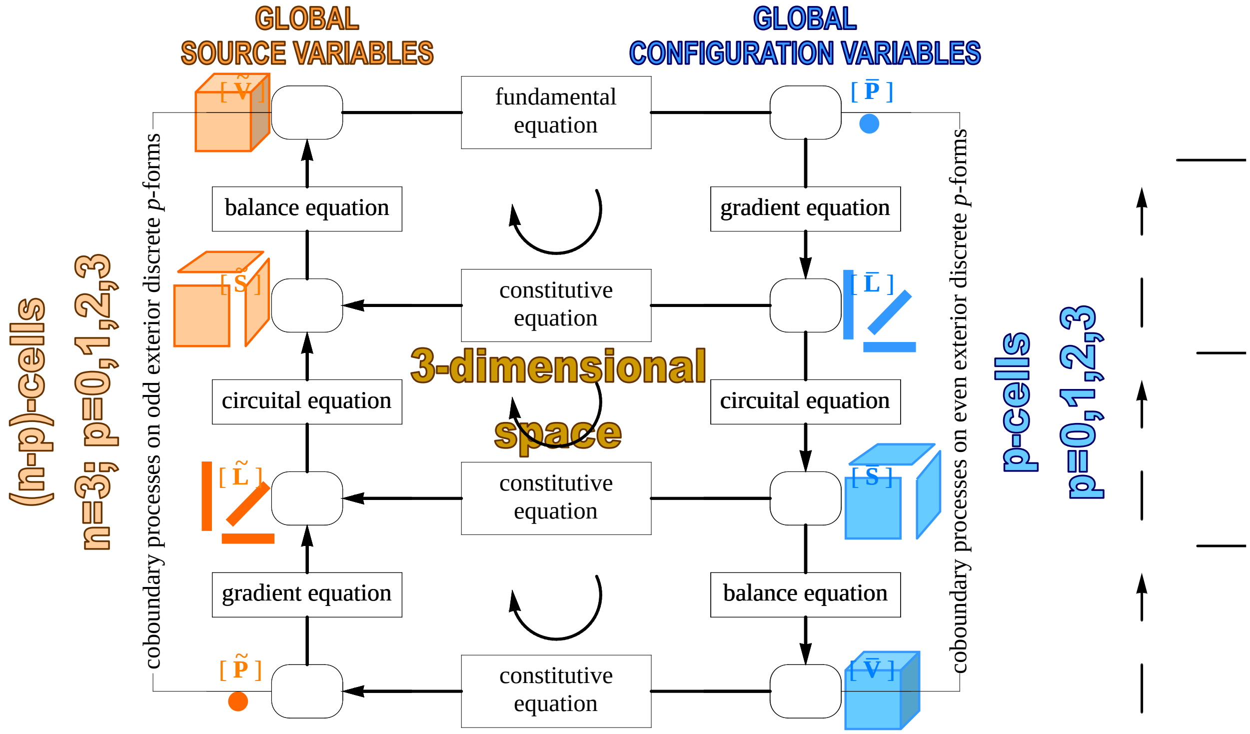

Fig. 8 The columns of the configuration and source variables in the classification diagram of the Cell Method

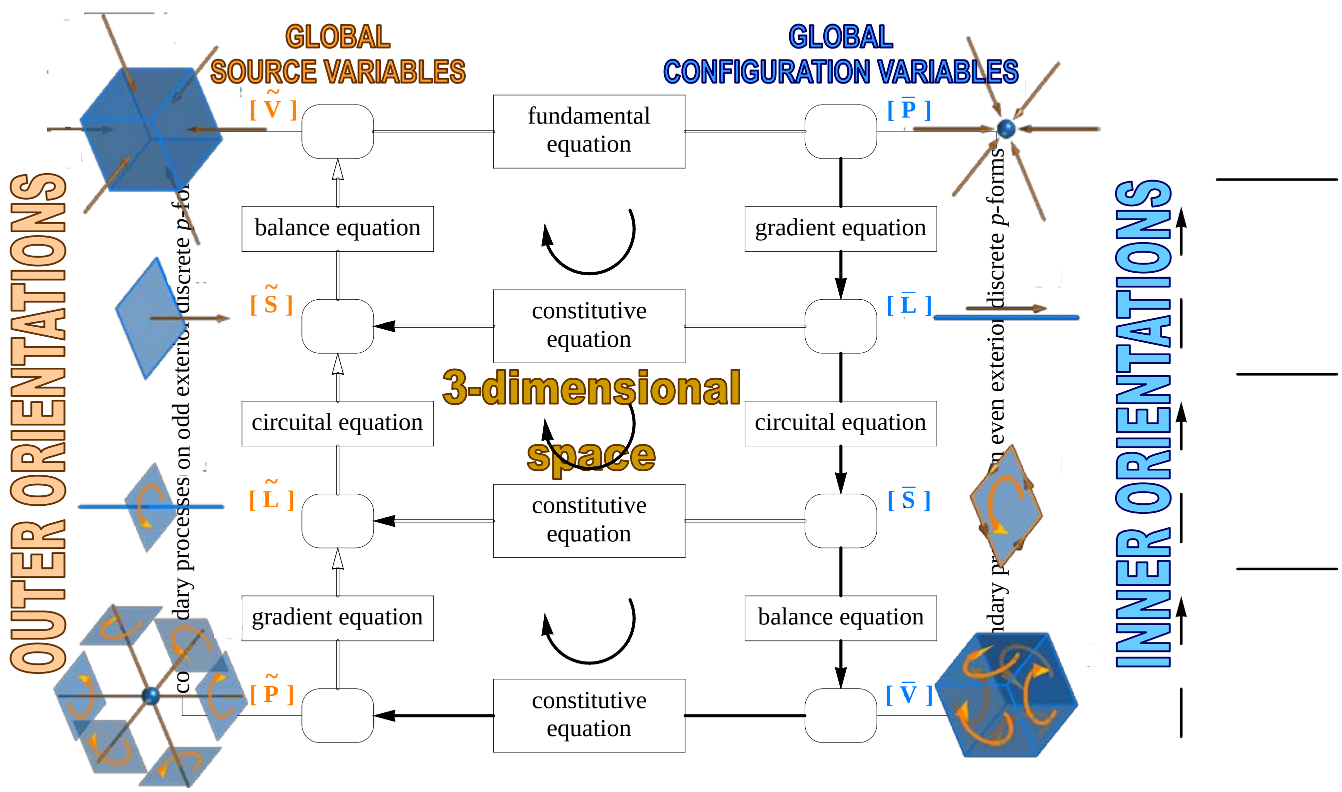

Fig. 9 Positive inner and outer orientations of the space elements associated with the global variables

always equals the dimension n of the space they are in (for the case of Fig. 8, $n = 3$). Consequently, the relationships between primal and dual variables at the same level of the classification diagram are the constitutive relations.

The space elements of both the primal and the dual cell-complexes may also be oriented (Fig. 9). We will define the inner orientation of a space element as the orientation that follows from choosing an order of traversal of its boundary. It is denoted as the inner orientation because, since we stay on the space element, this orientation does not depend on the dimension of the embedding space.

This definition immediately applies to the two-dimensional cells and can be easily extended to cells of dimensions 1 and 3. Finally, it can be extended also to points, where the (positive) inner orientation is the inward orientation, given by the incoming lines.

By providing the primal elements with an inner orientation, we also orientate the dual elements at the same level (Fig. 9). Moreover, since the dual elements depend on the dimension of the embedding space, this dual orientation depends on the space immersion. This is why we will denote the dual orientation as the outer orientation. This second time, we do not stay on the space element but we cross it.

A point, a line, a surface, and a volume endowed with outer orientations will be denoted by putting tildes over their symbols (Fig. 8, Fig. 9).

A point, a line, a surface, and a volume endowed with inner orientations will be denoted by putting bars over their symbols (Fig. 8, Fig. 9).

The structure of the classification diagram is the same for both the global and the field variables of every physical theory of the macrocosm. The importance of this diagram stands just in its ability of providing a concise description of physical variables, without distinguishing between the physical theories.

III. Some Basics of the Exterior Algebra

From the mathematical point of view, the algebra that provides an algebraic setting in which to answer geometric questions is the exterior algebra, which is the largest algebra that supports an alternating product on vectors. Its product is the exterior product, or wedge product.

The exterior product of any number k of vectors can be defined and is sometimes called a k-blade. It lives in a geometrical space known as the k-th exterior power, denoted by $\Lambda^k V$, which is the vector space of formal sums of k-multivectors. The magnitude of the k-blade is the volume of the k-dimensional parallelotope whose sides are the given vectors, just as the magnitude of the scalar triple product of vectors in three dimensions gives the volume of the parallelepiped spanned by those vectors.

In particular, the exterior product of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\wedge\mathbf{b}$, is a 2-vector, or bivector, and lives in a space called the exterior square, a geometrical vector space that differs from the original space of vectors. The magnitude of the bivector $\mathbf{a}\wedge\mathbf{b}$ can be interpreted as the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ (Fig. 10).

In three dimensions, the magnitude of $\mathbf{a}\wedge\mathbf{b}$ can also be computed by using the cross product of the two vectors.

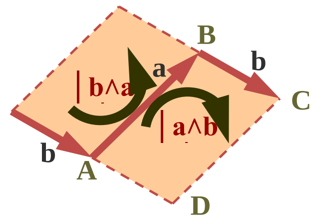

Fig. 10 Magnitudes and orientations of the bivectors $\mathbf{a}\wedge\mathbf{b}$ and $\mathbf{b}\wedge\mathbf{a}$

The senses of $\mathbf{a}$ and $\mathbf{b}$ orientate the sides of the parallelogram and define a sense of traversal of its boundary. In the case of Fig. 10, the traversal sense of $\mathbf{a}\wedge\mathbf{b}$ is a clockwise sense, which can be depicted by a clockwise arc.

