Wiener-Hopf Analysis of Sound Waves by a Rigid Cylindrical Pipe with External Impedance Surface

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Abstract**—**In the present work, diffraction of sound waves emanating from a ring source by a rigid cylindrical pipe with external impedance surface, is analyzed. This boundary-value problem is investigated using Wiener-Hopf technique. To obtain a couple of simultaneous Wiener-Hopf equations, Fourier transform is applied. The solution which involves two sets of infinitely many constants satisfying two infinite systems of linear algebraic equations, is found analytically by the application of saddle point technique.

Keywords—Wiener-Hopf, Fourier transform, saddle point technique, absorbent lining, duct acoustics.

I. Introduction

THE diffraction of waves problems have been subject to numerous past investigations due to their use in modelling many actual engineering problems of practical importance [1].

Reduction of noise is an important problem for researchers [2], [3]. Absorbent lining is one of the effective methods of reducing noise from ducts. As a first model of an aircraft engine, diffraction and radiation of sound waves from lined ducts has been extensively studied in the literature.

Rawlins who first considered the radiation of sound from an unflanged rigid cylindrical duct with an acoustically absorbing internal surface [4]. An exact solution including an infinite series was obtained for this mixed boundary value problem. Demir and Buyukaksoy then treated a similar problem now with a partial lining [5]. A hybrid method was applied successfully for the solution. In both studies above, some numerical results were also given graphically.

Absorbent lining has been used successfully to reduce the noise from ducts and it has been analyzed in depth in numerous articles, but those were usually lining from the inner side of the duct. Polat and Buyukaksoy distinctively examined the problem of diffraction of waves from a semi-infinite duct with certain wall thickness and different linings from inside, outside and end side [6].

In the present work, we analyze the effect of external lining of a rigid circular cylindrical pipe on diffraction phenomenon where the sound waves emanate from a ring source. This boundary-value problem related to the diffraction problem is reduced to Wiener-Hopf equation via Fourier transform. A solution is found analytically by solving the Wiener-Hopf equation. The integral encountered in finding the solution is evaluated via saddle point technique.

II. Analysis

A. Formulation of the Problem

A geometry is considered which consist of a semi-infinite duct. Duct walls are assumed to be infinitely thin and they occupy the region $\{r = a, z \in (-\infty, l)\}$ illuminated by a ring source located at $r = b > a, z = 0$ (Figure 1). The inner surface of the semi-infinite cylinder is assumed to be rigid, while the outer surface is assumed to be treated with an acoustically absorbing lining. The liner impedance is denoted by Z. We introduce a scalar potential $\psi(r,z,t)$ which defines the acoustic pressure and velocity by $p = -\rho_0(\partial/\partial t)\psi$ and $\vec{v} = grad\psi$ respectively, where ρ_0 the density of the undisturbed medium. All quantities are made dimensionless by b, ρ_0 and c_0 (speed of sound).

$$r, z \sim b \quad , \quad \rho \sim \rho_0 \quad , \quad t \sim b / c_0 \tag{1}$$

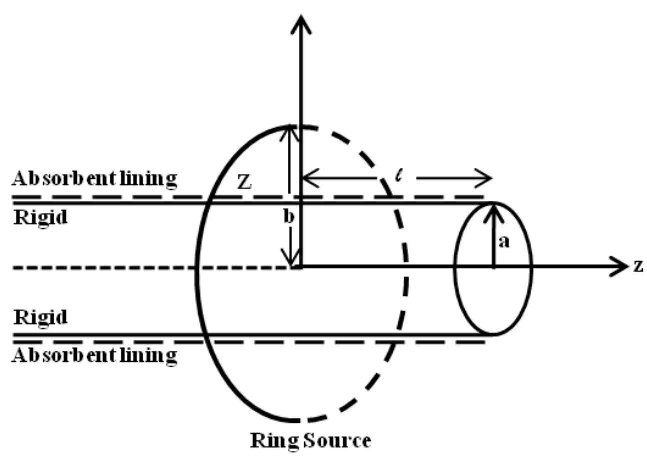

Fig. 1 Geometry of the problem

From the symmetry of the geometry of the problem and of the ring source, the total field will be independent of θ everywhere in circular cylindrical coordinate system (r, θ, z).

