Efficient and Accurate Scheme for Hyperbolic Conservation Laws

Mukkarum Hussain[1], Ihtram ul Haq[2], and Noor Fatima[3]

***Abstract*—**The progress of numerical techniques for scalar and one dimensional Euler equation has been a great interest of researchers in the field of computational fluid dynamics for decades. In 1983, Harten worked on non-oscillatory first order accurate scheme and modified its flux function to obtain a second order accurate total variation diminishing (TVD) explicit difference schemes for scalar and one dimensional Euler equation. Although, TVD schemes are low dissipative and high resolution schemes, but for explicit formulation they are bounded by stability criterion CFL<1. Stability criteria for explicit formulation limits time stepping and thus increase computational cost. Research in the field of efficient low dissipative high resolution scheme is an active ground. In 1986, Harten enhanced his TVD scheme and presented (2K+3) point explicit second order accurate schemes for scalar and one dimensional Euler equation which are TVD under CFL restriction K. Numerical experiments were made to demonstrate the performance of the schemes for several choices of K. His results depict that for increasing values of CFL total number of time steps are decreased which eventually decrease computational time. Computation of scalar problems depicts that Harten's large time step (LTS) scheme is a high resolution and efficient scheme. However, computations of hyperbolic conservation laws show some spurious oscillations in the vicinities of discontinuities for larger values of CFL. Zhan Sen Qian noticed that these spurious oscillations are due to the numerical formulation of the characteristic transformation used by Harten for extending the method for hyperbolic conservation laws. He suggested performing the inverse characteristic transformations by using the local right eigenvector matrix at each cell interface location to overcome these spurious oscillations. Large time step schemes developed by Harten and Qian have been tested with minmod limiter which is very dissipative. In present work, Qian MLTS TVD scheme is tested with more compressive limiters, namely, centralized MC and superbee. Shock tube problem for SOD boundary conditions is solved to understand the performance of MLTS TVD scheme with compressive limiters in the regions of discontinuities and strong shock waves.

***Keywords*—**CFL, Explicit scheme, Large time step, Shock tube problem, TVD scheme, 1D Euler equation.

I. Introduction

THEsystem of equation is called hyperbolic if it has all real and distinct Eigen values. Flow fields which are governed by hyperbolic equations are computed using marching solutions. A scheme for hyperbolic system of equations is started with the given initial conditions and successively computes the flow field in marching direction [*1*] [*2*]. Transient 1D Euler equation is hyperbolic, no matter whether the flow is locally subsonic or supersonic. The marching direction for 1D Euler equation is the time direction. Methods to solve hyperbolic system of equations are primarily derived for non-linear wave equation and then implemented on hyperbolic system of equations.

Lax in 1954, modified Euler's Forward Time Central Space (FTCS) method and presented first-order accurate method to solve nonlinear wave equation. Lax method is stable for Courant-Friedrichs-Lewy condition (CFL) less than 1 and predicts the location of moving discontinuity correctly [*1*] [*3*]. This method is very dissipative and smears discontinuities over several mesh points and become worse as CFL decreases. Lax-Wendroff proposed a second-order accurate method for non-linear wave equation. His method sharply defined discontinuity and also stable for CFL less than 1 but produce undesirable oscillations when discontinuities are encountered. Similar to Lax method quality of results computed by Lax-Wendroff method degrade as CFL decrease [*1*].

Lax and Lax-Wendroff central finite difference schemes are stable and converge if flow field is sufficiently smooth but produce unwanted oscillations when discontinuities are met. It is due to the fact that series expansion for obtaining a difference approximation is only valid for continuous functions and has continuous derivatives at least through the order of difference approximation [*4*] [*5*]. Godunov recognized this deficiency and proposed a finite volume scheme instead of a finite difference scheme to avoid the need of differentiability. He used exact Riemann problem solution for evaluating the flux term at the cell interface. Computation of nonlinear wave equation is easily accomplished by using Godunov method but this method is very inefficient and take long time when applied to system of equations [*1*] [*6*].To overcome this problem Roe suggested solving linear problem instead of actual nonlinear problem. Roe's approximate Riemann solver is efficient but cannot distinguish between expansion shock and compression shock. This is due to the violation of entropy condition and hence expansion shocks that are nonphysical may occur in computed results [*7*] [*8*]. A number of entropy fix have been recommended in literature to overcome this problem. Roe's upwind approximate Riemann solver capture physics in more appropriate way than Lax and Lax-Wendroff central schemes but is only first order accurate. Like second order central methods, higher order upwind

Mukkarum Hussain[1] from Institute of Space and Technology, Karachi, Pakistan; (mrmukkarum@yahoo.com).

Ihtram Ul Haq[2] from Institute of Space and Planetary Astrophysics (ISPA), University of Karachi, Karachi, Pakistan; (ihtram2010@hotmail.com).

Noor Fatima[3] from Department of Mathematics, University of Karachi, Karachi, Pakistan; (nfsiddique@uok.edu.pk).

methods have the same deficiencies and produce undesirable oscillations when discontinuities are encountered [9] [10].

Harten introduced the concept of Total Variation Diminishing (TVD) scheme. TVD schemes are monotonicity preserving schemes and therefore it must not create local extrema and the value of an existing local minimum must be non-decreasing and that of a local maximum must be non-increasing [1] [11] [12] [13]. He worked on non-oscillatory first order accurate scheme and modified its flux function to obtain a second order accurate total variation diminishing (TVD) explicit difference schemes for scalar and system of hyperbolic conservation laws. Numerical dissipation terms in TVD methods are nonlinear. The quantity varies from one grid point to another and usually consists of automatic feedback mechanisms to control the amount of numerical dissipation. After this break through a number of TVD scheme have been proposed and discussed in literature [14] [15] [16] [17] [18].