Since the exterior product is antisymmetric, $\mathbf{b}\wedge\mathbf{a}$, the exterior product between $\mathbf{b}$ and $\mathbf{a}$, is the negation of the bivector $\mathbf{a}\wedge\mathbf{b}$, producing the opposite orientation (Fig. 10).

The product of a k-multivector and an ℓ-multivector is a $(k+\ell)$-multivector. So, the direct sum:

$$\bigoplus_k \Lambda^k V \tag{1}$$

forms an associative algebra, which is closed with respect to the wedge product. This algebra, commonly denoted by ΛV, is called the exterior algebra of the vector space V.

The exterior algebra is one example of a bialgebra, meaning that it has a dual space that also possesses a product and this dual product is compatible with the exterior product. The dual algebra is precisely the algebra of alternating multi-linear forms on V and the pairing between the exterior algebra and its dual is given by the interior product.

Any vector space, V, has a corresponding dual vector space (or just dual space), V^*. Given any vector space V over a field F, the algebraic dual space V^*, also called the ordinary dual space, or simply the dual space, is defined as the set of all linear maps (linear functionals) from V to F:

$$\varphi: V \to F,\ v \mapsto \varphi(v). \tag{2}$$

The elements of the algebraic dual space V^* are sometimes called covectors, or 1-forms, and are denoted by bold, lowercase Greek. They are linear maps from V to its field of scalars.

If V is finite-dimensional, then V^* has the same dimension as V. Dual vector spaces for finite-dimensional vector spaces can be used for studying tensors.

The pairing of a functional φ in the dual space V^* and an element x of V is sometimes denoted by a bracket:

$$\varphi(x) = [\varphi, x] = \langle \varphi, x \rangle. \tag{3}$$

The pairing defines a non-degenerate bilinear mapping:

$$[\bullet,\bullet]: V^* \times V \to F. \tag{4}$$

Specifically, every non-degenerate bilinear form on a finite-dimensional vector space V gives rise to an isomorphism from V to V^*, $\langle\bullet,\bullet\rangle$. Then, there is a natural isomorphism:

$$V \to V^*,\ v \mapsto v^*, \tag{5}$$

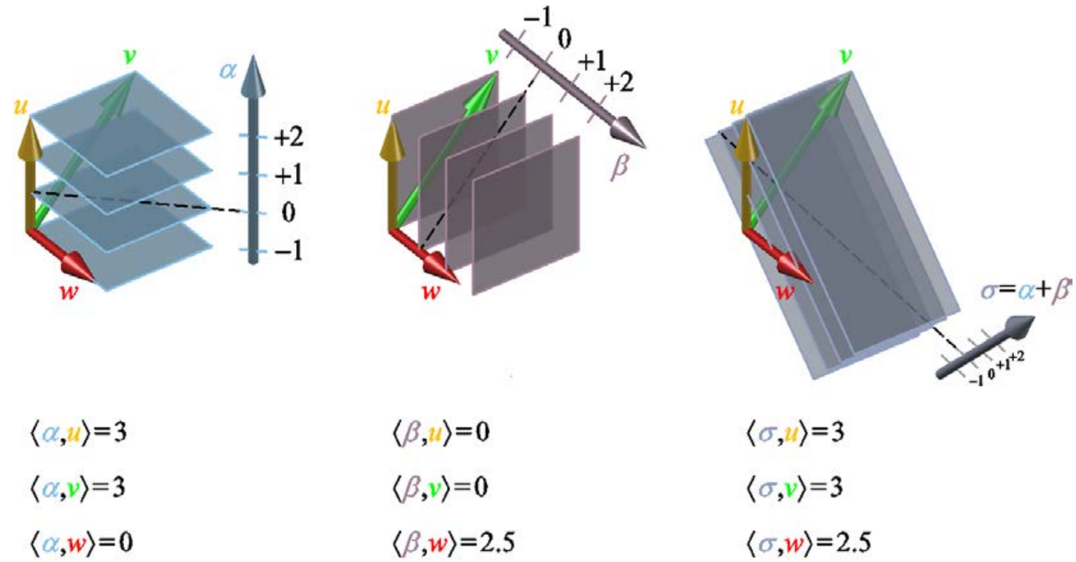

Fig. 11 Linear functionals (1-forms) α , β , their sum σ and vectors u , v , w , in 3d Euclidean space

given by:

$$v^*(w) := \langle v, w \rangle , \tag{6}$$

where $v^* \in V^*$ is said to be the dual vector of $v \in V$.

A topology on the dual space, X^* , of a topological vector space, X , over a topological field, $\mathbf{K}$, can be defined as the coarsest topology (the topology with the fewest open sets) such that the dual pairing $X^* \times X \to \mathbf{K}$ is continuous. This turns the dual space into a locally convex topological vector space. This topology is called the weak* topology, that is, a weak topology defined on the dual space X^* . In order to distinguish the weak topology from the original topology on X , the original topology is often called the strong topology. If X is equipped with the weak topology, then addition and scalar multiplication remain continuous operations and X is a locally convex topological vector space.

If V is a vector space of any (finite) dimension, then the level sets of a linear functional in V^* are parallel hyperplanes in V and the action of a linear functional on a vector can be visualized in terms of these hyperplanes, or p-planes , in the sense that the number of hyperplanes (1-forms) intersected by a vector equals the interior product between the covector and the vector.

In Fig. 11 we have depicted the level planes of two linear functionals in 3d Euclidean space. The pairing of the first linear functional and the vector u equals 3, because the vector u intersects the level planes three times.

Also the vector v intersects the level planes three times, whereas w does not intersect the planes. For the second linear functional, u and v do not intersect the level planes, whereas w intersects the level planes two and one half times.

The pairings between the linear functional that is the sum of the former two functionals and vectors u , v and w are given by the sums of the previous pairings (Fig. 11).

In multi-linear algebra, a multi-linear form, or k-form , is a map of the type:

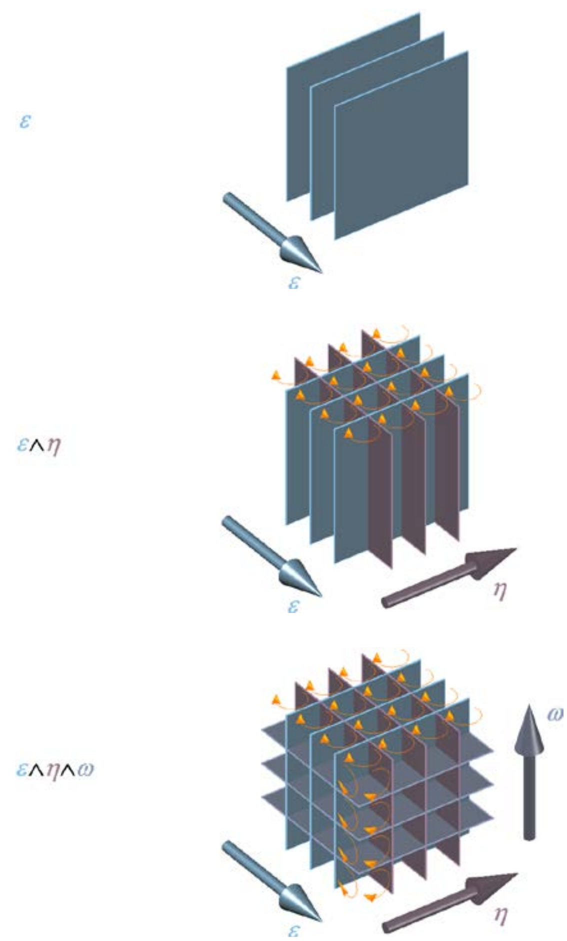

Fig. 12 Geometric interpretation for the exterior product of k 1-forms ($\boldsymbol{\varepsilon}$, $\boldsymbol{\eta}$, $\boldsymbol{\omega}$) to obtain an k-form ("mesh" of coordinate surfaces, here planes), for $k = 1, 2, 3$. The "circulations" show orientation

$$f : V^k \to \mathbf{K}, \tag{7}$$

where V is a vector space over the field $\mathbf{K}$, which is separately linear in each its k variables. The k-forms are generated by the exterior product on covectors.

The geometric interpretation for the exterior product of k covectors is that of mesh of k coordinate surfaces. In 3d Euclidean space, the coordinate surfaces are planes. Thus, we have a mesh of two planes when we perform the exterior product of 2 1-forms and a mesh of three planes when we perform the exterior product of 3 1-forms (Fig. 12).

The exterior algebra is contained in a wider algebra, the geometric algebra.

IV. An Insight into Geometric Algebra

The geometric algebra (GA) [57]–[63] is an approach alternative to vector algebra for providing additional algebraic structures on vector spaces, with geometric interpretations. The difference between vector algebra and geometric algebra is that vector algebra is specific to Euclidean three-dimensional space, whereas geometric algebra uses multi-linear algebra and applies in all dimensions. They are mathematically equivalent in three dimensions, though the approaches differ.

Geometric algebra gives emphasis on geometric interpretations and physical applications. A geometric algebra is the Clifford algebra $\mathcal{C}\ell(V,Q)$ of a vector space over the field of real numbers endowed with a quadratic form.

The distinguishing multiplication operation that defines the geometric algebra as a unital ring is the geometric product. Taking the geometric product among vectors can yield bivectors, trivectors, or general p-vectors. The addition operation combines these into general multi-vectors. This includes, among other possibilities, a well-defined sum of a scalar and a vector, an operation that is impossible by the traditional vector addition.

In the most general case, the geometric product is the sum between a scalar and a bivector. Actually, we may write the geometric product of any two vectors a and b as the sum of a symmetric product and an antisymmetric product:

$$ab = \frac{1}{2}(ab+ba) + \frac{1}{2}(ab-ba), \tag{8}$$

where the symmetric product is a real number, because it is a sum of squares:

$$a \cdot b := \frac{1}{2}(ab+ba) = \frac{1}{2}\left((a+b)^2 - a^2 - b^2\right), \tag{9}$$

and is not required to be positive definite. The symmetric product defines the inner product $a \cdot b$ of vectors a and b. It is not specifically the inner product on a normed vector space.

