# On the mathematical properties of the solutions in the models of fluid dynamics of the Ocean which involve heat and salinity transfer

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***Abstract*—**We investigate the properties of the solutions of PDE systems which describe fluid dynamics of the Ocean with heat and salinity transfer. We prove the existence and uniqueness theorem for a layer. We study the spectrum for the problems modelling the small inner oscillations of viscous rotating compressible three-dimensional fluid which consider the involvement of the heat and salinity transfer for different boundary value problems which include either kinematic viscosity or the combined kinematic and volume (bulk) viscosity. We prove that the essential spectrum for both operators consists of one real point which depends on the parameters of compressibility and viscosity. We also find the sector of the complex plane to which all the eigenvalues belong. We compare the obtained results with our previous study of the spectral properties for incompressible and inviscid rotating fluid. The results of this paper may find their application in the study, either theoretical or computational, of the Ocean and the Atmosphere of the Earth.

***Keywords*—**Eigenvalues of Differential Operators, Essential Spectrum, Fluid Dynamics, Inner Vibrations, Partial Differential Equations.

## I. INTRODUCTION

LET us consider a bounded domain $\Omega \subset R^3$ with the boundary $\partial\Omega$ of the class $C^1$ and the following system of fluid dynamics:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 - \nu \Delta u_1 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 - \nu \Delta u_2 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} - \nu \Delta u_3 + \dfrac{\partial p}{\partial x_3} - \gamma_T T - \gamma_S S = 0 \\ \alpha^2 \dfrac{\partial p}{\partial t} + \operatorname{div} \vec{u} = 0 \\ \dfrac{\partial T}{\partial t} - \nu \Delta T + \gamma_T u_3 = 0 \\ \dfrac{\partial S}{\partial t} - \nu \Delta S + \gamma_S u_3 = 0 \qquad x \in \Omega, \quad t \ge 0. \end{cases} \tag{1}$$

Here $\vec{u} = (u_1, u_2, u_3)$ is a velocity field, $p(x,t)$ is the scalar field of the dynamic pressure, $T(x,t)$ is the dynamic temperature of the fluid, $S(x,t)$ is the dynamic salinity, $\omega = \text{Const}$ is the Coriolis parameter, and $\gamma_T$ and $\gamma_S$ are constant positive stratification parameters.

For the compressibility coefficient $\alpha$ and the kinematic viscosity coefficient $\nu$ we assume $\alpha > 0$, $\nu > 0$.

The equations (1) are deduced, for example, in [1], [2]. The study of mathematical properties of different systems of fluid dynamics of rotating fluid was started in [3] – [5]. The spectral properties of operators generated by rotating and stratified compressible inviscid fluid, without accounting of the heat and salinity transfer, were studied in [6]- [12]. Particularly, it was proved in [6] that the essential spectrum of normal inner vibrations is the interval of the imaginary axis $[-i\omega, i\omega]$. The spectral properties of viscous compressible stratified fluid, also without the effects of heat and salinity transfer and rotation, were studied first in [13]-[15], where it was established that the essential spectrum consists of three real isolated points which tend to infinity for vanishing viscosity parameter. However, the spectral properties for the case of the equations (1), i.e., for the rotating compressible viscous fluid accounting the effects of heat and salinity transfer, has not been considered previously. The novelty of this problem, the consideration of different cases of the combination of kinematic and bulk viscosities, as well as the comparison of the obtained results with our previous works, the description of the spectral properties and their possible applications to the dynamics of the Ocean, either for theoretical fluid dynamics, or for computational fluid dynamics, as well as the explicit construction of the solution for the layer domain and the proof of its uniqueness, was the motivation of this paper.

We associate system (1) to the boundary conditions

$$\vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0 \tag{2}$$

where $\vec{n}$ is the exterior normal to the surface $\partial\Omega$. Let us consider the following problem of normal vibrations

$$\begin{aligned} \vec{u}(x,t) &= \vec{v}(x) e^{-\lambda t} \\ p(x,t) &= \frac{1}{\alpha} v_4(x) e^{-\lambda t} \\ T(x,t) &= v_5(x) e^{-\lambda t} \\ S(x,t) &= v_6(x) e^{-\lambda t} \quad , \quad \lambda \in C. \end{aligned} \tag{3}$$

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We denote $\tilde{v} = \left(\vec{v}, v_4, v_5, v_6\right)$ and write the system (1) in the matrix form

$$L\tilde{v} = 0 \tag{4}$$

where

$$L = M - \lambda I$$

and

$$M = \begin{pmatrix} -\nu\Delta & -\omega & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_1} & 0 & 0 \\ \omega & -\nu\Delta & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & 0 & 0 \\ 0 & 0 & -\nu\Delta & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & -\gamma_T & -\gamma_S \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 & 0 \\ 0 & 0 & \gamma_T & 0 & -\nu\Delta & 0 \\ 0 & 0 & \gamma_S & 0 & 0 & -\nu\Delta \end{pmatrix}. \tag{5}$$

We define the domain of the differential operator $M$ with the boundary condition (2) as follows.