Stability criteria for explicit formulation limits time stepping and thus increase computational cost. Similar to previously discussed schemes, explicit formulation of Harten and other TVD schemes are also stable only for Courant-Friedrichs-Lewy condition (CFL) less than 1. It is a challenging task to develop an explicit scheme which is stable for higher values of CFL number. In literature this kind of schemes are known as large time step (LTS) schemes and an active field of research for last three decades. Leveque described a method for approximating nonlinear interactions linearly which allows Godunov's method to be applied with arbitrarily large time steps [18] [19]. Harten extended Leveque work and proposed second-order accurate total variation diminishing large time step explicit schemes for the computation of hyperbolic conservation laws. Computation of nonlinear wave equation depicts that Harten's LTS scheme is a high resolution and efficient scheme [21]. However, computation of system of hyperbolic conservation laws show some spurious oscillations in the vicinities of discontinuities when CFL > 1. Zhan Sen Qian worked on Harten LTS TVD scheme and observed that these spurious oscillations are due to the numerical formulation of the characteristic transformation used by Harten for extending the method for hyperbolic conservation laws [22] [23] [24]. Zhan Sen Qian showed that if the inverse characteristic transformations are performed by using the local right eigenvector matrix at each cell interface location then these spurious oscillations are eliminated. His computations for shock tube problem confirm that the modified large time step total variation diminishing (MLTS TVD) scheme eliminate spurious oscillations for system of hyperbolic conservation laws without increasing the entropy fixing parameter.

Harten and Qian developed large time step schemes have been tested with minmod limiter which is very dissipative. In present work, Qian MLTS TVD scheme is tested with more compressive limiters, namely, centralized MC [25] and superbee [8]. Shock tube problem for SOD boundary conditions [26] is solved to understand the performance of MLTS TVD scheme with compressive limiters in regions of discontinuities and strong shock waves. Shock tube problem is often used by researcher to evaluate the performance of different schemes. Reasons of attraction in this test case are availability of analytical solution and at the same time presence of complex flow feature namely, expansion, shock wave, and contact discontinuities.

II. Numerical Method

In this paper 1D transient Euler equation in a conservation form is used:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{1}$$

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0 \tag{2}$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \quad ; \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)u \end{bmatrix} \tag{3}$$

$$A = \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma - 3)\frac{u^2}{2} & (3-\gamma) & (\gamma - 1) \\ (\gamma - 1)u^3 - \gamma u E & -\frac{3}{2}(\gamma - 1)u^2 + \gamma E & \gamma u \end{bmatrix} \tag{4}$$

equation (1) in numerical flux form can be written as:

$$U_i^{n+1} = U_i^n - \lambda\left(f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n\right) \tag{5}$$

where $\lambda = \frac{\Delta x}{\Delta t}$

Harten used Leveque's scheme [19] [20] and proposed large time step TVD scheme which is second order accurate using $(2K + 3)$ points explicit discretization for hyperbolic conservation laws and increasing the CFL restriction upto K [21]. The numerical flux for Harten's LTS TVD is given by:

$$f_{i+\frac{1}{2}} = \frac{1}{2}[F_{i+1} + F_i] + \frac{1}{2\lambda}\sum_{k=1}^{m} R_{i+\frac{1}{2}}^k\left(g_{i+1}^k + g_i^k\right)$$
$$-\frac{1}{\lambda}\sum_{k=1}^{m} R_{i+\frac{1}{2}}^k\left[\sum_{l=-K+1}^{K-1} C_l\left(v^k + \gamma^k\right)_{i+l+\frac{1}{2}}\alpha_{i+l+\frac{1}{2}}^k\right] \tag{6}$$

here;

$$v_{i+\frac{1}{2}}^k = \lambda a_{i+\frac{1}{2}}^k \tag{7}$$

$$\alpha_{i+\frac{1}{2}} = R^{-1}\Delta_{i+\frac{1}{2}}U \tag{8}$$

$$\tilde{g}_{i+\frac{1}{2}}^k = \frac{1}{2}\left\{Q\left(v_{i+\frac{1}{2}}^k\right) - \left(v_{i+\frac{1}{2}}^k\right)^2\right\}\alpha_{i+\frac{1}{2}}^k \tag{9}$$

$$\gamma_{i+\frac{1}{2}}^k = \begin{cases} \frac{\left(g_{i+1}^k - g_i^k\right)}{\alpha_{i+\frac{1}{2}}^k}, & \alpha_{i+\frac{1}{2}}^k \neq 0 \\ 0, & \alpha_{i+\frac{1}{2}}^k = 0 \end{cases} \tag{10}$$

$$\sigma(v) = \frac{K}{2}\left\{Q\left(\frac{v}{K}\right)\left[1 + \frac{K-1}{2}Q\left(\frac{v}{K}\right)\right] - \frac{K+1}{2}\left(\frac{v}{K}\right)^2\right\} \quad (11)$$

$$C_{\pm k}(v) = \begin{cases} c_k(\mu_{\mp}(v)), & 1 \le k \le K-1 \\ \frac{K}{2}Q\left(\frac{v}{K}\right), & k = 0 \end{cases} \quad (12)$$

$$\mu_{\pm}(v) = \frac{1}{2}\left[Q\left(\frac{v}{K}\right) \pm \frac{v}{K}\right] \quad (13)$$

$$Q(v) = \begin{cases} \frac{1}{2}\left(\frac{v^2}{\varepsilon} + \varepsilon\right) & |v| < \varepsilon \\ |v| & |v| \ge \varepsilon \end{cases} \quad (14)$$