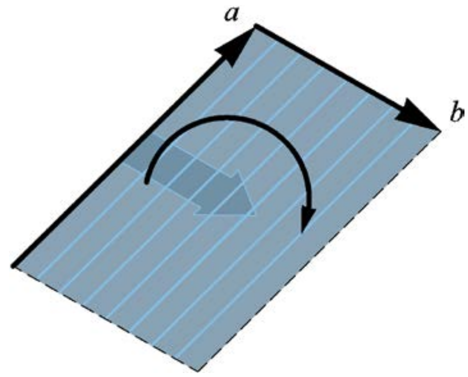

Fig. 13 The extension of vector a along vector b provides the geometric interpretation of $a \wedge b$

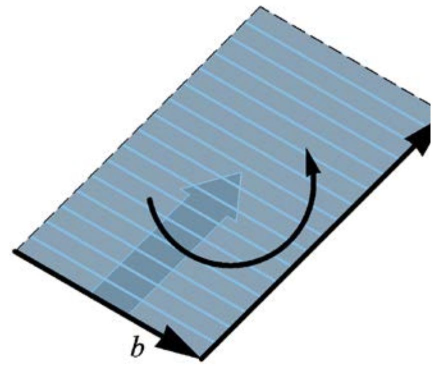

Fig. 14 The extension of vector b along vector a provides the geometric interpretation of the outer product $b \wedge a$

Moreover, the antisymmetric product in Eq. (8) is a bivector, equal to the exterior product $a \wedge b$ of the contained exterior algebra, and defines the outer product of vectors a and b:

$$a \wedge b := \frac{1}{2}(ab - ba). \tag{10}$$

Geometrically, the outer product $a \wedge b$ can be viewed by placing the tail of the arrow b at the head of the arrow a and extending vector a along vector b (Fig. 13). The resulting entity is a two-dimensional sub-space that has an area equal to the size of the parallelogram spanned by a and b.

The geometric interpretation of the outer product $b \wedge a$ is achieved by placing the tail of the arrow a at the head of the arrow b and extending vector b along vector a (Fig. 14). This reverses the circulation of the boundary, whereas it does not change the area of the parallelogram spanned by a and b.

In conclusion, the geometric product in Eq. (8) can be written as:

$$ab = a \cdot b + a \wedge b. \tag{11}$$

The scalar and the bivector are added by keeping the two entities separated, in the same way in which, in complex numbers, we keep the real and imaginary parts separated.

One can consider the Clifford algebra $\mathcal{C}\ell(V,Q)$ as an enrichment (or more precisely, a quantization) of the exterior algebra $\Lambda(V)$ on V with a multiplication that depends on Q. For non-zero Q there exists a canonical linear isomorphism between $\Lambda(V)$ and $\mathcal{C}\ell(V,Q)$, whenever the ground field K does not have characteristic two. That is, they are naturally isomorphic as vector spaces, but with different multiplications.

The p-vectors are charged with three attributes, or features: attitude, orientation, and magnitude. The second feature, taken singularly and combined with the first feature, gives rise to the two kinds of orientation in space, inner and outer orientations.

A. Inner Orientation of Space Elements

According to the definition given in Section II.B, the second feature of p-vectors, the orientation, is, more properly, an inner orientation, because it does not depend on the embedding space. The term "inner" refers to the fact that the circulations are defined for the boundaries of the elements, by choosing an order for the vertexes. Therefore, we move and stay on the boundaries of the elements, without going out from the elements themselves.

In GA, the inner orientation is the geometric interpretation of the exterior geometric product among vectors. In particular, the inner orientation of a plane surface can be viewed as the orientation of the exterior product between two vectors $\mathbf{u}$ and $\mathbf{v}$ (the bivector $\mathbf{u}\wedge\mathbf{v}$) of the plane on which the surface lies (Fig. 15). Analogously, the inner orientation of a volume can be viewed as the orientation of the exterior product between three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (the trivector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$) of the three-dimensional space containing the volume (Fig. 15).

By extending the geometrical interpretation of the bivector provided in Fig. 13 and Fig. 14, we can view the trivector as the extension of a bivector along a vector (Fig. 16). Its magnitude is equal to the volume spanned by the bivector and the vector.

The concept of inner orientation defined above did not apply to zero-dimensional vector spaces (points). However, since it is useful to be able to assign different inner orientations to a point, we extend the outer product to zero-dimensional vectors:

$$\mathbf{P}\wedge\mathbf{Q}\triangleq\mathbf{u}, \tag{12}$$

which has the geometrical meaning of point $\mathbf{P}$ extended toward point $\mathbf{Q}$. The extension of the outer product preserves the antisymmetric property of the product, since $\mathbf{Q}\wedge\mathbf{P}$ (point $\mathbf{Q}$ extended toward point $\mathbf{P}$) is the negation of $\mathbf{P}\wedge\mathbf{Q}$:

$$\mathbf{Q}\wedge\mathbf{P}=-\mathbf{u}. \tag{13}$$

Analogously, a point extended by a vector results in an oriented length (Fig. 16), which can be represented by the vector itself. Consequently, we can see a bound vector with origin in $\mathbf{P}$ as the outer product between $\mathbf{P}$ and the free

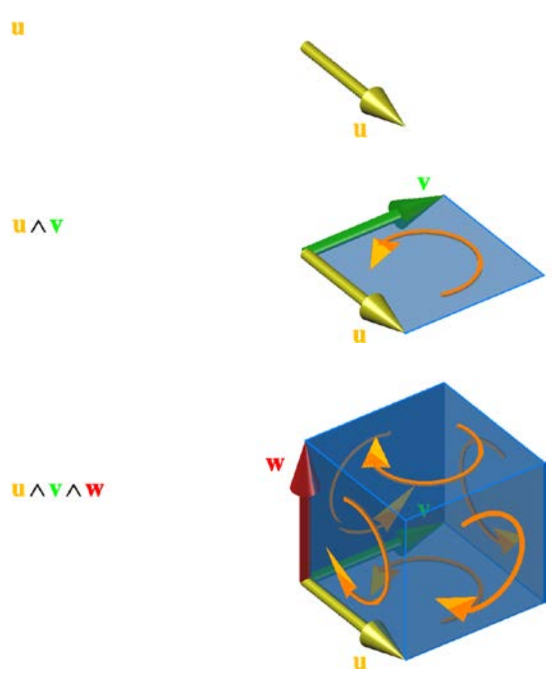

Fig. 15 Geometric interpretation for the exterior product of p vectors to obtain a p-vector, where $p=1,2,3$. The "circulations" show the inner orientation

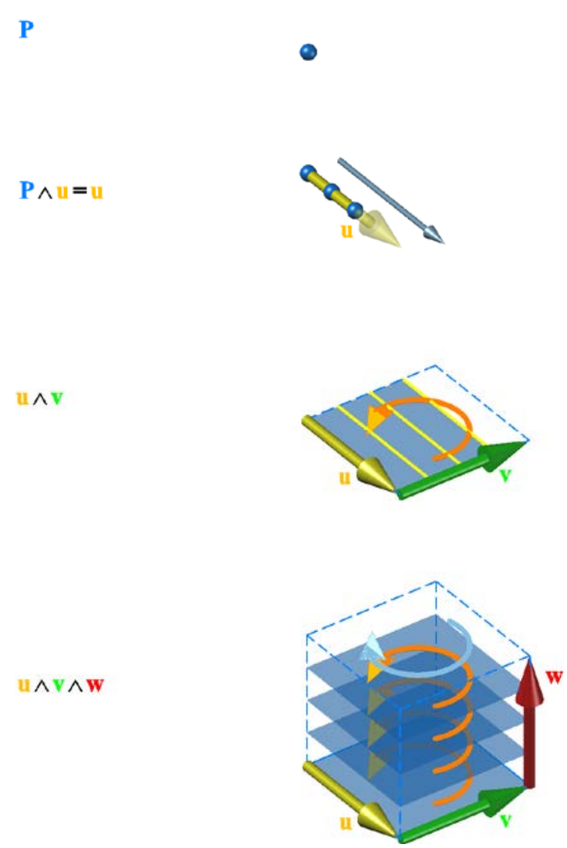

Fig. 16 The inner orientation of a p-space element is induced by the $(p-1)$-space elements on its boundary

vector $\mathbf{u}$:

$$\mathbf{P} \wedge \mathbf{u} = \mathbf{u} . \tag{14}$$

Then, since the bound vector $(\mathbf{P}, \mathbf{u})$ is often denoted by simply $\mathbf{u}$, as its free vector, we can also write

$$\mathbf{P} \wedge \mathbf{u} = \mathbf{u} . \tag{15}$$

For consistency, we must therefore define the outer product between the vector $\mathbf{u}$ and the point $\mathbf{P}$ as the negation of $\mathbf{u}$:

$$\mathbf{u} \wedge \mathbf{P} \triangleq -\mathbf{u} . \tag{16}$$

This allows us to find the k-vectors and their inner orientations inductively, from the elements of the zero-dimensional space to the elements of a space of any dimension (Fig. 16).

In analogy to the direction of the vector product $\mathbf{u} \times \mathbf{v}$, which is orthogonal both to $\mathbf{u}$ and $\mathbf{v}$, the result of the operation $\mathbf{P} \wedge \mathbf{Q}$, defined in $\mathbb{R}^1$ on elements of $\mathbb{R}^0$, has the direction of a line that is orthogonal both to $\mathbf{P}$ and $\mathbf{Q}$. In three-dimensional space, where we can define infinite sub-spaces of dimension 1, each provided with its own basis, this operation produces elements in the direction of any line of the three-dimensional space. Being orthogonal to each direction of the three-dimensional space, the point is orthogonal to the three-dimensional space itself and to each volume of the space.