For the sake of analytical convenience we will assume that the surrounding medium is slightly lossy and ω has a small negative imaginary part. The lossless case can be obtained by letting $Im\omega \to 0$ at the end of the analysis. For analysis purposes the total diffracted field can be written in different regions as:

$$\psi^T(r,z,t) = \begin{cases} \psi_1(r,z)exp(i\omega t) & , \quad r>1 \\ \psi_2(r,z)exp(i\omega t) & , \quad r\in(h,1) \\ \psi_3(r,z)exp(i\omega t) & , \quad r\in(0,h) \end{cases} \tag{2}$$

where ω is the dimensionless angular frequency (Helmholtz number) and $h = a/b$. Time dependency is taken $e^{i\omega t}$. Also for conformity,

$$J(Z,u) = iJ_0(\lambda\omega h)/Z + \lambda J_1(\lambda\omega h) \tag{3}$$

$$Y(Z,u) = iY_0(\lambda\omega h)/Z + \lambda Y_1(\lambda\omega h) \tag{4}$$

$$H(Z,u) = iH_0^{(2)}(\lambda\omega h)/Z + \lambda H_0^{(2)}(\lambda\omega h) \tag{5}$$

would be appropriate to define.

B. Derivation of the Wiener-Hopf System

The unknown velocity potentials $\psi_1(r,z)$, $\psi_2(r,z)$ and $\psi_3(r,z)$ satisfy the Helmholtz equation for $z\in(-\infty,\infty)$.

$$\left[\frac{1}{r}\frac{\partial}{\partial \mathrm{r}}\left(r\frac{\partial}{\partial \mathrm{r}}\right) + \frac{\partial^2}{\partial \mathrm{z}^2} + \omega^2\right]\psi_{1,2,3}(r,z) = 0 \tag{6}$$

Full range Fourier transform of these equations along z enables us to write solutions as inverse Fourier integrals, such that

$$\psi_1(r,z) = \frac{\omega}{2\pi}\int_L A(u)H_0^{(2)}(\lambda\omega r)e^{-i\omega uz}du \tag{7}$$

$$\psi_2(r,z) = \frac{\omega}{2\pi}\int_L [B(u)J_0(\lambda\omega r) + C(u)Y_0(\lambda\omega r)]e^{-i\omega uz}du \tag{8}$$

$$\psi_3(r,z) = \frac{\omega}{2\pi}\int_L D(u)J_0(\lambda\omega r)e^{-i\omega uz}du \tag{9}$$

where L is a suitable inverse Fourier transform integration contour along or near the real axis in the complex $u-$domain (Figure 2). J_0 and Y_0 are the Bessel and Neumann functions of order zero, $H_0^{(2)} = J_0 - iY_0$ is the Hankel function of the second type. λ is square root function which is defined as

$$\lambda(u) = \sqrt{1-u^2} \tag{10}$$

Branch cuts for λ is taken on the line from 1 to ∞ and from $-\infty$ to -1.

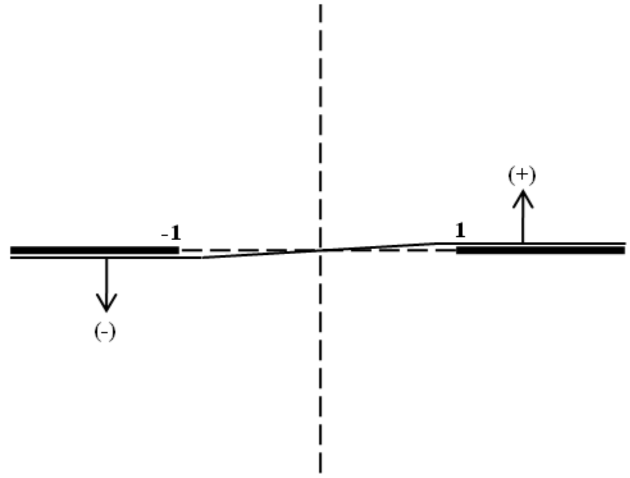

Fig. 2 Complex u - plane with Fourier contour and branch cut

Equations for the unknown spectral coefficients $A(u), B(u), C(u)$ and $D(u)$ will be obtained below from boundary conditions and relations of continuity.