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u & u+c & u-c \\ \frac{u^2}{2} & \frac{u^2}{2} + uc + \frac{c^2}{(\gamma-1)} & \frac{u^2}{2} - uc + \frac{c^2}{(\gamma-1)} \end{bmatrix} \quad (15)$$

$$R^{-1} = \begin{bmatrix} 1 - \frac{(\gamma-1)u^2}{2c^2} & (\gamma-1)\frac{u}{c^2} & -\frac{(\gamma-1)}{c^2} \\ -\frac{u}{2c} + \frac{(\gamma-1)u^2}{4c^2} & \frac{1}{2c} - \frac{(\gamma-1)u}{2c^2} & \frac{(\gamma-1)}{2c^2} \\ \frac{u}{2c} + \frac{(\gamma-1)u^2}{4c^2} & -\frac{1}{2c} - \frac{(\gamma-1)u}{2c^2} & \frac{(\gamma-1)}{2c^2} \end{bmatrix} \quad (16)$$

the limiter function g_i can be expressed as

Minmod:

$$g_i^k = \text{minmod}\left(\tilde{g}_{i+\frac{1}{2}}^k, \tilde{g}_{i-\frac{1}{2}}^k\right) \quad (17)$$

Centralized MC:

$$g_i^k = S.\max\left(0, min\left(2\left|\tilde{g}_{i+\frac{1}{2}}^k\right|, S.\tilde{g}_{i-\frac{1}{2}}^k\right), min\left(\left|\tilde{g}_{i+\frac{1}{2}}^k\right|, 2S.\tilde{g}_{i-\frac{1}{2}}^k\right)\right) \quad (18)$$

Superbee:

$$g_i^k = \text{minmod}\left(2\tilde{g}_{i+\frac{1}{2}}^k, 2\tilde{g}_{i-\frac{1}{2}}^k, \frac{1}{2}\left(\tilde{g}_{i+\frac{1}{2}}^k + \tilde{g}_{i-\frac{1}{2}}^k\right)\right) \quad (19)$$

The extreme limiting is permissible in minmod limiter within the TVD region. Therefore it is reasonably dissipative and smears out discontinuities. Least limiting and maximum steepening is applied in superbee limiter within the TVD region. Therefore it is very compressive and in some cases suffers from excessive sharpening of slopes as a result. The centralized MC limiter is a compromise between Superbee and minmod limiters.

Computation of scalar problems depicts that Harten's LTS scheme is a high resolution and efficient scheme. However, computations of hyperbolic conservation laws show some spurious oscillations in the vicinities of discontinuities for larger values of CFL. Qian modified Harten's LTS TVD scheme to eliminate spurious oscillations [*22*] [*23*]. He suggested performing the inverse characteristic transformations by using the local right eigenvector matrix at each cell interface location to overcome these spurious oscillations. The numerical flux for Qian's MLTS TVD is given by:

$$f_{i+\frac{1}{2}} = \frac{1}{2}[F_{i+1} + F_i] + \frac{1}{2\lambda}\sum_{k=1}^{m} R_{i+\frac{1}{2}}^k \left(g_{i+1}^k + g_i^k\right)$$
$$-\frac{1}{\lambda}\sum_{l=-K+1}^{K-1}\left[\sum_{k=1}^{m} R_{i+l+\frac{1}{2}}^k C_l\left(v^k + \beta^k\right)_{i+l+\frac{1}{2}} \alpha_{i+l+\frac{1}{2}}^k\right] \quad (20)$$

Table 1: Cl(x) at different K

K	C_1	C_2	C_3
2	x^2		
3	$x^2(3-x)$	x^3	
4	$x^2(6-4x+x^2)$	$2x^3(2-x)$	x^4

III. Test Case Description

Shock tube is one of the few 1D problem for which analytical solution is possible to obtain and hence it is often used as a test case for validation of numerical schemes. SOD shock tube problems are used for present study and analysis. SOD boundary conditions used in present computation are described in Table 2.The size of computational domain is $0 \le x \le 1$ and number of grids are 1000. CFL is taken 0.8, 1.8, 2.8, and 3.8 for 1, 2, 3, 4 values of K respectively. Simulations are run for 0.15 physical time while entropy fix parameter ε is taken 0.1 for all computations. Initial discontinuity is centered on $x = x_o$ and $t = 0$ has following conditions:

$$U(x,t) = \begin{cases} U_L, & x < x_o \\ U_R, & x \ge x_o \end{cases}$$

where, x_o=0.5.

Table 2: SOD Boundary Condition

p_R	ρ_R	u_R	p_R	ρ_R	u_R
0.1	0.125	0.0	1.0	1.0	0.0

IV. Result and Discussion

In present work, Qian MLTS TVD scheme [*22*] [*23*] is tested with more compressive limiters, namely, centralized MC [*25*] and superbee [*8*]. Shock tube problem for SOD boundary conditions [*26*] are solved to understand the performance of MLTS TVD scheme with compressive limiters in regions of expansion fan, discontinuities and strong shock waves.