It is worth noting that the inner orientation of a surface is not positive or negative in itself. Neither choosing the sign of the inner orientation can be considered an arbitrary convention. Providing the inner orientation of a surface with a sign makes sense only when the surface is "watched" by an external observer, that is, only when the surface is studied in an embedding space of dimension greater than 2, the dimension of the surface.

The six faces of the positive trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ in Fig. 15 have a negative inner orientation when they are watched by an external observer, whereas they have a positive inner orientation when they are watched by a local observer that is inside the volume. This happens since the inner volume of the trivector is the intersection of the six positive half-spaces, that is, the half-spaces of the six observers that watch the positive surfaces originated by the trivector. By relating the sign of the inner orientation to the external observer also in this second case, the positive inner orientation of a volume is the one watched by the external observer. As a consequence, the inner orientation of a volume is positive when the inner orientations of all its faces are negative, as in Fig. 15 and Fig. 9.

Also for the point we can define two inner orientations, the outward and the inward orientations (Fig. 17). In the first case, the point is called a source, whereas, in the second case, is called a sink.

By making use of the notion of observer in this latter case too, each incoming line can be viewed as the sense along which the external observer watches the point.

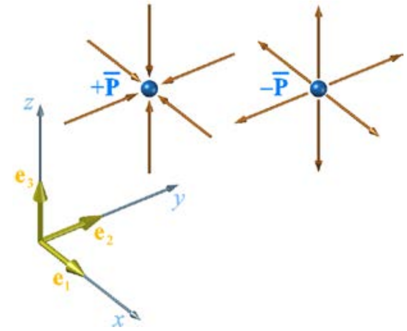

Fig. 17 Positive and negative inner orientations of a point

In this sense, a sink is a point with a positive inner orientation, whereas a source is a point with a negative inner orientation (Fig. 17). This also explains why we have chosen as positive inner orientation of the point in Fig. 9 the inward orientation.

In conclusion, the positive or negative inner orientation of a p-space element is induced by the positive or negative inner orientation of the $(p-1)$-space elements on its boundary. This allows us to extend the procedure for finding the inner orientation of the space elements to spaces of any dimension.

B. *Outer Orientation of Space Elements*

The attitude is part of the description of how p-vectors are placed in the space they are in. Thus, the notion of attitude is related to the notion of embedding of a p-vector in its space, or space immersion. In particular, a vector in three dimensions has an attitude given by the family of straight lines parallel to it (possibly specified by an unoriented ring around the vector), a bivector in three dimensions has an attitude given by the family of planes associated with it (possibly specified by one of the normal lines common to these planes), and a trivector in three dimensions has an attitude that depends on the arbitrary choice of which ordered bases are positively oriented and which are negatively oriented.

Between a p-vector and its attitude there exists the same kind of relationship that exists between an element a of a set X and the equivalence class of a in the quotient set of X by a given equivalence relationship. In the special case of the attitude of a vector in the three-dimensional space, the set is that of the straight lines and the equivalence relationship is that of parallelism between lines. One of the invariants of the equivalence relation of parallelism is the family of planes that are normal to the lines in a given equivalence class. Since we can choose any of the parallel planes for representing the invariant, we can speak both in terms of family of parallel planes and in terms of one single plane.

Similar considerations may also be applied to the relationship between bivectors and their attitudes, or trivectors and their attitudes. Thus, we can describe the attitude of a p-vector either in terms of its equivalence class (the family of

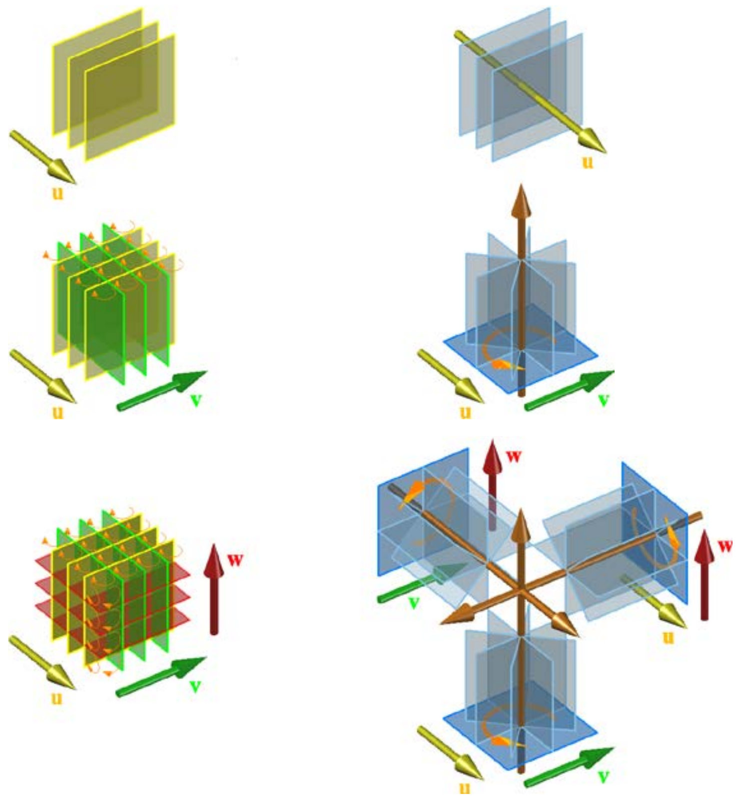

Fig. 18 Geometric interpretation of the attitude of a p-vector in terms of class invariants

parallel lines, when the p-vector is a vector), or in terms of its class invariant (the family of normal planes, when the p-vector is a vector), that is, the equivalence class of its orthogonal complement. In particular, the attitude of a vector $\mathbf{u}$ can be viewed as a family of normal planes (Fig. 18), each one originated by the translation of a plane normal to $\mathbf{u}$, along the direction of $\mathbf{u}$ (the planes span the direction of $\mathbf{u}$). Equivalently, the attitude of $\mathbf{u}$ can be represented by an arbitrary plane of the family of normal planes.

Analogously, the attitude of a bivector $\mathbf{u} \wedge \mathbf{v}$ can be viewed as two families of parallel planes (Fig. 18), the first family normal to $\mathbf{u}$ and the second family normal to $\mathbf{v}$ (the planes span both the directions of $\mathbf{u}$ and $\mathbf{v}$). Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent in their common plane, the planes that span both the directions of $\mathbf{u}$ and $\mathbf{v}$ originate all the planes normal to $\mathbf{u} \wedge \mathbf{v}$, that is, all the planes parallel to $\mathbf{u} \times \mathbf{v}$. These planes can be represented by the line of intersection between an arbitrary plane of the first family and an arbitrary plane of the second family (Fig. 18). The intersection line is parallel to all planes of the two families and to vector $\mathbf{u} \times \mathbf{v}$. Finally, the attitude of a trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ can be viewed as three families of parallel planes (Fig. 18), provided that the three families are normal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, respectively (the planes span the three directions of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$). If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, then the three families originate all the plane of the three-dimensional space.

A possible representation of all the planes of the three-dimensional space, under the equivalence relation of parallelism, is achieved by choosing a point of the space and considering the set of all the planes that contain the point. Being common to all the planes, the point can be used for representing the whole set of planes, which, in turn, represents all the planes of the three-dimensional space.

In conclusion, as for the inner orientation of the p-vectors, also the attitude of the p-vectors is defined inductively, starting from the 1-vector. This allows us to define the attitude of the p-vectors even in dimension greater than 3.

The same family of parallel planes represents both the set of planes that are normal to $\mathbf{u}$ and the set of hyperplanes of $\mathbf{u}^*$, the dual vector of $\mathbf{u}$. Consequently, the attitude of the class invariant of a vector $\mathbf{u}$ equals the attitude of the covector $\mathbf{u}^*$.

This is ultimately a consequence of the Riesz representation theorem, which allows us to represent a covector by its related vector [64]. Thus, there exists a bijective correspondence between the attitude of the orthogonal complement of a vector $\mathbf{u}$ and the attitude of its covector, $\mathbf{u}^*$.

The bijective correspondence extends also to the second feature, that is, the orientations of a vector and its covector, since the order of the hyperplanes is determined by the sense of $\mathbf{u}$. This allows us to define a second type of orientation for the covector $\mathbf{u}^*$, which we call the outer orientation since it is induced by the (inner) orientation of $\mathbf{u}$ and has the geometrical meaning of sense of traversal of the hyperplanes of $\mathbf{u}^*$. In doing so, we have established a bijective correspondence between the inner orientation of a vector and the outer orientation of its covector. On the other hand, since it is always possible to define an inner orientation for $\mathbf{u}^*$ (by choosing a basis bivector for $\mathbf{u}^*$), the duality between vectors and covectors will result in an outer orientation for $\mathbf{u}$, induced by the inner orientation of $\mathbf{u}^*$.

Therefore, the inner orientation of a covector induces an outer orientation on its vector. Moreover, since the equivalence classes of $\mathbf{u}^*$ are in bijective correspondence with the attitude of $\mathbf{u}$, to fix the inner orientation of $\mathbf{u}^*$ is also equivalent to fixing an orientation, which is an inner orientation, for the attitude of $\mathbf{u}$. In doing so, the attitude of $\mathbf{u}$ becomes an attitude vector and its inner orientation equals the outer orientation of $\mathbf{u}$. Therefore, by providing the attitude with an orientation, we establish an isomorphism between the orthogonal complement and the dual vector space of any sub-set of vectors. This means that the pairing between the geometric algebra and its dual can be described by the invariants of the equivalence relation of parallelism.

In conclusion, we can define the orientation of a vector by providing either its inner orientation or the inner orientation of its attitude vector (which is also the outer orientation of the vector). The latter, in turn, is equal to the inner orientation of the covector.