The boundary condition on the absorbent surface can be given in terms of the potential function ψ_2.

$$\frac{\partial}{\partial r}\psi_2(h,z) = \frac{i\omega}{Z}\psi_2(h,z) \quad , \quad z<l \tag{11}$$

Then the inner duct wall is rigid, so that

$$\frac{\partial}{\partial r}\psi_3(h,z) = 0 \quad , \quad z<l \tag{12}$$

Consider now the continuity conditions related to total fields at $r=h, z>l$ which are given by

$$\frac{\partial}{\partial r}\psi_2(h,z) = \frac{\partial}{\partial r}\psi_3(h,z) \quad , \quad z>l \tag{13}$$

$$\psi_2(h,z) = \psi_3(h,z) \quad , \quad z>l \tag{14}$$

By the definition of the ring source given as

$$\frac{\partial}{\partial r}\psi_1(1,z) - \frac{\partial}{\partial r}\psi_2(1,z) = \delta(z-0), z\in(-\infty,\infty) \tag{15}$$

$$\psi_1(1,z) = \psi_2(1,z) \quad , \quad z\in(-\infty,\infty) \tag{16}$$

In addition to these boundary conditions and continuity relations, we assume that the field propagates outward to infinity and does not reflect backward.

To ensure the uniqueness of the mixed boundary-value problem stated by (6) and (11-16), one has to take into account the following edge condition.

$$\psi_{2,3}(h,l) = cons. \tag{17}$$

$$\frac{\partial}{\partial r}\psi_{2,3}(h,l) = \mathcal{O}\ (z^{-1/2}) \tag{18}$$

Applying the boundary conditions on $r = h$

$$\frac{\partial}{\partial r}\psi_2(h,z) = -\frac{\omega}{2\pi}\int_L [B(u)\lambda\omega J_1(\lambda\omega h) + \lambda\omega C(u) Y_1(\lambda\omega h)]e^{-i\omega uz}du \tag{19}$$

$$\frac{\partial}{\partial r}\psi_3(h,z) = -\frac{\omega}{2\pi}\int_L D(u)\lambda\omega J_1(\lambda\omega h)e^{-i\omega uz}du \tag{20}$$

from (11,12) and taking Fourier transforms gives

$$\omega\left[B(u)J(Z,u) + C(u)Y(Z,u)\right] = e^{i\omega ul}\phi_1^+(u) \tag{21}$$

$$-D(u)\lambda\omega J_1(\lambda\omega h) = e^{i\omega ul}\phi_2^+(u) \tag{22}$$

Continuity of pressure at $r = h$ yields

$$-D(u)\lambda\omega J_1(\lambda\omega h) + B(u)\lambda\omega J_1(\lambda\omega h) + C(u)\lambda\omega Y_1(\lambda\omega h) = e^{i\omega ul}\phi_1^-(u) \tag{23}$$

$$D(u)J_0(\lambda\omega h) - B(u)J_0(\lambda\omega h) - C(u)Y_0(\lambda\omega h) = e^{i\omega ul}\phi_2^-(u) \tag{24}$$

where $\phi_{1,2}^+$ and $\phi_{1,2}^-$ are a function analytic at the upper and lower half-plane and defined as

$$\phi_1^+(u) = \int_l^\infty \left[\frac{i\omega}{Z}\psi_2(h,z) - \frac{\partial}{\partial r}\psi_2(h,z)\right]e^{i\omega u(z-l)}dz \tag{25}$$

$$\phi_2^+(u) = \int_l^\infty \frac{\delta}{\delta r}\psi_3(h,z)e^{i\omega u(z-l)}dz \tag{26}$$

$$\phi_1^-(u) = \int_{-\infty}^l \left[\frac{\partial}{\partial r}\psi_3(h,z) - \frac{\partial}{\partial r}\psi_2(h,z)\right]e^{i\omega u(z-l)}dz \tag{27}$$

$$\phi_2^-(u) = \int_{-\infty}^l \left[\psi_3(h,z) - \psi_2(h,z)\right]e^{i\omega u(z-l)}dz \tag{28}$$