Figure 1-16 depicts the comparison of computed results of density profile after 0.15 physical time at shock, contact and expansion fan regions along with analytical results for SOD case taking K = 1, 2, 3, and 4 for 0.8, 1.8, 2.8, and 3.8 values of CFL, respectively. Oscillation free results near shock are observed with all three limiters for different values of K and

Courant number. Slight oscillation is present near contact discontinuity for centralized MC and superbee limiters. Oscillations become worse as Courant number increases. Extant of spurious oscillation is greater for centralized MC limiter results as compare to super bee limiter. For smaller values of Courant number (CFL ≦ 2) minmod limiter produces oscillation free results across shock and contact discontinuity. Although for larger values of Courant number (CFL > 2) minmod limiter also produces oscillation near contact discontinuity but the extant of oscillation produces by minmod limiter is not as much as compare to other two limiters. Results computed using centralized MC and superbee limiters near shock and contact discontinuities are less dissipative as compare to minmod limiter.

Higher pressure side of expansion fan is found oscillation free with all three limiters for different values of K and Courant number. Slight oscillation is present near lower pressure side of expansion fan. Oscillations become worse as Courant number increases. Extant of spurious oscillation is greater for centralized MC limiter results as compare to super bee limiter. Minmod limiter produces minimum oscillatory results near lower pressure side of expansion fan.

Results for super bee limiter taking K = 1, 2, 3, and 4 for 0.8, 1.8, 2.8, and 3.8 values of Courant number respectively are also compared and analyzed in Figure 17-20. Computed results of density profile are plotted at shock, contact and expansion fan regions along with analytical results. Results near shock discontinuity are oscillation free for all values of Courant number. For K=1, predicted shock discontinuity is behind the analytically calculated shock discontinuity. Predicted shock discontinuity travel in the direction of shock as K increases. For K=4, predicted shock discontinuity surpasses analytically calculated shock discontinuity and it is in front of it. Spurious oscillation is noticed near contact discontinuity. Oscillation is found to be a function of Courant number and it increases as Courant number increases. Results also depict that dissipation across expansion fan widen as Courant number increases.

Computed results near shock wave region for a particular limiter with a specific value of Courant number are superior to contact discontinuity. This is due to the fact that characteristic lines near shock wave are convergent while near contact discontinuity they are parallel to each other. Convergent nature of characteristic lines minimizes dissipation near shock region. The computed results are in good agreement with analytical results with all three limiters for different values of K and Courant number apart from slight oscillations near contact discontinuity for large values of K.

Computed results depict that the difference between analytical and numerical results near expansion fan, contact and shock discontinuities increase for larger values of Courant number. Increase in discrepancy might be due to the increase in truncation error. Sine truncation error strongly depends on step size and time step size increase as Courant number increase.