$$D(M) = \left\{ \begin{array}{l} \vec{v} \in \left(\overset{0}{W}{}_2^1(\Omega)\right)^3, v_5 \in \overset{0}{W}{}_2^1(\Omega), v_6 \in \overset{0}{W}{}_2^1(\Omega) \\ v_4 \in L_2(\Omega) \,:\, M\tilde{v} \in \left(L_2(\Omega)\right)^6 \end{array} \right\},$$

where $\overset{0}{W}{}_2^1(\Omega)$ is a closure of the functional space $C_0^\infty(\Omega)$ in the norm

$$\|f\| = \left( \int_\Omega \left[ |\nabla f|^2 + f^2 \right] dx \right)^{1/2}.$$

We also will consider the system with volume (bulk) viscosity coefficient $\beta > 0$:

$$\begin{cases} \frac{\partial u_1}{\partial t} - \omega u_2 - \nu\Delta u_1 - \nu\beta\frac{\partial}{\partial x_1}\left(\mathrm{div}\vec{u}\right) + \frac{\partial p}{\partial x_1} = 0 \\ \frac{\partial u_2}{\partial t} + \omega u_1 - \nu\Delta u_2 - \nu\beta\frac{\partial}{\partial x_2}\left(\mathrm{div}\vec{u}\right) + \frac{\partial p}{\partial x_2} = 0 \\ \frac{\partial u_3}{\partial t} - \nu\Delta u_3 - \nu\beta\frac{\partial}{\partial x_3}\left(\mathrm{div}\vec{u}\right) + \frac{\partial p}{\partial x_3} - \gamma_T T - \gamma_S S = 0 \\ \alpha^2\frac{\partial p}{\partial t} + \mathrm{div}\vec{u} = 0 \\ \frac{\partial T}{\partial t} - \nu\Delta T + \gamma_T u_3 = 0 \\ \frac{\partial S}{\partial t} - \nu\Delta S + \gamma_S u_3 = 0 \qquad x \in \Omega, \quad t \ge 0. \end{cases} \tag{6}$$

The equations (6) are deduced, for example, in [16]. We consider the system (6) with the Dirichlet boundary conditions

$$\left.\left(\vec{u}, T, S\right)\right|_{\partial\Omega} = 0\,. \tag{7}$$

After applying the separation of variables (3) we write the system (6) in the form (4) and denote the corresponding matrix as $M_1$:

$$M_1 = \begin{pmatrix} -\nu\Delta - \nu\beta\frac{\partial^2}{\partial x_1^2} & -\omega - \nu\beta\frac{\partial^2}{\partial x_1 \partial x_2} & -\nu\beta\frac{\partial^2}{\partial x_1 \partial x_3} & \frac{1}{\alpha}\frac{\partial}{\partial x_1} & 0 & 0 \\ \omega - \nu\beta\frac{\partial^2}{\partial x_1 \partial x_2} & -\nu\Delta - \nu\beta\frac{\partial^2}{\partial x_2^2} & -\nu\beta\frac{\partial^2}{\partial x_2 \partial x_3} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & 0 & 0 \\ -\nu\beta\frac{\partial^2}{\partial x_1 \partial x_3} & -\nu\beta\frac{\partial^2}{\partial x_2 \partial x_3} & -\nu\Delta - \nu\beta\frac{\partial^2}{\partial x_3^2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & -\gamma_T & -\gamma_S \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 & 0 \\ 0 & 0 & \gamma_T & 0 & -\nu\Delta & 0 \\ 0 & 0 & \gamma_S & 0 & 0 & -\nu\Delta \end{pmatrix}.$$

For the operator $M_1$ we define the same domain as for the operator $M$.

From the physical point of view, the separation of variables (3) serves as a tool to establish the possibility to represent every non-stationary process described by (1) as a linear superposition of the normal vibrations. The knowledge of the spectrum of normal vibrations may be very useful for studying the stability of the flows. Also, the spectrum of operators $M, M_1$ is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operator $M$, which we consider in this paper.

For small variation of $x_3$, $u_3$ we also consider the corresponding system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 - \nu \Delta u_1 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 - \nu \Delta u_2 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial p}{\partial x_3} - \gamma_T T - \gamma_S S = 0 \\ \operatorname{div} \vec{u} = 0 \\ \dfrac{\partial T}{\partial t} - \nu \Delta T + \gamma_T u_3 = 0 \\ \dfrac{\partial S}{\partial t} - \nu \Delta S + \gamma_S u_3 = 0 \qquad x \in \Omega, \quad t \geq 0, \end{cases} \tag{8}$$

$$\Omega = \left\{ (x_1, x_2, x_3) : (x_1, x_2) \in R^2, 0 < x < h \right\},$$

with the following initial and boundary conditions:

$$\begin{cases} u_i \big|_{t=0} = u_i^0(x), i = 1,2 \,; \\ T\big|_{t=0} = T^0(x); \;\; S\big|_{t=0} = S^0(x) \\ \dfrac{\partial u_i}{\partial x_3}\Big|_{x_3=0,h} = 0, i = 1,2 \,; \; u_3\big|_{x_3=0,h} = 0 \\ T\big|_{x_3=0,h} = 0; \;\; S\big|_{x_3=0,h} = 0 \end{cases} , \tag{9}$$

where

$$u_1^0, u_2^0, T^0, S^0 \in W_1^4(\Omega), \; \int_0^h \left[ \frac{\partial u_1^0}{\partial x_1} + \frac{\partial u_2^0}{\partial x_2} \right] dx_3 = 0 \,,$$

and the initial and boundary conditions correspond each other.