The spectral coefficients $A(u), B(u)$ and $C(u)$ are related to each other by the definition of the ring source given in (15, 16), application of the boundary conditions on $r = 1$

$$\frac{\partial}{\partial r}\psi_1(1,z) = -\frac{\omega}{2\pi}\int_L A(u)\lambda\omega H_1^{(2)}(\lambda\omega)e^{-i\omega uz}du \tag{29}$$

$$\frac{\partial}{\partial r}\psi_2(1,z) = -\frac{\omega}{2\pi}\int_L [B(u)\lambda\omega J_1(\lambda\omega) + \lambda\omega C(u)Y_1(\lambda\omega)]e^{-i\omega uz}du \tag{30}$$

the Fourier transform of which provides

$$\lambda\omega A(u)H_1^{(2)}(\lambda\omega) = \lambda\omega B(u)J_1(\lambda\omega) + \lambda\omega C(u)Y_1(\lambda\omega) - 1 \tag{31}$$

$$A(u)H_0^{(2)}(\lambda\omega) = B(u)J_0(\lambda\omega) + C(u)Y_0(\lambda\omega) \tag{32}$$

The elimination of $C(u)$ between (31) and (32) yields

$$B(u) = A(u) + \frac{\pi}{2}Y_0(\lambda\omega) \tag{33}$$

Similarly, we obtain

$$C(u) = -iA(u) - \frac{\pi}{2}J_0(\lambda\omega) \tag{34}$$

From (21) and (22) we find that

$$A(u) = \frac{e^{i\omega ul}\phi_1^+(u)}{\omega H(Z,u)} - \frac{\pi}{2H(Z,u)}[Y_0(\lambda\omega)J(Z,u) - J_0(\lambda\omega)Y(Z,u)] \tag{35}$$

$$D(u) = -\frac{e^{i\omega ul}\phi_2^+(u)}{\lambda\omega J_1(\lambda\omega h)} \tag{36}$$

leading to

$$\begin{aligned} &e^{i\omega ul}\phi_2^+(u) + \frac{\lambda H_1^{(2)}(\lambda\omega h)}{H(Z,u)}e^{i\omega ul}\phi_1^+(u) - \frac{\lambda\omega\pi}{2} \\ &\times\frac{H_1^{(2)}(\lambda\omega h)}{H(Z,u)}\left[Y_0(\lambda\omega)J(Z,u) - J_0(\lambda\omega)Y(Z,u)\right] \\ &+\frac{\lambda\omega\pi}{2}\left[Y_0(\lambda\omega)J_1(\lambda\omega h) - J_0(\lambda\omega)Y_1(\lambda\omega h)\right] \\ &= e^{i\omega ul}\phi_1^-(u) \end{aligned} \tag{37}$$

$$\begin{aligned} &-\frac{J_0(\lambda\omega h)}{\lambda\omega J_1(\lambda\omega h)}e^{i\omega ul}\phi_2^+(u) - \frac{H_0^{(2)}(\lambda\omega h)}{\omega H(Z,u)}e^{i\omega ul}\phi_1^+(u) \\ &+\frac{\pi}{2}\frac{H_0^{(2)}(\lambda\omega h)}{H(Z,u)}\left[Y_0(\lambda\omega)J(Z,u) - J_0(\lambda\omega)Y(Z,u)\right] \\ &+\frac{\pi}{2}\left[J_0(\lambda\omega)Y_0(\lambda\omega h) - Y_0(\lambda\omega)J_0(\lambda\omega h)\right] \\ &= e^{i\omega ul}\phi_2^-(u) \end{aligned} \tag{38}$$

and note the following Wronskian-type relations

$$\begin{aligned} &-\frac{\lambda\omega\pi}{2}\frac{H_1^{(2)}(\lambda\omega h)}{H(Z,u)}[Y_0(\lambda\omega)J(Z,u) \\ &- J_0(\lambda\omega)Y(Z,u)] + \frac{\lambda\omega\pi}{2}[Y_0(\lambda\omega)J_1(\lambda\omega h) \\ &- J_0(\lambda\omega)Y_1(\lambda\omega h)] = \frac{i}{hZ}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)} \end{aligned} \tag{39}$$

$$\frac{\pi}{2}\frac{H_0^{(2)}(\lambda\omega h)}{H(Z,u)}\left[Y_0(\lambda\omega)J(Z,u)-J_0(\lambda\omega)Y(Z,u)\right] + \frac{\pi}{2}\left[J_0(\lambda\omega)Y_0(\lambda\omega h)-Y_0(\lambda\omega)J_0(\lambda\omega h)\right] = \frac{1}{\omega h}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)} \tag{40}$$