The relationship between the inner and outer orientations and the related notion of orthogonal complement (or dual element) are implicit, both in mathematics and physics. They are given by the right-hand rule, which is equivalent to the right-hand grip rule and the right-handed screw rule. We make them explicit in this paper because they are at the basis of the CM description of physics.

The dual of a p-dimensional space element has dimension $n-p$, in the n-dimensional space. This means that the outer orientation depends on the dimension of the embedding space, whereas the inner orientation does not.

C. Relationship between Cell Method and Geometric Algebra

Due to the geometrical interpretation of k-vectors provided by the geometric algebra, we can associate the elements of a vector space and its dual space with the geometrical elements of two cell-complexes, where the elements of the second cell-complex are the orthogonal complements of the corresponding elements in the first cell-complex (Fig. 19). As a consequence, by providing the elements of the first cell-complex with an inner (or an outer) orientation, we induce an outer (or an inner) orientation on the second cell-complex.

It is true that the inner orientation of the elements of a vector space also induces an outer orientation on the elements of the same vector space and this may allow us to think that a single cell-complex would be sufficient. Nevertheless, the association between the two orientations of the same cell-complex is not automatic. There are always two possible criteria for establishing the correspondence between the two orientations, which depend on the orientation of the embedding space. Conversely, the relationship between inner (or outer) orientation of a cell-complex and outer (or inner) orientation of its dual cell-complex is derived from the Riesz representation theorem and does not depend on the orientation of the embedding space. Therefore, choosing to use two cell-complexes, the one the dual of the other, instead of one single cell-complex, is motivated by the need to provide a description of vector spaces that is independent of the orientation of the embedding space. This means that a proper description of a given physical phenomenon requires to use two cell-complexes in relation of duality, not just one, as usually was done in computational physics before the introduction of the CM.

We will denote the first cell-complex as the primal cell-complex, or primal complex, and the second cell-complex as the dual cell-complex, or dual complex.

In algebraic topology, the cell-complexes are viewed as generalizations of the oriented graphs. Therefore, all the properties of the dual graphs naturally extend to the dual cell-

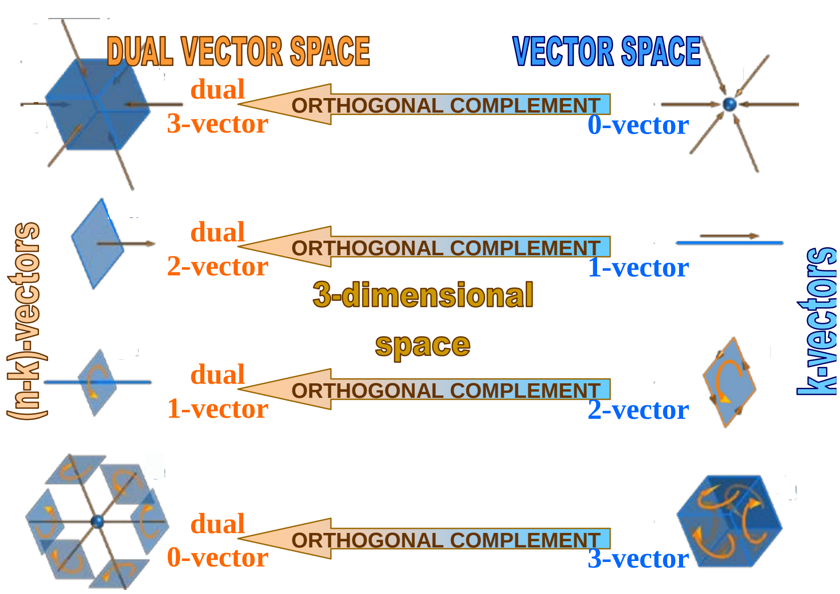

Fig. 19 Association between the elements of a vector space and the elements of the dual space in 3-dimensional space

complexes. In particular, the dual graphs depend on a particular embedding. Since even the orthogonal complements (that is, the isomorphic dual vectors) and the outer orientation depend on the embedding, we will associate the outer orientation with the dual cell-complex and will retain the inner orientation for the primal cell-complex.

The most natural way for building the two cell complexes is starting from a primal cell complex made of simplexes and providing this first cell-complex with an arbitrary inner orientation. The set of the dual elements can then be chosen as any arbitrary set of staggered elements whose outer orientations provide the (known) inner orientations of the primal p-cells. In this sense, we can say that the outer orientations of the dual p-cells are induced by the inner orientations of the primal p-cells.

Moreover, the dimension of the dual of a k-vector depends on the dimension n of the space in which it is embedded and is equal to $n-k$. Thus, in a three-dimensional space the dual of a 0-vector has dimension 3, where the correspondence also extends to the inner and outer orientations due to the Riesz representation theorem (Fig. 19). Analogously, the dual of an 1-vector has dimension 2, the dual of a 2-vector has dimension 1, and the dual of a 3-vector has dimension 0.

This is exactly the same correspondence we have in the classification diagram of the physical variables (Fig. 9). Therefore, now we are able to recognize in the classification diagram of the CM a structure of bialgebra.

Moreover, since the global source variables require outer orientations, we have gained the mathematical explanation of why the global source variables must be associated with the dual p-cells.

In conclusion, by associating the global configuration variables of the CM with the primal p-cells:

- The set of topological equations between global configuration variables defines a geometric algebra on the space of global configuration variables, provided with a geometric product.
- The operators of these topological equations are generated by the outer product of the geometric algebra, which is equal to the exterior product of the enclosed exterior algebra.
- The dual algebra of the enclosed exterior algebra is the space of global source variables, associated with the dual p-cells, and is provided with a dual product that is compatible with the exterior product of the exterior algebra.
- The topological equations between global source variables arise from the adjoint operators of the primal operators.
- The pairing between the exterior algebra and its dual gives rise to the energetic variables, by the interior product.
- Since the reversible constitutive relations may be written in terms of energetic variables, because energy is the potential of the reversible constitutive relations, the reversible constitutive relations realize the pairing between the exterior algebra and its dual.

V. Inner and Outer Orientations of Time Elements in the CM

When the physical phenomenon evolves in time, we have so many classification diagrams as the time instants are. Since it is not possible to draw a classification diagram for each time instant, we simply double the diagram and shift it to the rear, along the time axis (Fig. 20).

Finding the orientations of the time elements could be viewed in the same way that finding the inner and outer orientations of the space elements in a one-dimensional space. In fact, the time axis defines a one-dimensional cell-complex (Fig. 21), where the time instants, $\mathbf{I}$, are the primal nodes and the time intervals, $\mathbf{T}$, are the primal sides (the time instants are the boundaries, or the faces, of the time intervals).

Moreover, in a one-dimensional space the dual (orthogonal complement) of a point is a line segment and the dual of a line segment is a point. Consequently, the nodes of the dual cell-complex are the middle points of the primal sides. They define

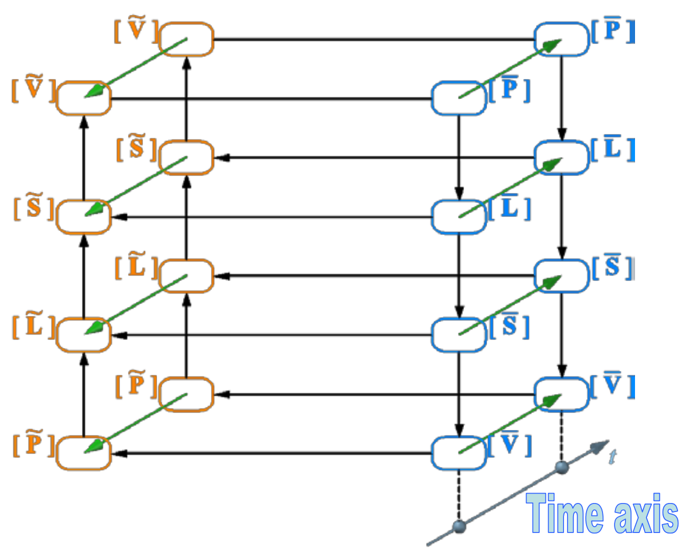

Fig. 20 Space–time classification diagram of the physical variables

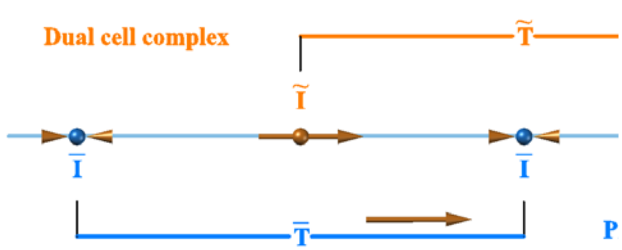

Fig. 21 Time elements and their duals

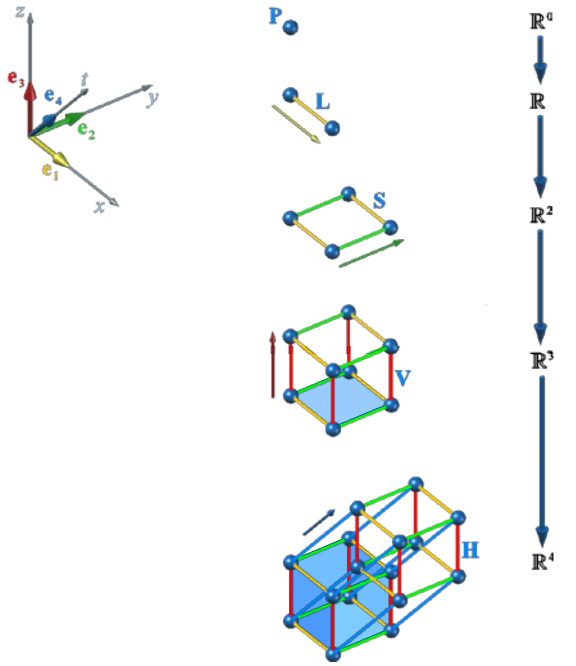

Fig. 22 Inductive construction of a 4D hyperprism from dimension 0 to dimension 4, by adding one dimension at a time

the dual time instants and are the duals of the primal sides.