For the problem (8), (9) we will construct the solution and prove the existence and uniqueness theorem.

## II. Problem formulation

Let us consider first the problem (8), (9).

We will construct the solution using the Laplace transform with respect to $t$, the Fourier transform with respect to $x_1, x_2$ and finite sine- and cosine- integral transforms with respect to $x_3$. For the first, the second and the fourth equation of the system (8) we apply finite cosine-transform with respect to $x_3$, and for the rest of the equations of the system (8), we apply the corresponding sine-transform. For that, we multiply the first, the second and the fourth equation by $\cos \lambda_n x_3$; the rest of the equations we multiply by $\sin \lambda_n x_3$, and integrate over the interval $x_3 \in [0, h]$, where $\lambda_n = \dfrac{\pi n}{h}$. The general idea of construction of such solution in a layer is taken from [17].

We introduce the following notations:

$$(\hat{u}_i, \hat{p})(x', n, t) = \int_0^h (u_i, p)(x', x_3, t) \cos(\lambda_n x_3) dx_3, i = 1,2,$$

$$\left(\hat{u}_3, \hat{T}, \hat{S}\right)(x', n, t) = \int_0^h (u_3, T, S)(x', x_3, t) \sin(\lambda_n x_3) dx_3,$$

$$\left(\hat{u}_i, \hat{T}, \hat{S}\right)(x', n, t)\Big|_{t=0} = \left(\hat{u}_i^0, \hat{T}^0, \hat{S}^0\right)(x', n), \; i = 1,2.$$

In this way, we can reduce the problem (8), (9) to the following problem

$$\begin{cases} \dfrac{\partial \hat{u}_1}{\partial t} - \omega \hat{u}_2 - \nu \Delta_2 \hat{u}_1 + \nu \lambda_n^2 \hat{u}_1 + \dfrac{\partial \hat{p}}{\partial x_1} = 0 \\ \dfrac{\partial \hat{u}_2}{\partial t} + \omega \hat{u}_1 - \nu \Delta_2 \hat{u}_2 + \nu \lambda_n^2 \hat{u}_2 + \dfrac{\partial \hat{p}}{\partial x_2} = 0 \\ -\lambda_n \hat{p} = \gamma_T \hat{T} + \gamma_S \hat{S} \\ \dfrac{\partial \hat{u}_1}{\partial x_1} + \dfrac{\partial \hat{u}_2}{\partial x_2} + \lambda_n \hat{u}_3 = 0 \\ \dfrac{\partial \hat{T}}{\partial t} - \nu \Delta_2 \hat{T} + \nu \lambda_n^2 \hat{T} + \gamma_T \hat{u}_3 = 0 \\ \dfrac{\partial \hat{S}}{\partial t} - \nu \Delta_2 \hat{S} + \nu \lambda_n^2 \hat{S} + \gamma_S \hat{u}_3 = 0 \\ \left(\hat{u}_i, \hat{T}, \hat{S}\right)\Big|_{t=0} = \left(\hat{u}_i^0, \hat{T}^0, \hat{S}^0\right)(x', n) \;, \; i = 1,2 \end{cases} \tag{10}$$

For Laplace transform with respect to $t$ and for Fourier transform with respect to $x' = (x_1, x_2)$, we will use the notations:

$$F_{x' \to \xi'} L_{t \to \lambda} \left[ \hat{\vec{u}}, \hat{p}, \hat{T}, \hat{S} \right](x', n, t) = \left( \tilde{\vec{u}}, \tilde{p}, \tilde{T}, \tilde{S} \right)(\xi', n, \lambda),$$

$$F_{x' \to \xi'} \left[ \hat{u}_i^0, \hat{T}^0, \hat{S}^0 \right](x', n) = \left( \tilde{u}_i^0, \tilde{T}^0, \tilde{S}^0 \right)(\xi', n,), \; i = 1,2.$$

Thus, from (10) we obtain the system of algebraic equations

$$\begin{cases} \left(\lambda + \nu\left(|\xi'|^2 + \lambda_n^2\right)\right)\tilde{u}_1 - \omega \tilde{u}_2 + i\xi_1 \tilde{p} = \tilde{u}_1^0 \\ \left(\lambda + \nu\left(|\xi'|^2 + \lambda_n^2\right)\right)\tilde{u}_2 + \omega \tilde{u}_1 + i\xi_2 \tilde{p} = \tilde{u}_2^0 \\ \lambda_n \tilde{p} + \gamma_T \tilde{T} + \gamma_S \tilde{S} = 0 \\ i\xi_1 \tilde{u}_1 + i\xi_2 \tilde{u}_2 + \lambda_n \tilde{u}_3 = 0 \\ \left(\lambda + \nu\left(|\xi'|^2 + \lambda_n^2\right)\right)\tilde{T} + \gamma_T \tilde{u}_3 = \tilde{T}^0 \\ \left(\lambda + \nu\left(|\xi'|^2 + \lambda_n^2\right)\right)\tilde{S} + \gamma_S \tilde{u}_3 = \tilde{S}^0, \end{cases} \tag{11}$$