Then equation (37) and (38) can be written as follows

$$\phi_2^+(u)+\frac{\lambda H_1^{(2)}(\lambda\omega h)}{H(Z,u)}\phi_1^+(u) + \frac{ie^{-i\omega ul}}{hZ}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)}=\phi_1^-(u) \tag{41}$$

$$-\frac{J_0(\lambda\omega h)}{\lambda\omega J_1(\lambda\omega h)}\phi_2^+(u)-\frac{H_0^{(2)}(\lambda\omega h)}{\omega H(Z,u)}\phi_1^+(u) + \frac{e^{-i\omega ul}}{\omega h}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)}=\phi_2^-(u) \tag{42}$$

$H_0^{(2)}(\lambda\omega)/H(Z,u)$ can be eliminated from equation (41) and (42), leading to

$$\phi_1^+(u)+L(u)\phi_2^+(u)=\left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right] \tag{43}$$

By eliminating $\phi_2^+(u)$ we find finally from equation (42) and (43)

$$M(u)\phi_1^+(u)+\frac{e^{-i\omega ul}}{h}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)} = \left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right]\frac{J_0(\lambda\omega h)}{J(Z,u)}+\omega\phi_2^-(u) \tag{44}$$

These are the two coupled Wiener-Hopf equations to be solved, $L(u)$ and $M(u)$ are kernel functions which have to be factorized. They are defined by

$$L(u)=\frac{J(Z,u)}{\lambda J_1(\lambda\omega h)} \tag{45}$$

$$M(u)=\frac{J_0(\lambda\omega h)}{J(Z,u)}-\frac{H_0^{(2)}(\lambda\omega h)}{H(Z,u)} \tag{46}$$

C. Solution of the Wiener-Hopf Equation

Consider the first equation in (43) and rearrange it in the following form

$$\frac{\lambda J_1(\lambda\omega h)}{J(Z,u)}L_+(u)\phi_1^+(u)+L_+(u)\phi_2^+(u) = \frac{1}{L_-(u)}\left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right] \tag{47}$$

Here, $L_+(u)$ and $L_-(u)$ are the split functions regular and free of zeros in the upper $(Imu>0 \ or \ Imu=0 \ and \ Reu<0)$ and lower $(Imu<0 \ or \ Imu=0 \ and \ Reu>0)$ half planes, respectively, resulting from the Wiener-Hopf factorization of $L(u)$

$$L(u)=L_+(u)L_-(u) \tag{48}$$

Now, the right hand side is a function analytic in the lower half of the complex u-plane and the left hand side is a function analytic in the upper half of the complex u-plane, except for the zeros of the function $J(Z,u)$.

If the infinite sum of poles is subtracted from both sides of the equation we have

$$\frac{\lambda J_1(\lambda\omega h)}{J(Z,u)}L_+(u)\phi_1^+(u)-\sum_{p=1}^{\infty}\frac{d_p^+}{u-\xi_p^-}+L_+(u)\phi_2^+(u) = \frac{1}{L_-(u)}\left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right]-\sum_{p=1}^{\infty}\frac{d_p^+}{u-\xi_p^-} \tag{49}$$

where

$$d_p^+=L_+(\xi_p^-)\phi_1^+(\xi_p^-)\lim_{u\to\xi_p^-}\frac{\lambda J_1(\lambda\omega h)}{\frac{d}{du}J(Z,u)} \tag{50}$$

In this expression $L_+(u)$ and $L_-(u)$ stand for the split functions regular and free of zeros in the upper and lower half-planes, respectively. The explicit expressions of the split function $L_+(u)$ is given [7]

$$L_+(u)=\left[\frac{iJ_0(\omega h)/Z+J_1(\omega h)}{J_1(\omega h)}\right]^{1/2} \times\prod_{p=1}^{\infty}\frac{\left(1+u/\xi_p^-\right)e^{-u/\xi_p^-}}{\left(1+u/\chi_p^-\right)e^{-u/\chi_p^-}} \tag{51}$$

$$L_-(u)=L_+(-u) \tag{52}$$

and χ_p^- and ξ_p^- are the roots of the following equations:

$$\sqrt{1-(\chi_p^-)^2}J_1\left(\sqrt{1-(\chi_p^-)^2}\omega h\right)=0 \tag{53}$$

$$\frac{i}{Z}J_0\left(\sqrt{1-(\xi_p^-)^2}\omega h\right) +\sqrt{1-(\xi_p^-)^2}J_1\left(\sqrt{1-(\xi_p^-)^2}\omega h\right)=0 \tag{54}$$