As far as the inner orientation is concerned, all the time instants, both those along the positive semi-axis and those along the negative semi-axis, are sinks. Thus, they have an inward inner orientation (Fig. 21). Finally, we can decide that the inner orientation of the time intervals is the same as the orientation of the time axis.

After a more detailed analysis, however, it is clear that building a cell-complex in time makes no sense in itself [65]. In fact, in physics time has not importance in itself. It is just a variable, useful for describing how a physical phenomenon evolves in space. Consequently, the time axis must always be related to one or more axes in space.

The perception itself of time is linked to bodies. Therefore, a cell-complex in time must be two-dimensional, at least. In particular, for studying three-dimensional bodies in time, we have to add a time axis to a three-dimensional cell-complex where the cell of greater dimension has been originated by a trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$.

The extension of a trivector along a forth direction gives rise to a 4-vector (Fig. 22), the tesseract, which is the basic unit for building a four-dimensional space/time cell-complex.

In multi-linear algebra, the tesseract is a further element of the (graded) exterior algebra on a vector space. It is the four-dimensional analog of the cube, in the sense that it is to the cube as the cube is to the square. Just as the surface of the cube consists of six square faces, the hyper-surface of the tesseract consists of eight cubical cells. Each edge of a tesseract is of the same length and there are three cubes folded together around every edge.

For representing a tesseract in the plane, we can unfold the tesseract in its eight constituents cubes, use of one of its shadows in 2 dimensions, or employ one-point perspective for drawing the fourth dimension and axonometric projection for drawing the remaining three dimensions. In Fig. 23 we have shown the most commonly used representation of the tesseract in the plane, where the centre of projection is inside the tesseract. According to this representation, the attention of the observer is focused on the body, which changes dimension with time because the relative position between body and observer changes with time.

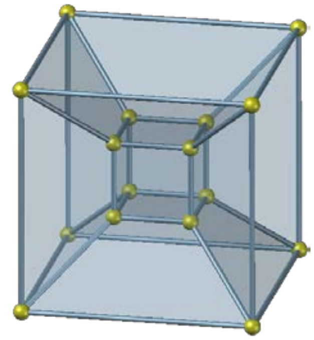

Fig. 23 The 8 cubical cells of the tesseract folded, in threes, around the same edge

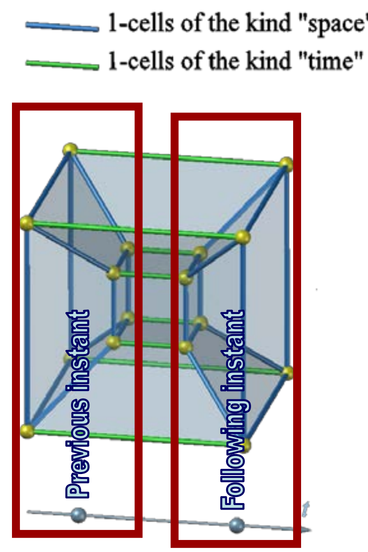

Fig. 24 Different kinds of 1-cells in a space/time tesseract

In the following, we will adopt the representation shown in Fig. 23 but with a different association between types of projection and dimensions of the tesseract, since we will plot the fourth dimension of time along one of the three directions of axonometric projection. Specifically, we will associate the left cube of Fig. 23 with the previous time instant and the right cube with the following time instant (Fig. 24). Using one direction of axonometric projection for representing time means that we are focusing our attention on time rather than on the three space dimensions of the body.

The p-cells of the space/time 4-vector are of different nature, since some p-cells are associated with a variation of the space variables, some other p-cells are associated with a variation of the time variables, and the remaining p-cells are associated with a variation of both the space and time

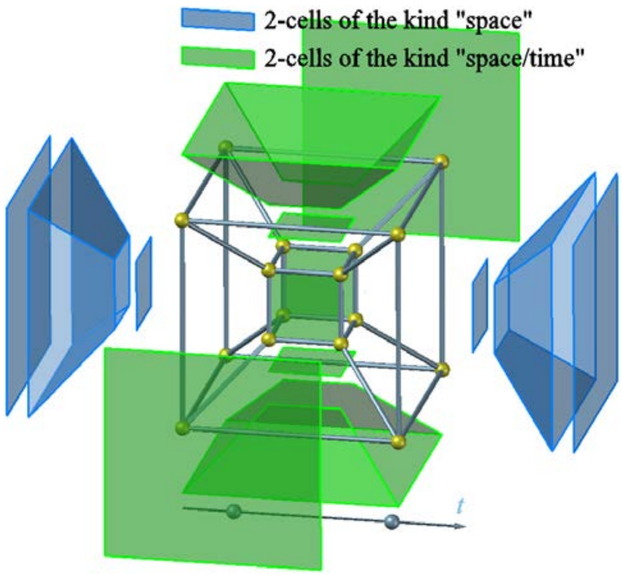

Fig. 25 Different kinds of 2-cells in a space/time tesseract

variables.

In particular, the points are associated with a variation of both the space and time variables. Consequently, we can say that there exists just one kind of 0-cells.

As far as the others p-cells are concerned, on the contrary, we can define two different kinds of p-cells for each $p = 1, 2, 3$. Therefore, we have two kinds of 1-cells (Fig. 24):

- 1-cells of the kind "space," which connect points associated with the same time instant, that is, the edges of the cube (the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$) at the previous instant and the edges of the cube at the following instant.
- 1-cells of the kind "time," which connect points associated with two adjacent time instants, that is, the time intervals.

Analogously, we have two kinds of 2-cells (Fig. 25):

- 2-cells of the kind "space," which connect edges associated with the same time instant, that is, the faces of the cubes at a given instant.
- 2-cells of the kind "space/time," which connect edges associated with two adjacent time instants. The area of one of these faces is given by the product between the time interval and one edge of the space trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$.

Finally, we have two kinds of 3-cells (Fig. 26):

- 3-cells of the kind "space," which connect faces associated with the same time instant, that is, the volume of the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ at a given instant.
- 3-cells of the kind "space/time," which are enclosed within faces associated with two adjacent time instants. The volume of one of these 3-cells is given by the product between the time interval and two edges of the space trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$.

Each cube of the tesseract has both an inner and an outer orientation. For the sake of simplicity, in Fig. 27 we have shown just the inner orientations of the eight cubes. In order to comply with the natural time sequence, from past to future, we will use the orientations of the tesseract just for the two cubes of the kind space and will not orientate the remaining six cubes (Fig. 28).

Moreover, we will provide the eight edges of the kind "time" with the inner orientation from past to future (Fig. 28), in the same orientation of the time axis. Finally, since the same point of a four-dimensional space denotes both a point in space and a point in time (a time instant), it follows that the time instants have an inward inner orientation, that is, they are sinks.

Thus, we will treat the time dimension differently from the three space dimensions. This is exactly the same thing that happens in spacetime, the four-dimensional Minkowski continuum, whose metric treats the time dimension differently from the three spatial dimensions. Consequently, spacetime is not an Euclidean space.

In conclusion, by exploiting the geometrical associations provided by the Cell Method when we associate the global

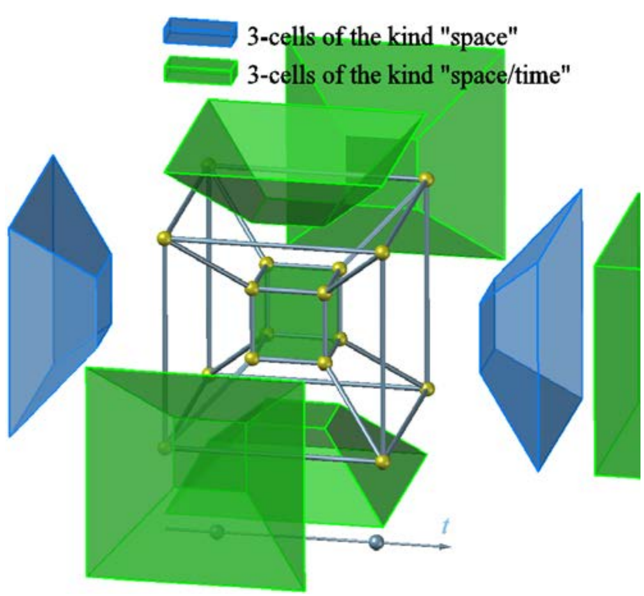

Fig. 26 Different kinds of 3-cells in a space/time tesseract

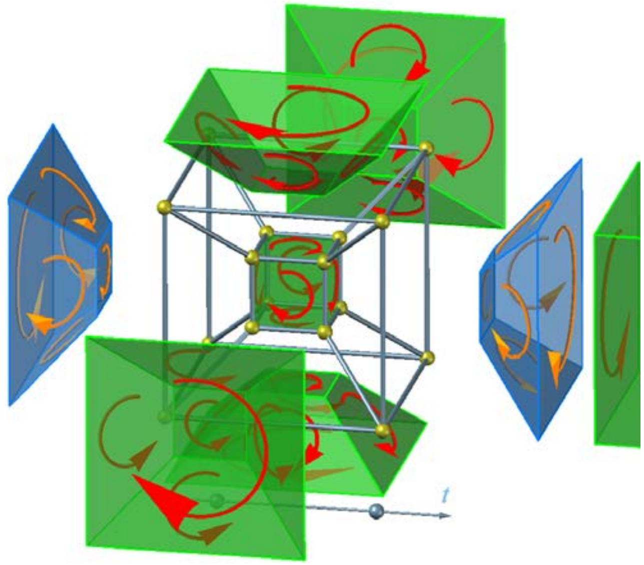

Fig. 27 Inner orientations on the 2-cells of the 4-vector

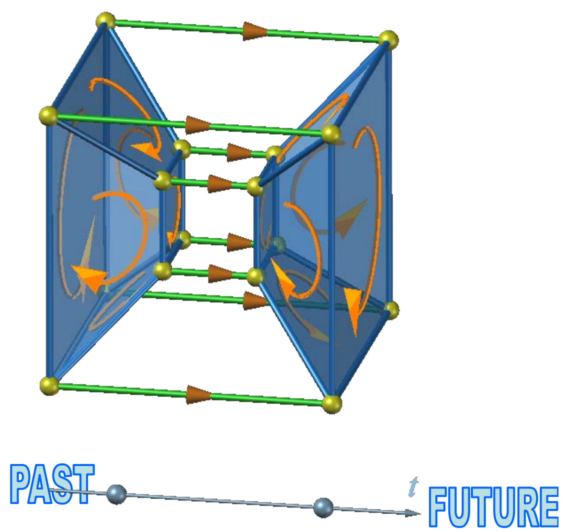

Fig. 28 The CM tesseract: inner orientations on the 3-cells of the kind space and the 1-cells of the kind time

space and time variables with the oriented elements of a 4-vector, we have obtained the algebraic version of spacetime.