$$\xi' = (\xi_1, \xi_2) \in R^2, \; |\xi'|^2 = \xi_1^2 + \xi_2^2.$$

After solving (11) and applying the inverse Fourier and Laplace transforms $F^{-1}_{\xi' \to x'} L^{-1}_{\lambda \to t}$, we can represent the solution of the problem (10) as

$$\hat{u}_k\left(x',n,t\right)=\frac{1}{\left(2\pi\right)^2}\iint_{R^2} e^{i(x',\xi')}\left\{\tilde{u}_k^0 e^{Ht}-\left(\gamma\xi_k^2+\omega^2\lambda_n^2\right)\tilde{u}_k^0\Psi_0-\right.$$
$$-\left(-1\right)^k\left[\lambda_n^2\omega\Psi_1+\left(-1\right)^k\xi_1\xi_2\gamma\Psi_0\right]\tilde{u}_{3-k}^0+$$
$$\left.+\lambda_n\left[i\xi_k\Psi_1-\left(-1\right)^k i\xi_{3-k}\omega\Psi_0\right]\tilde{U}_3^0\right\}d\xi'\ k=1,2,$$

$$\hat{u}_3\left(x',n,t\right)=\frac{1}{\left(2\pi\right)^2}\iint_{R^2} e^{i(x',\xi')}\begin{bmatrix}\lambda_n\left(\omega\tilde{U}_2^0\Psi_1-\tilde{U}_1^0\Psi_2\right)+\\+\left|\xi'\right|^2\tilde{U}_3^0\Psi_1\end{bmatrix}d\xi',$$

$$\hat{p}\left(x',n,t\right)=\frac{1}{\left(2\pi\right)^2}\iint_{R^2} e^{i(x',\xi')}\left[\gamma\left(\omega\tilde{U}_2^0\Psi_0-\tilde{U}_1^0\Psi_1\right)-\right.$$
$$\left.-\lambda_n\left(\omega^2\Psi_0+\Psi_2\right)\tilde{U}_3^0\right]d\xi',$$

$$\hat{T}\left(x',n,t\right)=\frac{1}{\left(2\pi\right)^2}\iint_{R^2} e^{i(x',\xi')}\left[\tilde{T}^0e^{Ht}+\left|\xi'\right|^2\Psi_0\left(\gamma_S\tilde{U}_4^0-\gamma\tilde{T}^0\right)+\right.$$
$$\left.+\gamma_T\lambda_n\left(\tilde{U}_1^0\Psi_1-\omega\tilde{U}_2^0\Psi_0\right)\ \right]d\xi',$$

$$\hat{S}\left(x',n,t\right)=\frac{1}{\left(2\pi\right)^2}\iint_{R^2} e^{i(x',\xi')}\left[\tilde{S}^0e^{Ht}-\left|\xi'\right|^2\Psi_0\left(\gamma_T\tilde{U}_4^0+\gamma\tilde{S}^0\right)+\right.$$
$$\left.+\gamma_S\lambda_n\left(\tilde{U}_1^0\Psi_1-\omega\tilde{U}_2^0\Psi_0\right)\ \right]d\xi',$$

where

$$\tilde{U}_1^0\left(\xi',n\right)=i\xi_1\tilde{u}_1^0+i\xi_2\tilde{u}_2^0\ ,\ \tilde{U}_2^0\left(\xi',n\right)=i\xi_1\tilde{u}_2^0-i\xi_2\tilde{u}_1^0,$$
$$\tilde{U}_3^0\left(\xi',n\right)=\gamma_T\tilde{T}^0+\gamma_S\tilde{S}^0\ ,\ \tilde{U}_4^0\left(\xi',n\right)=\gamma_S\tilde{T}^0-\gamma_T\tilde{S}^0,$$

$$H=-\nu\left(\left|\xi'\right|^2+\lambda_n^2\right)\ ,\ \gamma=\gamma_T^2+\gamma_S^2\ ,$$

$$\Psi_0\left(\xi',n,t\right)=\frac{2e^{Ht}}{\Lambda^n}\sin^2\left(\frac{\Lambda t}{2\lambda_n}\right)\ ,$$

$$\Psi_1\left(\xi',n,t\right)=\frac{2e^{Ht}}{\lambda_n\Lambda}\sin\left(\frac{\Lambda t}{\lambda_n}\right)\ ,$$

$$\Psi_2\left(\xi',n,t\right)=\frac{2e^{Ht}}{\lambda_n^2}\cos\left(\frac{\Lambda t}{\lambda_n}\right)\ ,$$

$$\Lambda=\sqrt{\omega^2\lambda_n^2+\gamma\left|\xi'\right|^2}.$$

In this way, the solution of the problem (8), (9) can be represented as follows:

$$\left(u_i,p\right)\left(x,t\right)=\frac{1}{h}\left(\hat{u}_i,\hat{p}\right)\left(x',0,t\right)+\frac{2}{h}\sum_{n=1}^{\infty}\left(\hat{u}_i,\hat{p}\right)\left(x',n,t\right)\cos\left(\lambda_n x_3\right),$$
$$i=1,2, \qquad (12)$$

$$\left(u_3,T,S\right)\left(x,t\right)=\frac{2}{h}\sum_{n=1}^{\infty}\left(\hat{u}_3,\hat{T},\hat{S}\right)\left(x',n,t\right)\sin\left(\lambda_n x_3\right).$$