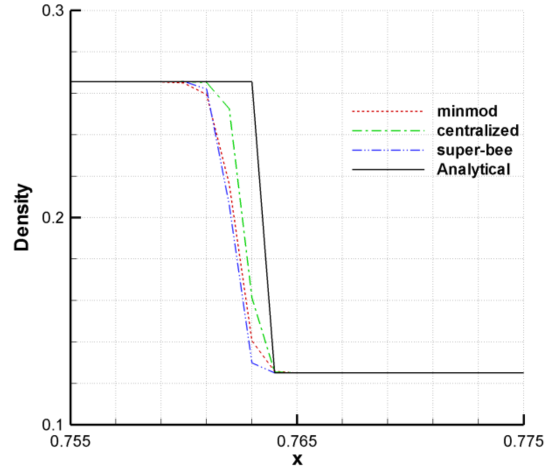

Figure 1: NearShock Region, K=1, CFL=0.8

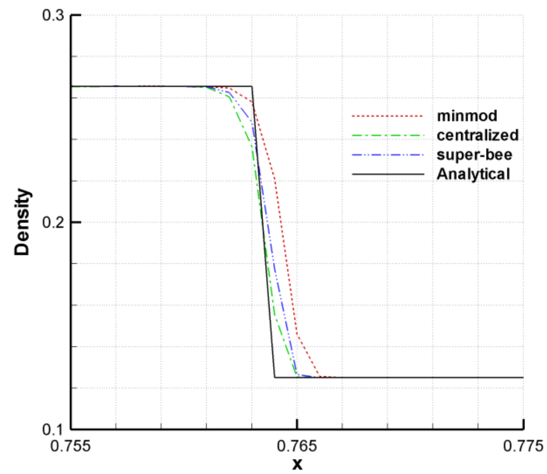

Figure 2: Near Shock Region, K=2, CFL=1.8

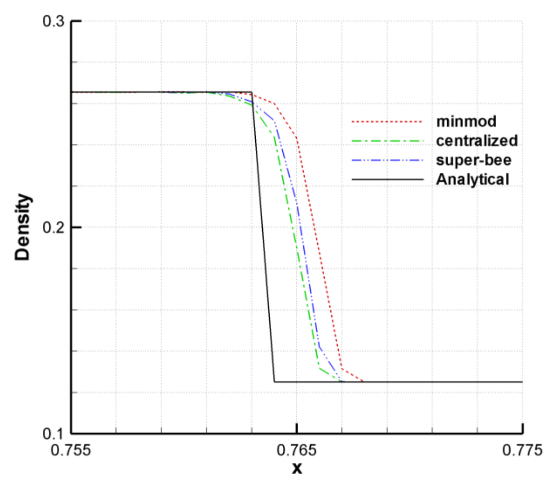

Figure 3: Near Shock Region, K=3, CFL=2.8

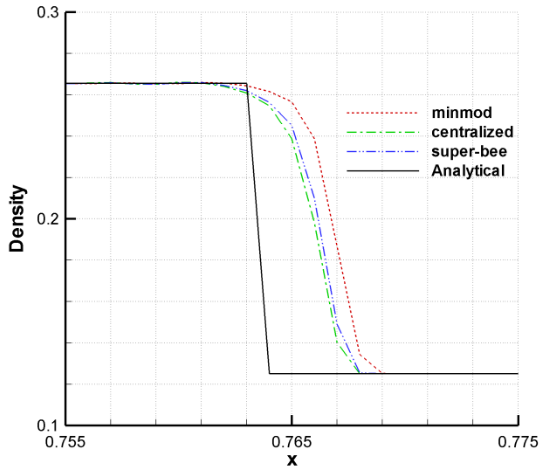

Figure 4: Near Shock Region, K=4, CFL=3.8

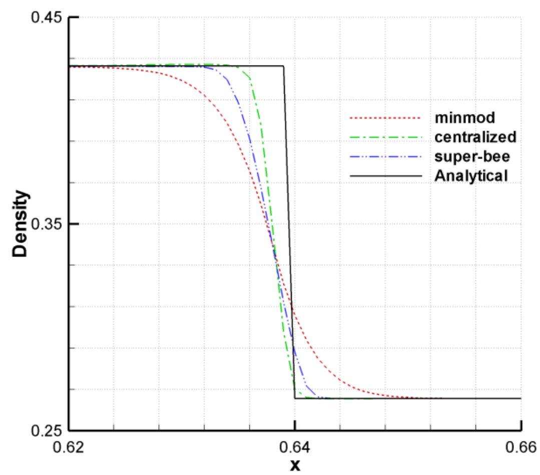

Figure 5: Near Contact Region, K=1, CFL=0.8

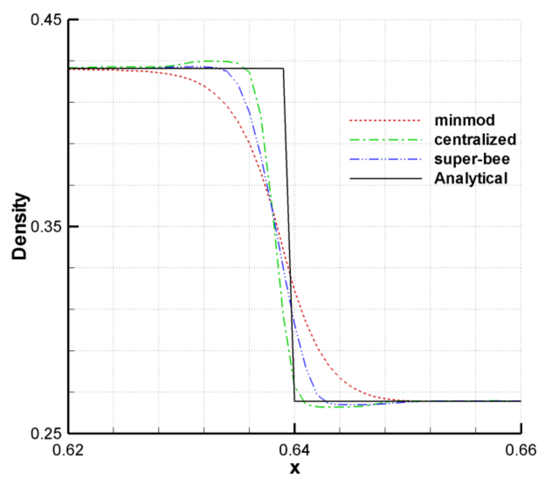

Figure 6: Near Contact Region, K=2, CFL=1.8

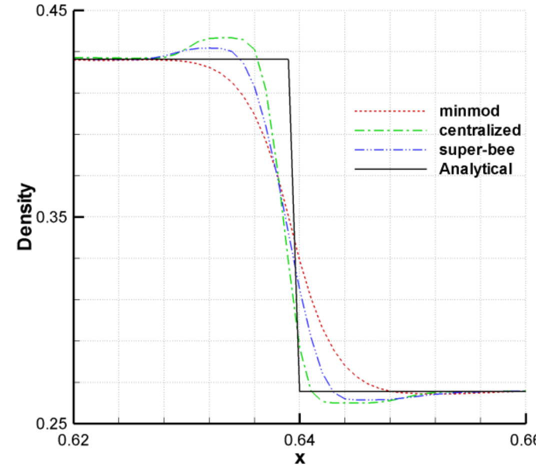

Figure 7: Near Contact Region, K=3, CFL=2.8

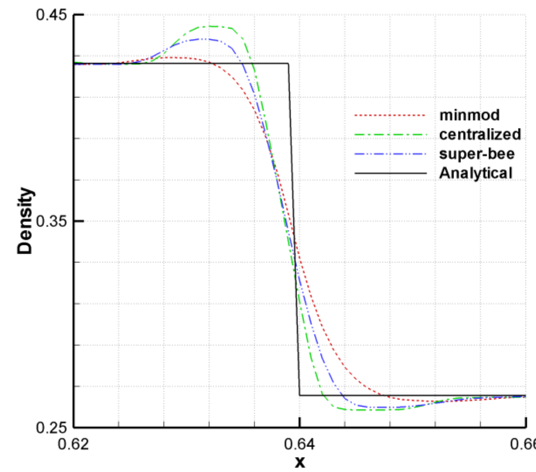

Figure 8: Near Contact Region, K=4, CFL=3.8

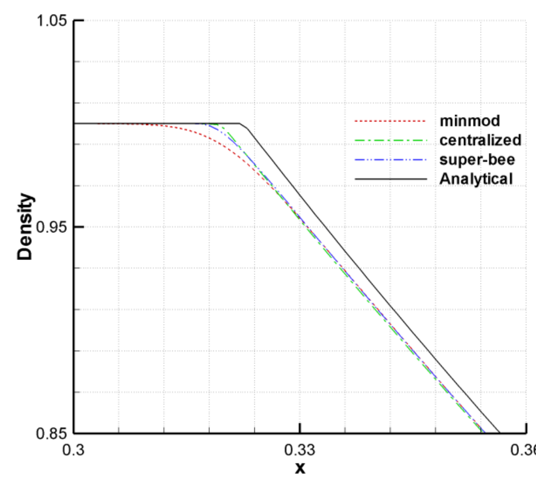

Figure 9: Near Start of Expansion Wave Region, K=1, CFL=0.8

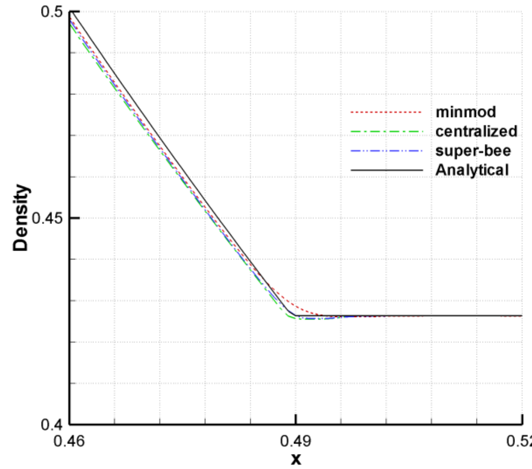

Figure 10: Near End of Expansion Wave Region, K=1, CFL=0.8

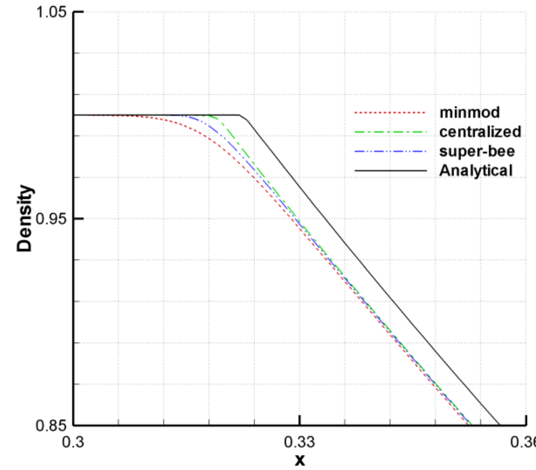