VI. Conclusion

In this paper, we have shown that the geometric interpretations of the operations on vectors, provided by both the exterior and geometric algebra, and the notions of extension of a vector by another vector, multi-vector (or p-vector), dual vector space, covector, and bialgebra are of special importance for understanding the mathematical foundations of the Cell Method (CM).

The geometric approach allowed us to view the space elements and the time elements as p-vectors of a geometric algebra, all inductively generated by the outer product of the geometric algebra. From the attitude and orientation of p-vectors, we have then derived the two kinds of orientation for p-vectors, inner and outer orientations, which apply to both the space and the time elements. We have also discussed how the orientation of a p-vector is induced by the orientation of the $(p-1)$-vectors on its boundary and how the inner orientation of the attitude vector of a vector equals the outer orientation of its covector. This establishes an isomorphism between the orthogonal complement and the dual vector space of any sub-set of vectors.

By exploiting the geometrical interpretations of p-vectors, we have gained an insight into the mathematical foundations of the Cell Method. Specifically, we have seen that distinguishing between configuration and source variables, which is at the basis of the classification diagram, has an additional meaning beyond the mere classification of variables. Actually, the global configuration variables, with their topological equations, define a bialgebra on a vector space, which is denoted as the primal vector space, and the global source variables, with their topological equations, define a dual algebra on the dual vector space.

The operators of the topological equations are generated by the outer product of the geometric algebra, for the primal vector space, and by the dual product of the dual algebra, for the dual vector space.

Being expressed as topological equations in two different vector spaces, compatibility and balance can be enforced at the same time, with compatibility enforced on the primal cell-complex and equilibrium enforced on the dual cell-complex.

Moreover, choosing to use two cell-complexes, the one the dual of the other by the Riesz representation theorem, allows us to provide a description of vector spaces whose orientations are independent of the orientation of the embedding space.

Finally, by extending the primal and dual cell-complexes along the fourth dimension of a time axis and treating the time dimension differently from the three spatial dimensions, we obtain the algebraic version of the four-dimensional

Minkowski continuum, useful for studying spacetime.

REFERENCES

[1] E. Tonti, *The mathematical structure of classical and relativistic physics*. Birkhäuser, 2013.

[2] A. Askarova, S. Bolegenova, S. Bolegenova, A. Bekmukhamet, V. Maximov, M. Beketayeva, "Numerical experimenting of burning high-ash content Ekibastuz coal in the real boiler of CHP," in *Recent Advances in Fluid Mechanics and Heat & Mass Transfer, Recent Advances in Mechanical Engineering Series*, no. 3, WSEAS Press, 2013, pp. 138–147.

[3] F. Behrouzi, N. A. B. Che Sidik, M. Nakisa, A. Witri, "Numerical prediction of wind flow around the high-rise buildings by two equations turbulence models for urban street canyon," in *Computational Methods in Science & Engineering, Mathematics and Computers in Science and Engineering Series*, no. 10, WSEAS Press, 2013, pp. 152–156.

[4] M. A. Bennani, A. El Akkad, A. Elkhalfi, "Mixed finite element method for linear elasticity in a cracked domain," in *Advances in Applied and Pure Mathematics, Mathematics and Computers in Science and Engineering Series*, no. 27, WSEAS Press, 2014, pp. 328–339.

[5] M. Calbureanu, R. Malciu, D. Tutunea, A. Ionescu, M. Lungu, "Finite element modeling of a spark ignition engine piston head," in *Recent Advances in Fluid Mechanics and Heat & Mass Transfer, Recent Advances in Mechanical Engineering Series*, no. 3, WSEAS Press, 2013, pp. 61–64.

[6] R. Duddu, "Numerical modeling of corrosion pit propagation using the combined extended finite element and level set method," *Comput. Mech*., vol. 54, no. 3, pp. 613–627, 2014.

[7] C. Dumitru, "Numerical problems in 3D magnetostatic FEM analysis," in *Advances in Automatic Control, Modelling & Simulation, Recent Advances in Electrical Engineering Series*, no. 13, WSEAS Press, 2013, pp. 385–390.

[8] N. Dumitru, R. Malciu, M. Calbureanu, "Contributions to the elastodynamic analysis of mobile mechanical systems using finite element method," in *Recent Advances in Robotics, Aeronautical & Mechanical Engineering, Recent Advances in Mechanical Engineering Series*, no. 4, WSEAS Press, 2013, pp. 116–121.

[9] D. Foti, A. Romanazzi, "Numerical modeling of the shape of cells for the optimization of the mechanical properties of brick blocks," in *Recent Researches in Information Science & Applications, Recent Advances in Computer Engineering Series*, no. 9, WSEAS Press, 2013, pp. 179–183.

[10] K. Frydrýšek, R. Janco, H. Gondek, "Report about the solutions of beams and frames on elastic foundation using FEM," in *Recent Advances in Mathematical Methods & Computational Techniques in Modern Science, Mathematics and Computers in Science and Engineering Series*, no. 11, WSEAS Press, 2013, pp. 143–146.

[11] K. Frydrýšek, M. Kvíčala, "Report about the temperature stress-strain states in continuously cast bloom (FEM modelling and experiments)," in *Mathematical Applications in Science & Mechanics, Mathematics and Computers in Science and Engineering Series*, no. 14, WSEAS Press, 2013, pp.259–262.

[12] K. Frydrýšek, M. Nikodým, "Report about solutions of beam on nonlinear elastic foundation," in *Mathematical Applications in Science & Mechanics, Mathematics and Computers in Science and Engineering Series*, no. 14, WSEAS Press, 2013, pp. 263–266.

[13] I. Ignatov, "Numerical study of friction induced vibrations of a rail," Recent Advances in Continuum Mechanics, Hydrology and Ecology, in *Energy, Environmental and Structural Engineering Series*, no. 14, WSEAS Press, 2013, pp. 76–79.

[14] B. B. Kanbur, S. O. Atayilmaz, H. Demir, A. Koca, Z. Gemici, "Investigating the thermal conductivity of different concrete and reinforced concrete models with numerical and experimental methods," in *Recent Advances in Mechanical Engineering Applications, Recent Advances in Mechanical Engineering Series*, no. 8, WSEAS Press, 2013, pp. 95–101.

[15] K. N. Kiousis, A. X. Moronis, E. D. Fylladitakis, "Finite element analysis method for detection of the corona discharge inception voltage in a wire-cylinder arrangement," in *Recent Advances in Finite Differences and Applied & Computational Mathematics, Mathematics and Computers in Science and Engineering Series*, no. 12, WSEAS Press, 2013, pp. 188–193.

[16] M. Y. S. Kuan, N. C. Green, D. M. Espino, "Application of fluid-structure interaction to investigate a malformed biological heart valve: a three dimensional study of the bicuspid aortic valve," in *Recent Researches in Mechanical Engineering, Recent Advances in Mechanical Engineering Series*, no. 2, WSEAS Press, 2013, pp. 113–115.

[17] P. Lehner, P. Konecny, P. Ghosh, "Finite element analysis of 2-D chloride diffusion problem considering time-dependent diffusion coefficient model," in *Recent Advances in Applied & Theoretical Mathematics, Mathematics and Computers in Science and Engineering Series*, no. 20, WSEAS Press, 2013, pp. 93–96.

[18] F. Lizal, J. Elcner, J. Jan, M. Jicha, "Investigation of flow in a model of human airways using constant temperature anemometry and numerical simulation," in *Recent Advances in Fluid Mechanics and Heat & Mass Transfer, Recent Advances in Mechanical Engineering Series*, no. 3, WSEAS Press, 2013, pp. 35–40.

[19] R. Manimaran, R. T. K. Raj, "Numerical investigations of spray droplet parameters on combustion and emission characteristics in a direct injection diesel engine using 3-zone extended coherent flame model," in *Advances in Modern Mechanical Engineering, Advances in Mechanical Engineering Series*, no. 6, WSEAS Press, 2013, pp. 47–67.

[20] M. Pirulli, M. Barbero, F. Barpi, M. Borri-Brunetto, O. Pallara, "The contribution of continuum-mechanics based numerical models to the design of debris flow barriers," in *Latest Trends in Engineering Mechanics, Structures, Engineering Geology, Mathematics and Computers in Science and Engineering Series*, no. 26, WSEAS Press, 2014, pp. 13–21.

[21] A.-M. Pop, D. Grecea, A. Ciutina, "Numerical vs. experimental behaviour of bolted dual-steel T-stub connections," in *Recent Advances in Civil and Mining Engineering, Mathematics and Computers in Science and Engineering Series*, no. 18, WSEAS Press, 2013, pp. 192–199.