It is easy to see that the solution (12) satisfies the problem (8), (9), and that it belongs to the class

$$\left(u_i,S,T\right)\in C_{x,t}^{2,1}\left(\Omega\times\left(0,t\right)\right)\ ,\ i=1,2,$$
$$\left(u_3,p\right)\in C_{x,t}^{1,0}\left(\Omega\times\left(0,t\right)\right). \qquad (13)$$

The uniqueness of the solution (12) follows from the energy integral

$$\frac{1}{2}\int_{\Omega}\left(\sum_{i=1}^{2}u_i^2+T^2+S^2\right)dx+\nu\int_0^t\int_{\Omega}\left\{\sum_{i=1}^{2}\sum_{j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2+\right.$$
$$\left.+\sum_{i=1}^{3}\left[\left(\frac{\partial T}{\partial x_i}\right)^2+\left(\frac{\partial S}{\partial x_i}\right)^2\right]\right\}dxdt=0, \qquad (14)$$

which is obtained by multiplying the system (8) by $\left(\vec{u},T,S\right)$ and integrating by parts.

Summing up the above results, we can state the following statement.

<u>Theorem 1.</u>

There exist a solution of the problem (8), (9), which is represented by (12). The solution is unique in the class of functions (13).

Now, let us investigate the spectral properties of the differential operators generated by the system (1). We observe that the above defined operator $M$ is a closed operator, and its domain is dense in $\left(L_2\left(\Omega\right)\right)^6$.

Let us denote by $\sigma_{ess}\left(N\right)$ the essential spectrum of a closed linear operator $N$. We recall that the essential spectrum

$$\sigma_{ess}\left(N\right)=\left\{\lambda\in C:\left(N-\lambda I\right)\text{ is not of Fredholm type}\right\},$$

is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity (see [18] ,[19]).

In this way, every spectral point which does not belong to the essential spectrum, is an eigenvalue of finite multiplicity. To find the essential spectrum of the operator $M$, we will use the following property (see [20]):

$$\sigma_{ess}\left(M\right)=Q\cup S\ ,$$

where

$$Q=\left\{\begin{matrix}\lambda\in C:\left(M-\lambda I\right)\text{is not elliptic}\\ \text{in sense of Douglis-Nirenberg}\end{matrix}\right\}$$

and

$$S=\left\{\begin{matrix}\lambda\in C\setminus Q:\text{the boundary conditions of }\left(M-\lambda I\right)\\ \text{do not satisfy Lopatinski conditions}\end{matrix}\right\}.$$

We recall the following two definitions.

<u>Definition 1.</u>

Let us consider a differential matrix operator

$$L=\begin{pmatrix}l_{11}&\dots&l_{1N}\\ \dots&\dots&\dots\\ l_{N1}&\dots&l_{NN}\end{pmatrix},\quad l_{ij}=\sum_{|\alpha|\le n_{ij}}a_{ij}^{(\alpha)}D^{\alpha},$$
$$\alpha=\left(\alpha_1,\dots,\alpha_n\right),\quad \left|\alpha\right|=\alpha_1+\dots+\alpha_n,$$

$$D_j = \frac{\partial}{\partial x_j}, \quad D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

Let $\{s_i\}_{i=1}^N$ , $\{t_j\}_{j=1}^N$ be two sets of integer numbers such that, if $l_{ij} \neq 0$, then $n_{ij} = \deg l_{ij} \le s_i + t_j$. In case $l_{ij} = 0$, we do not require any condition for the sum $s_i + t_j$. Now, we construct the main symbol of *L(D)* as follows.

$$\tilde{L}(D) = \begin{pmatrix} \tilde{l}_{11}(D) & ... & \tilde{l}_{1N}(D) \\ ... & ... & ... \\ \tilde{l}_{N1}(D) & ... & \tilde{l}_{NN}(D) \end{pmatrix},$$

$$\tilde{l}_{ij} = \begin{cases} 0 \text{ if } l_{ij}(D) = 0 \text{ or } \deg l_{ij}(D) < s_i + t_j \\ \sum\limits_{|\alpha| = s_i + t_j} a_{ij}^{(\alpha)} D^\alpha \text{ if } \deg l_{ij}(D) = s_i + t_j. \end{cases}$$

If there exist the sets *s* and *t* which satisfy the above conditions and if the following condition holds,

$$\det \tilde{L}(\xi) \neq 0 \quad \text{for all} \quad \xi \in R^n \setminus \{0\},$$

then the operator *L(D)* is called *elliptic in sense of Douglis-Nirenberg* (see [21]).

Definition 2.

Let us consider $\xi = (\xi_1, \xi_2, \xi_3)$, $\tilde{\xi} = (\xi_1, \xi_2)$, $L^*(\xi)$ - the matrix of the algebraic complements of the main symbol matrix $\tilde{L}(\xi)$, $G(\xi)$ is the main symbol of the matrix *G(D)* which defines the boundary conditions, $M^+(\tilde{\xi}, \tau) = \prod (\tau - \tau_j(\tilde{\xi}))$, $\tau_j(\tilde{\xi})$ are the roots of the equation $\det \tilde{L}(\tilde{\xi}, \tau) = 0$ with positive imaginary part.