Figure 13: Near Start of Expansion Wave Region, K=3, CFL=2.8

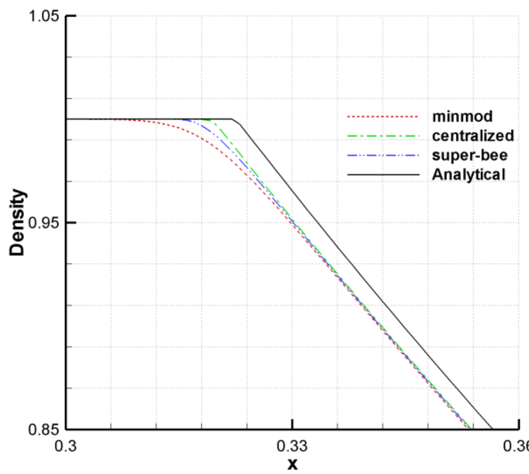

Figure 11: Near Start of Expansion Wave Region, K=2, CFL=1.8

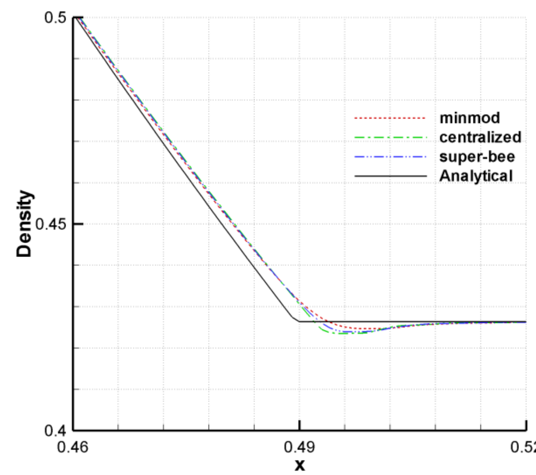

Figure 14: Near End of Expansion Wave Region, K=3, CFL=2.8

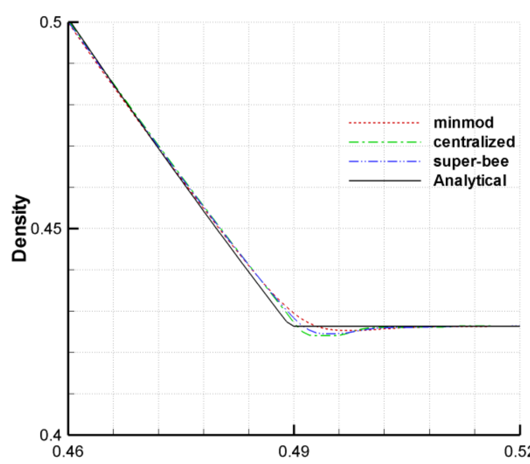

Figure 12: Near End of Expansion Wave Region, K=2, CFL=1.8

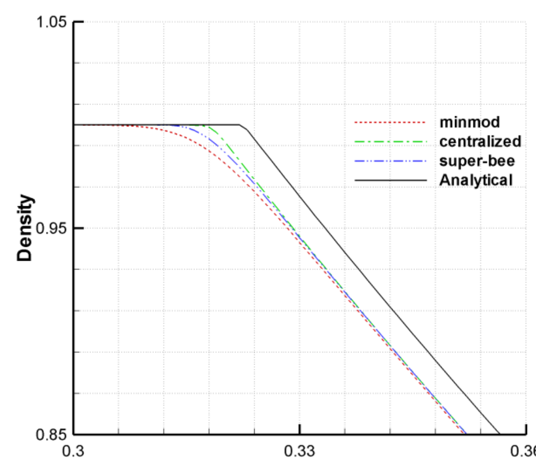

Figure 15: Near Start of Expansion Wave Region, K=4, CFL=3.8

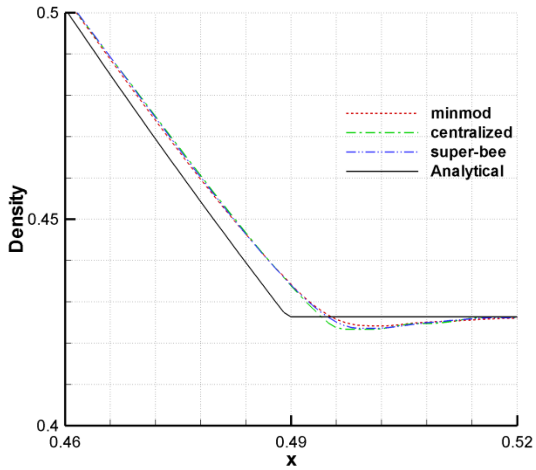

Figure 16: Near End of Expansion Wave Region, K=4, CFL=3.8

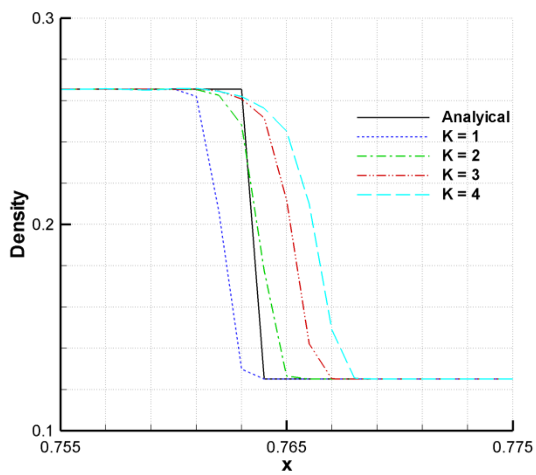

Figure 17: Near Shock Region, superbee limiter

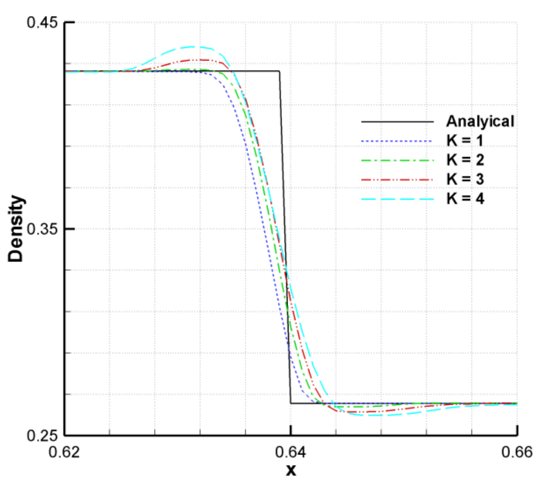

Figure 18: Near Contact Region, superbee limiter

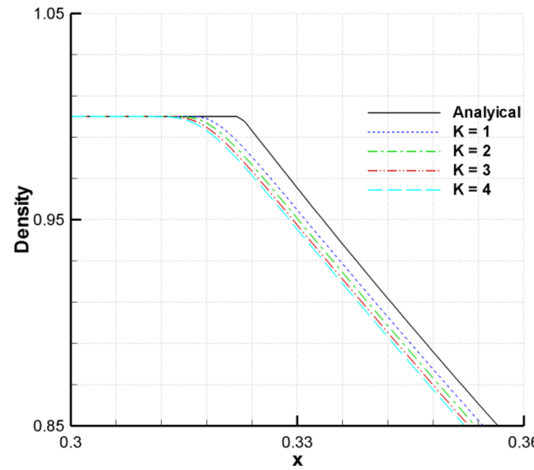

Figure 19: Near Start of Expansion Wave Region, superbee limiter

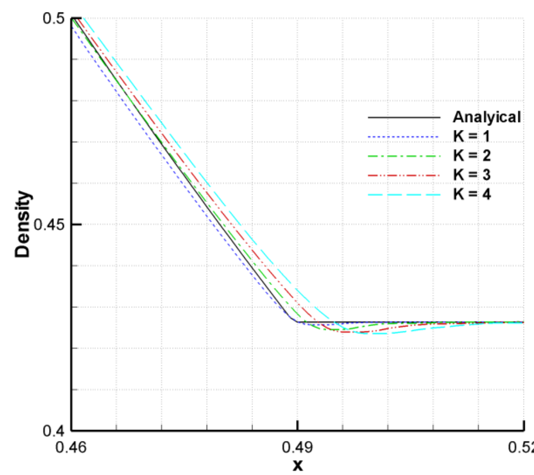

Figure 20: Near End of Expansion Wave Region, superbee limiter

V. CONCLUSION

In present work Qian MLTS TVD scheme is tested with more compressive limiters, namely, centralized MC and superbee. Shock tube problem for SOD boundary conditions is solved to understand the performance of MLTS TVD scheme with compressive limiters in the regions of discontinuities and strong shock waves. Recent results suggested that MLTS TVD scheme is remain stable for compressive limiter. For all three limiters some oscillations are found near contact and lower pressure side of expansion fan. As expected, it is noticed that minmod limiter produces least oscillation while oscillations are larger for centralized MC limiter as compare to super bee limiter. Results also depicts that centralized MC and super bee limiters are less dissipative as compare to minmod limiter, which is due to the compressive nature of former two limiters.