[22] L. Porojan, F. Topală, S. Porojan, "Finite element method applied for the evaluation of teeth restored with custom made post-and-core systems," in *Advances in Applied and Pure Mathematics, Mathematics and Computers in Science and Engineering Series*, no. 27, WSEAS Press, 2014, pp. 258–264.

[23] A. Puskas, A. Chira, "Numerical investigations on a wide reinforced concrete beam subjected to fire," in *Mathematical Models in Engineering & Computer Science*, WSEAS Press, 2013, pp. 169–174.

[24] I. R. Răcănel, "Design of bridge shallow foundations using finite element method," in *Recent Advances in Civil and Mining Engineering, Mathematics and Computers in Science and Engineering Series*, no. 18, WSEAS Press, 2013, pp. 23–29.

[25] I. R. Răcănel, G. Stoicescu, "Analysis of the stress state for rails with holes near joints," in *Recent Advances in Civil and Mining Engineering, Mathematics and Computers in Science and Engineering Series*, no. 18, WSEAS Press, 2013, pp. 129–135.

[26] N. G. Radu, I. Comanescu, "The analysis of the state of stress and strain of a loaded portal crane, using FEM, with respect to the strength, stiffness and stability," in *Mathematical Models in Engineering & Computer Science*, WSEAS Press, 2013, pp. 143–147.

[27] T. Rymarczyk, J. Sikora, "Solving inverse problems by connection of level set method, gradient technique and finite or boundary elements," in *Advances in Applied and Pure Mathematics, Mathematics and Computers in Science and Engineering Series*, no. 27, WSEAS Press, 2014, pp. 202–205.

[28] E. Scutelnicu, "Simulation of thermo-mechanical effects induced by submerged double-arc welding process in pipelines," in *Recent Advances in Mechanical Engineering Applications, Recent Advances in Mechanical Engineering Series*, no. 8, WSEAS Press, 2013, pp. 111–116.

[29] R. L. Sharma, "Viscous incompressible flow simulation using penalty finite element," in *Latest Trends in Engineering Mechanics, Structures, Engineering Geology, Mathematics and Computers in Science and Engineering Series*, no. 26, WSEAS Press, 2014, pp. 22–29.

[30] I. Skotnicová, P. Tymová, Z. Galda, L. Lausová, "Numerical analysis and experimental validation of the thermal response of light weight timber frame structures," in *Recent Advances in Applied & Theoretical*

Mathematics, Mathematics and Computers in Science and Engineering Series, no. 20, WSEAS Press, 2013, pp. 178–182.

[31] O. Sucharda, J. Kubosek, “Analysing the slabs by means of the finite difference method,” in *Recent Advances in Applied & Theoretical Mathematics, Mathematics and Computers in Science and Engineering Series*, no. 20, WSEAS Press, 2013, pp. 268–274.

[32] S. Voloaca, G. Fratila, “Theoretical and experimental researches of Brake Discs’ thermal stress,” in *Advances in Automatic Control, Modelling & Simulation, Recent Advances in Electrical Engineering Series*, no. 13, WSEAS Press, 2013, pp. 57–62.

[33] Y.-C. Yoon, J.-H. Song, “Extended particle difference method for moving boundary problems,” *Comput. Mech*., vol. 54, no. 3, pp. 723–743, 2014.

[34] E. Ferretti, “Crack propagation modeling by remeshing using the cell method (CM),” *CMES-Comp. Model. Eng.*, vol. 4, no. 1, pp. 51–72, Feb. 2003.

[35] E. Ferretti, “A Cell Method (CM) code for modeling the pullout test step-wise,” *CMES-Comp. Model. Eng.*, vol. 6, no. 5, pp. 453–476, Nov. 2004.

[36] E. Ferretti, “Crack-path analysis for brittle and non-brittle cracks: A cell method approach,” *CMES-Comp. Model. Eng.*, vol. 6, no. 3, pp. 227–244, Sep. 2004.

[37] E. Ferretti, “A discrete nonlocal formulation using local constitutive laws,” *Int. J. Fracture* (Letters section), vol. 130, no. 3, pp. L175–L182, 2004.

[38] E. Ferretti, “A local strictly nondecreasing material law for modeling softening and size-effect: A discrete approach,” *CMES-Comp. Model. Eng.*, vol. 9, no. 1, pp. 19–48, Jul. 2005.

[39] E. Ferretti, “On nonlocality and locality: Differential and discrete formulations,” in *Proc. ICF11 - 11th International Conference on Fracture*, vol. 3, 2005, pp. 1728–1733.

[40] E. Ferretti, “Cell method analysis of crack propagation in tensioned concrete plates,” *CMES-Comp. Model. Eng.*, vol. 54, no. 3, pp. 253–281, Dec. 2009.

[41] E. Ferretti, “The cell method: An enriched description of physics starting from the algebraic formulation,” *CMC: Comput. Mater. Con*., vol. 36, no.1, pp. 49-71, Jul. 2013.

[42] E. Ferretti, “A Cell Method stress analysis in thin floor tiles subjected to temperature variation,” *CMC: Comput. Mater. Con.*, vol. 36, no. 3, pp. 293–322, Aug. 2013.

[43] E. Ferretti, “The cell method as a case of bialgebra,” in *Recent Advances in Applied Mathematics, Modelling and Simulation, Mathematics and Computers in Science and Engineering Series*, no. 34, WSEAS Press, 2014, pp. 322–331.

[44] E. Ferretti, “The assembly process for enforcing equilibrium and compatibility with the CM: a coboundary process,” *CMES-Comp. Model. Eng.*, to be published.

[45] E. Ferretti, “Similarities between cell method and non-standard calculus,” in *Recent Advances in Computational Mathematics, Mathematics and Computers in Science and Engineering Series*, no. 39, WSEAS Press, 2014, pp. 110–115.

[46] E. Ferretti, “The algebraic formulation: Why and how to use it,” *Curved and Layer. Struct.*, vol. 2, pp. 106–149, 2015.

[47] E. Ferretti, “The cell method: An overview on the main features,” *Curved and Layer. Struct.*, vol. 2, pp. 195–243, 2015.

[48] E. Ferretti, E. Casadio, A. Di Leo, “Masonry walls under shear test: A CM modeling,” *CMES-Comp. Model. Eng.*, vol. 30, no. 3, pp. 163–189, Jun. 2008.

[49] E. Viola, F. Tornabene, E. Ferretti, N. Fantuzzi, “Soft core plane state structures under static loads using GDQFEM and Cell Method,” *CMES-Comp. Model. Eng.*, vol. 94, no. 4, pp. 301–329, Aug. 2013.

[50] E. Viola, F. Tornabene, E. Ferretti, N. Fantuzzi, “GDQFEM numerical simulations of continuous media with cracks and discontinuities,” *CMES-Comp. Model. Eng.*, vol. 94, no. 4, pp. 331–369, Aug. 2013.

[51] E. Viola, F. Tornabene, E. Ferretti, N. Fantuzzi, “On static analysis of composite plane state structures via GDQFEM and Cell Method,” *CMES-Comp. Model. Eng.*, vol. 94, no. 5, pp. 421–458, Aug. 2013.

[52] E. Ferretti, “Some new findings on the mathematical structure of the cell method,” *International Journal of Mathematical Models and Methods in Applied Sciences*, submitted for publication.

[53] R. Bott, L. W. Tu, *Differential forms in algebraic topology*. Springer-Verlag, Berlin, New York, 1982.

[54] F. H. Jr. Branin, “The algebraic topological basis for network analogies and the vector calculus,” in *Proc*. *Symp. on Generalized Networks*, Brooklyn Polit., 1966, pp. 453–487.

[55] P. Duhem, *The aim and structure of physical theory*. Atheneum, New York, 1977.

[56] A. G. Sumedrea, “A topological approach to tensional psychological construct,” in *Advances in Automatic Control, Modelling & Simulation, Recent Advances in Electrical Engineering Series*, no. 13, WSEAS Press, 2013, pp. 226–231.

[57] Md. S. Alam, M.H. Ahsan, “Comparative study of geometric product and mixed product,” in *Statistics and its Applications (ICMSA2010) - Proc. 6th IMT-GT Conference on Mathematics*, Kuala Lumpur (Malaysia), 2003, pp. 110–114.

[58] Md. S. Alam, S. Bauk, “Mixed number and Clifford algebra,” in *Recent Researches in Applied & Computational Mechanics*, 2011, pp. 15–17.

[59] G. R. Franssens, “Clifford analysis formulation of electromagnetism,” in *Proc. 9th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering (MACMESE ‘07)*, Trinidad and Tobago, 2007, pp. 51–57.

[60] B. Jefferies, “Spectral theory for systems of matrices,” in *Proc. 2nd WSEAS Multiconference on Applied and Theoretical Mathematics*, Cairns (Queensland, Australia), 2001, pp. 5511–5515.

[61] K. Sato, “A formula of an orthogonal matrix with Clifford algebra,” in *Proc. WSEAS International Conference on Applied Mathematics*, Corfu (Grece), 2004, pp. 1–3.

[62] V. Skala, “Computation in projective space,” in *Mathematical Methods, System Theory and Control, Mathematics and Computers in Science and Engineering*, WSEAS Press, 2009, pp. 152–157.

[63] E. Zupan, M. Saje, D. Zupan, “Dinamics of spatial beams in quaternion description based on the Newmark integration scheme,” *Comput. Mech*., vol. 51, no. 1, pp. 47–64, 2013.

[64] L. Demkowicz, J. Li, “Numerical simulations of cloaking problems using a DPG Method,” *Comput. Mech*., vol. 51, no. 5. pp. 661–672, 2013.

[65] E. Ferretti, *The cell method: A purely algebraic computational method in physics and engineering*. Momentum Press, 2014.