If the rows of the matrix $G(\tilde{\xi}, \tau) L^*(\tilde{\xi}, \tau)$ are linearly independent with respect to the module $M^+(\tilde{\xi}, \tau)$ for $|\tilde{\xi}| \neq 0$, then we will say that the *conditions of Lopatinski are satisfied* (see [20]).

We will find the essential spectrum of the operators $M$ and localize the sector of the complex plane to which all the eigenvalues belong.

## III. Problem Solution

Theorem 2.

The essential spectrum of the operator $M$ is composed of one real point $\sigma_{ess}(M) = \left\{ \frac{1}{\nu \alpha^2} \right\}$.

Proof.

We observe that, according to the Definition 1, we can choose

$$s_1 = s_2 = s_3 = s_5 = s_6 = 0, \; s_4 = -1,$$
$$t_1 = t_2 = t_3 = t_5 = t_6 = 2, \; t_4 = 1,$$

so that the main symbol of the operator $L = M - \lambda I$ will be expressed as:

$$\tilde{L}(\xi) = \begin{pmatrix} -\nu|\xi|^2 & 0 & 0 & \frac{1}{\alpha}\xi_1 & 0 & 0 \\ 0 & -\nu|\xi|^2 & 0 & \frac{1}{\alpha}\xi_2 & 0 & 0 \\ 0 & 0 & -\nu|\xi|^2 & \frac{1}{\alpha}\xi_3 & 0 & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu|\xi|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\nu|\xi|^2 \end{pmatrix}.$$

We calculate the determinant of the last matrix:

$$\det\left(\widetilde{M - \lambda I}\right)(\xi) = \frac{\nu^3}{\alpha^2}|\xi|^8 \left(\nu\lambda\alpha^2 - 1\right),$$

and thus we can see that for only one point $\lambda = \frac{1}{\nu\alpha^2}$ the operator $L = M - \lambda I$ is not elliptic in sense of Douglis-Nirenberg. Now we will show, additionally, that the conditions of Lopatinski are satisfied.

The boundary condition (2) can be written in a matrix form

$$G\tilde{v}\big|_{\partial\Omega} = 0 \quad , \quad G = \begin{pmatrix} n_1 & n_2 & n_3 & 0 & 0 & 0 \end{pmatrix}.$$

If we denote $\tilde{\xi} = (\xi_1, \xi_2)$ , $\xi_3 = \tau$ ; then

$$\det\left(\widetilde{M - \lambda I}\right)(\tilde{\xi}, \tau) = \frac{\nu^3}{\alpha^2}\left(|\tilde{\xi}|^2 + \tau^2\right)^4 \left(\nu\lambda\alpha^2 - 1\right),$$

and thus the equation $\det\left(\widetilde{M - \lambda I}\right)(\tilde{\xi}, \tau) = 0$ for $\lambda \neq \frac{1}{\nu\alpha^2}$ has one root $\tau = i|\tilde{\xi}|$ of multiplicity four in the upper half of the complex plane.

In this way, $M^+(\tilde{\xi}, \tau) = \left(\tau - i|\tilde{\xi}|\right)^4$. Since the elements of the matrices $\widetilde{M - \lambda I}$ and $G$ are homogeneous functions with respect to $\tilde{\xi}, \tau$, then it is sufficient to verify the Lopatinski conditions for unitary vectors $\tilde{\xi}$. Let us choose a local system of coordinates so that $\xi_1 = 1$, $\xi_2 = 0$.

Then, we have $M^+(\tilde{\xi}, \tau) = (\tau - i)^4$, and the corresponding matrix will have the following form:

$$\left(\widetilde{M - \lambda I}\right)(\tau) =$$

$$\begin{pmatrix} -\nu\left(1+\tau^2\right) & 0 & 0 & \frac{1}{\alpha} & 0 & 0 \\ 0 & -\nu\left(1+\tau^2\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\nu\left(1+\tau^2\right) & \frac{\tau}{\alpha} & 0 & 0 \\ \frac{1}{\alpha} & 0 & \frac{\tau}{\alpha} & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu\left(1+\tau^2\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\nu\left(1+\tau^2\right) \end{pmatrix}$$

For the matrix $\left(\widetilde{M-\lambda I}\right)$ we construct first the adjoint matrix $\left(M-\lambda I\right)^*$ (which is composed of algebraic complements of the original matrix), then we multiply $\left(M-\lambda I\right)^*$ by the boundary conditions matrix $G$ and thus obtain the following.

$$G\left(M-\lambda I\right)^* = \left(n_1 B^3\left[B\lambda + \frac{\tau^2}{\alpha^2}\right], 0, -n_3 B^3 \frac{\tau}{\alpha^2}, 0, 0, 0\right),$$

where $B = -\nu\left(1+\tau^2\right)$.

Since $G\left(M-\lambda I\right)^*$ is a vector row, then, evidently, the Lopatinski conditions are satisfied and thus the theorem is proved.

Theorem 3.

The essential spectrum of the operator $M_1$ is composed of one real point $\sigma_{ess}\left(M\right) = \left\{\frac{1}{\nu\alpha^2\left(\beta+1\right)}\right\}$.

Proof.