VI. REFERENCES

[1] Tennehill John C., Anderson Dale A., and Pletcher H., *Computational Fluid Mechanics and Heat Transfer, Second Edition*. United States of America: Taylor & Francis, 1984.

[2] Culbert B. Laney, *Computational Gasdynamics*. University of Colorado, New York, United States of America: Cambridge University Press, 1998.

[3] Peter D. Lax, *Hyberbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves.* Courant Institute of Mathematical Sciences, New York University, USA.: CBMS-NSF Regional Conference Series in Applied Mathematics. Supported by the National Science Foundation and published by SIAM., 1973.

[4] Klaus A. Hoffmann and Steve T. Chiang, *Computational Fluid Dynamics,Fourth Edition,Volume I*. Wichita,Kansas,USA.: Engineering Education System(EES), August 2000.

[5] Klaus A. Hoffmann and Steve T.Chiang, *Computational Fluid Dynamics, Fourth Edition,Volume II*. Wichita, Kansas,USA: Engineering Education System(EES), August 2000.

[6] JR. John D. Anderson, *Computational Fluid Dynamics: The Basics with Applications*. University of Maryland ,United States of America. : McGraw-Hill, Inc. , 1976.

[7] P. L. Roe, "Approximate Riemann Solvers, Parameter Vectors,and Difference Schemes.," *Journal of Computational Physics 43, (1981)*, pp. 357-372(16), March 30, 1981.

[8] P. L. Roe, "Characteristic Based Scheme for the Euler Equation.," *College of Aeronautics, Cranfield Institute of Technology,Cranfield MK43 OAL, England.*, p. 29, 1986.

[9] H.C. Yee, "Upwind and Symmetric Shock-Capturing Schemes," *NASA Technical Memorandum 89464*, p. 130, May 1987,Ames Research Center, Moffett Field, California,USA.

[10] H.C. Yee, G.H. Klopfer, and J.L. Montagn, "High-Resolution Shock-Capturing Schemes for lnviscid and Viscous Hypersonic Flows.," *NASA Technical Memorandum 100097*, p. 38, April 1988.

[11] Ami Harten, "High Resolution Schemes for Hyperbolic Conservation Laws.," *Journal of Computational Physics 135, 260–278 (1997), Article No. CP975713*, p. 19, June 23, 1982,New York.

[12] Ami Harten, "The Artificial Compression Method for Computation of Shocks and Contact Discontinuities: III. Self-Adjusting Hybrid Schemes.," *Mathematics of Computation, Vol. 32, No. 142*, pp. 363-389(27), April 1978.

[13] H K Versteeg and W Malalasekera, *An Introduction to Computational Fluid Dynamics,The Finite Volume Method,Second Edition*. England, United Kingdom.: PEARSON Prentice Hall, 2007.

[14] H. C. Yee, "A Class of High-Resolution Explicit and Implicit Shock-Capturing Methods.," *NASA Technical Memorandum 101088* , p. 228, February 1989,Ames Research Center, Moffett Field, California,USA.

[15] H. C. Yee and Harten Ami., "Implicit TVD Schemes for Hyperbolic Conservation Laws in Curvilinear Coordinates.," *AIAA Journal Vol. 25, No. 2,* pp. 266-274(9), 1985.

[16] H.C. Yee, "Linearized Form of Implicit TVD Schemes for Multidimensional Euler and Navier-Stokes Equations.," *Computational Fluid Dynamics Branch. MS 202A-1. NASA ,Ames Research Center, Moffett Field. CA 94035. U.S.A.*, pp. Vol 12A. No, 45. pp J.13-432., 1986.

[17] P. K. Sweby, "High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws.," *SIAM Journal on Numerical Analysis, Vol. 21, No. 5.*, p. 19, (Oct., 1984), pp. 995-1011.

[18] Professor Eleuterio F. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics,A Practical Introduction,Third Edition.* University of Trento,Italy: Springer Dordrecht Heidelberg London New York, 2009.

[19] J. Leveque Randall, "Large Time Step Shock Capturing Techniques for Scalar Conservation Laws," *SIAM J. Numerical Analysis.*, pp. Vol. 19, No.6, December 1982.

[20] Randall J. Leveque, "A Large Time Step Generalization of Godunov's Method for Systems of Conservation Laws," *SIAM J. Numerical Analysis.*, pp. Vol. 22, No, 6, December 1985.

[21] Ami Harten, "On a Large Time-Step High Resolution Scheme.," *Mathematics of Computation,Volume 46, Number 174*, pp. Pages 379-399(21), April 1986.

[22] ChunHian Lee ZhanSen Qian, "A class of Large Time Step Godunov Scheme for Hyperbolic Conservation Laws and Applications," *Journal of Computational Physics*, 2011.

[23] Chun-Hian Lee ZhanSen Qian, "On Large Time Step TVD Scheme for Hyperbolic Conservation Laws and its Efficiency Evaluation," *Journal of Computational Physics*, 2012.

[24] Huang Huang, Chunhian Leey, Haitao Dongz, and Jinbai Zhang, "Modification and Applications of a Large Time-Step High Resolution TVD Scheme.," *51st AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition,AIAA 2013-0077*, p. 8, 07 - 10 January 2013, Grapevine (Dallas/Ft. Worth Region), Texas.

[25] Bram Van Leer, "Towards the Ultimate Conservative Difference Scheme.," *Journal of Computational Physics*, pp. 276-299,(23), 1977,University Observatory, Leiden, The Netherlands.

[26] Gary A. Sod, "A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws," *Journal of Computational Physics*, vol. 27, 1978.