We can choose the same sets $s$ , $t$ as in the proof of theorem 2:

$$s_1 = s_2 = s_3 = s_5 = s_6 = 0,\ s_4 = -1,$$
$$t_1 = t_2 = t_3 = t_5 = t_6 = 2,\ t_4 = 1.$$

In this way, the main symbol of the operator $L = M_1 - \lambda I$ will be expressed as:

$$\tilde{L}\left(\xi\right) = \begin{pmatrix} -\nu\left|\xi\right|^2 - \nu\beta\xi_1^2 & -\nu\beta\xi_1\xi_2 & -\nu\beta\xi_1\xi_3 & \frac{1}{\alpha}\xi_1 & 0 & 0 \\ -\nu\beta\xi_1\xi_2 & -\nu\left|\xi\right|^2 - \nu\beta\xi_2^2 & -\nu\beta\xi_2\xi_3 & \frac{1}{\alpha}\xi_2 & 0 & 0 \\ -\nu\beta\xi_1\xi_3 & -\nu\beta\xi_2\xi_3 & -\nu\left|\xi\right|^2 - \nu\beta\xi_3^2 & \frac{1}{\alpha}\xi_3 & 0 & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu\left|\xi\right|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\nu\left|\xi\right|^2 \end{pmatrix}.$$

It can be easily seen that the determinant of the last matrix is the following:

$$\det\left(\widetilde{M_1-\lambda I}\right)\left(\xi\right) = \frac{\nu^4}{\alpha^2}\left|\xi\right|^8\left(\nu\lambda\alpha^2\left(\beta+1\right)-1\right),$$

and thus we can conclude that for only one point $\lambda = \frac{1}{\nu\alpha^2\left(\beta+1\right)}$ the operator $L = M_1 - \lambda I$ is not elliptic in sense of Douglis-Nirenberg. The proof that the conditions of Lopatinski are satisfied is analogous to theorem 2.

Theorem 4.

The spectrum of operators $M, M_1$ is symmetrical with respect to the real axis. All the eigenvalues of operator $M$ are in the following sector of the complex plane: $Z = \left\{\lambda \in C:\ \operatorname{Re}\lambda \ge 0,\ \left|\operatorname{Im}\lambda\right| \le A + \frac{3\operatorname{Re}\lambda}{\nu\alpha^2 A}\right\}$, where

$$A = \max\left\{\omega, \gamma_T, \gamma_S\right\}.$$

For operator $M_1$ , the eigenvalues belong to the sector

$$Z = \left\{\lambda \in C:\ \operatorname{Re}\lambda \ge 0,\ \left|\operatorname{Im}\lambda\right| \le A + \frac{3\operatorname{Re}\lambda}{\nu\alpha^2\beta A}\right\}.$$

Proof.

We consider first the case of the operator $M$.

Let us denote $\begin{cases} \tilde{v} = \left(v_1, v_2, v_3, v_4, v_5, v_6\right) \\ \breve{v} = \left(v_1, v_2, v_3, v_5, v_6\right) \end{cases}$ and

$$K\tilde{v} = \begin{pmatrix} 0 & -\omega & 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_T & -\gamma_S \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_T & 0 & 0 & 0 \\ 0 & 0 & \gamma_S & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}.$$

Then, the system $\left(M-\lambda I\right)\left\{\tilde{v}\right\} = 0$ can be written in the form

$$\begin{cases} -\lambda\breve{v} + K\breve{v} - \nu\Delta\breve{v} + \frac{1}{\alpha}\nabla v_4 = 0 \\ -\lambda v_4 + \frac{1}{\alpha}\operatorname{div}\vec{v} = 0 \end{cases}.$$

Now we multiply the last system by $\overline{\left\{\tilde{v}\right\}}$ and then integrate by parts in $\Omega$. In this way, we obtain the following equations:

$$-\lambda\left\|\breve{v}\right\|^2 + \left(K\breve{v}, \breve{v}\right) + \nu\sum_{k=1,2,3,5,6}\left\|\nabla v_k\right\|^2 - \frac{1}{\alpha}\left(v_4, \operatorname{div}\vec{v}\right) = 0$$
$$-\lambda\left\|v_4\right\|^2 + \frac{1}{\alpha}\left(\operatorname{div}\vec{v}, v_4\right) = 0$$

We sum up these two equations

$$-\lambda\left(\|\tilde{v}\|^2+\|v_4\|^2\right)+\left(K\tilde{v},\tilde{v}\right)+\nu\sum_{k=1,2,3,5,6}\|\nabla v_k\|^2+$$
$$+\frac{1}{\alpha}\left[\left(\operatorname{div}\vec{v},v_4\right)-\left(v_4,\operatorname{div}\vec{v}\right)\right]=0$$

and then separate the real and the imaginary parts, keeping in mind the fact that, since $K$ is skew-symmetric matrix, then the expression $\left(K\tilde{v},\tilde{v}\right)$ is imaginary.

$$\operatorname{Re}\lambda=\frac{\nu\sum\limits_{k=1,2,3,5,6}\|\nabla v_k\|^2}{\|\tilde{v}\|^2+\|v_4\|^2}\geq 0,$$

$$\left|\operatorname{Im}\lambda\right|=-i\frac{\left(K\tilde{v},\tilde{v}\right)+\frac{1}{\alpha}\left[\left(\operatorname{div}\vec{v},v_4\right)-\left(v_4,\operatorname{div}\vec{v}\right)\right]}{\|\tilde{v}\|^2+\|v_4\|^2}.$$

Let us introduce the value $A=\max\left\{\omega,\gamma_T,\gamma_S\right\}$.

In this way, we can draw the following two estimates:

$$\left|\operatorname{Im}\lambda\right|\leq\frac{A\|\tilde{v}\|^2+\frac{2}{\alpha}\|\operatorname{div}\vec{v}\|\|v_4\|}{\|\tilde{v}\|^2+\|v_4\|^2}\leq\frac{A\|\tilde{v}\|^2+A\|v_4\|^2+\frac{\|\operatorname{div}\vec{v}\|^2}{\alpha^2 A}}{\|\tilde{v}\|^2+\|v_4\|^2}.$$

Here we used the inequalities

$$\left(f,g\right)_{L_2}\leq\|f\|_{L_2}\|g\|_{L_2},\qquad 2\frac{a}{\sqrt{A}}b\sqrt{A}\leq\frac{a^2}{A}+b^2A.$$

From the relations

$$\|\operatorname{div}\vec{v}\|^2\leq 3\sum_{k=1}^{3}\|\nabla v_k\|^2\leq 3\sum_{k=1,2,3,5,6}\|\nabla v_k\|^2,$$

$$\frac{\operatorname{Re}\lambda}{\nu}=\frac{\sum\limits_{k=1,2,3,5,6}\|\nabla v_k\|^2}{\|\tilde{v}\|^2+\|v_4\|^2},\qquad\left|\operatorname{Im}\lambda\right|\leq A+\frac{\frac{1}{\alpha^2 A}\|\operatorname{div}\vec{v}\|^2}{\|\tilde{v}\|^2+\|v_4\|^2},$$

we finally obtain

$$\left|\operatorname{Im}\lambda\right|\leq A+\frac{3\operatorname{Re}\lambda}{\nu\alpha^2 A}.$$

For the operator $M_1$ we have the estimates

$$\frac{\operatorname{Re}\lambda}{\nu\alpha^2\beta A}=\frac{\frac{1}{\alpha^2\beta A}\sum\limits_{k=1,2,3,5,6}\|\nabla v_k\|^2+\frac{1}{\alpha^2 A}\|\operatorname{div}\vec{v}\|^2}{\|\tilde{v}\|^2+\|v_4\|^2},$$

$$\left|\operatorname{Im}\lambda\right|\leq A+\frac{3\operatorname{Re}\lambda}{\nu\alpha^2\beta A}.$$

Now, it remains to prove that the spectrum is symmetrical with respect to the real axis. For that purpose, we apply the complex-conjugation to the original system of $M-\lambda I=0$:

$$\begin{cases}-\bar{\lambda}\bar{\tilde{v}}+K\bar{\tilde{v}}-\nu\Delta\bar{\tilde{v}}+\dfrac{1}{\alpha}\nabla\bar{v}_4=0\\ -\bar{\lambda}\bar{v}_4+\dfrac{1}{\alpha}\operatorname{div}\bar{\vec{v}}=0\end{cases},$$

from which we can see that, if $\lambda$ is an eigenvalue of $M$, then $\bar{\lambda}$ is also an eigenvalue of operator $M$. For operator $M_1$ the proof is analogous, there will only appear the additional term of $-\nu\beta\operatorname{div}\bar{\vec{v}}$, and thus the theorem is proved.

## IV. Conclusion

For the inviscid case of compressible rotating fluid, it seems natural to put $\gamma_T=\gamma_S=0$ and not to take into account the functions $T$, $S$. In this way, the system (1) turns into

$$\begin{cases}\dfrac{\partial u_1}{\partial t}-\omega u_2+\dfrac{\partial p}{\partial x_1}=0\\ \dfrac{\partial u_2}{\partial t}+\omega u_1+\dfrac{\partial p}{\partial x_2}=0\\ \dfrac{\partial u_3}{\partial t}+\dfrac{\partial p}{\partial x_3}=0\\ \alpha^2\dfrac{\partial p}{\partial t}+\operatorname{div}\vec{u}=0\end{cases}.$$

For the last system, we proved, for example, in [6]-[8], the essential spectrum of inner vibrations is the interval of the imaginary axis $\left(\left[-i\omega,i\omega\right]\right)$.

Comparing these results with the compressible viscous case, we can conclude that the considered problems and the results of Theorems 2, 3, and 4 are remarkable and interesting due to the special property that, for the viscous fluid, the points of the essential spectrum $\frac{1}{\nu\alpha^2}$, $\frac{1}{\nu\alpha^2\left(\beta+1\right)}$, move to infinity for $\nu\to 0$; while the essential spectrum of the inviscid fluid contains an interval of the imaginary axis. That property is analogous to the fact that the explicit form for the solutions of the inviscid fluid cannot be obtained from the solutions of viscous fluid by merely putting the viscosity parameter equal to zero.

Additionally, as we can see, the previous results obtained for the inviscid fluid $\left(\left[-i\omega,i\omega\right]\right)$, correspond to the statement of Theorem 2 if we put $\operatorname{Re}\lambda=0,\ \gamma_S=\gamma_T=0,\ A=\omega$:

$$\left(\operatorname{Re}\lambda=0,\ \left|\operatorname{Im}\lambda\right|\leq\omega\right)$$

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