From analytic continuation principle and Liouville's theorem it follows that the left and right hand sides define the same analytic function. Anticipating smooth enough behaviour at $z=0$ this function is zero, leading to

$$\left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right]=L_-(u)\sum_{p=1}^{\infty}\frac{d_p^+}{u-\xi_p^-} \tag{55}$$

and

$$\frac{\lambda J_1(\lambda\omega h)}{J(Z,u)}L_+(u)\phi_1^+(u)=-L_+(u)\phi_2^+(u)+\sum_{p=1}^{\infty}\frac{d_p^+}{u-\xi_p^-} \tag{56}$$

Next we consider equation (44). By using the classical factorization and decomposition procedure, so that we arrive at the equation

$$\begin{aligned}M_+(u)\phi_1^+(u)&=\left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right]\frac{J_0(\lambda\omega h)}{J(Z,u)}M_-(u)\\&\quad+\omega\phi_2^-(u)M_-(u)-\frac{e^{-i\omega ul}}{h}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)}M_-(u)\end{aligned} \tag{57}$$

the split functions $M_+(u)$ and $M_-(u)$, result from the factorization of $M(u)$ (Appendix A), regular and free of zeros in the upper and lower half planes, respectively, as follows

$$M(u)=\frac{M_+(u)}{M_-(u)} \tag{58}$$

The regularity of the right hand side is violated by the zeros on the lower half plane, namely at ξ_p^+. Subtraction of the infinite sum of poles from both sides, yields

$$\begin{aligned}&M_+(u)\phi_1^+(u)-Q_+(u)-\sum_{p=1}^{\infty}\frac{d_p^-}{u-\xi_p^+}\\&=\left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right]\frac{J_0(\lambda\omega h)}{J(Z,u)}M_-(u)\\&\quad-\sum_{p=1}^{\infty}\frac{d_p^-}{u-\xi_p^+}+\omega\phi_2^-(u)M_-(u)+Q_-(u)\end{aligned} \tag{59}$$

The solution to the second Wiener-Hopf equation, using the analytic continuation principle and Liouville's theorem, is then eventually:

$$M_+(u)\phi_1^+(u)=Q_+(u)+\sum_{p=1}^{\infty}\frac{d_p^-}{u-\xi_p^+} \tag{60}$$

and

$$\begin{aligned}\omega\phi_2^-(u)M_-(u)&=-\left[\phi_1^-(u)-\frac{i\omega}{Z}\phi_2^-(u)\right]\\&\quad\times\frac{J_0(\lambda\omega h)}{J(Z,u)}M_-(u)-Q_-(u)+\sum_{p=1}^{\infty}\frac{d_p^-}{u-\xi_p^+}\end{aligned} \tag{61}$$

where

$$d_p^-=M_-(\xi_p^+)\left[\phi_1^-(\xi_p^+)-\frac{i\omega}{Z}\phi_2^-(\xi_p^+)\right]\lim_{u\to\xi_p^+}\frac{J_0(\lambda\omega h)}{\frac{d}{du}J(Z,u)} \tag{62}$$

$$Q(u)=-\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)}e^{-i\omega ul}M_-(u)=Q_+(u)+Q_-(u) \tag{63}$$

and $Q_+(u)$, $Q_-(u)$ stand for the split functions regular and free of zeros in the upper and lower half-planes, respectively. Their explicit forms are obtained in a similar manner as is done in (Appendix B).

D. Determining the Coefficient d_p^+ and d_p^-

The solution is not yet known until we have found d_p^+ and d_p^- coefficients. We can set up an infinite system of linear equations by evaluating the related equations at the respective values of u.

From expression (55) and (60) for $u=\xi_r^+$ and $u=\xi_r^-$ gives

$$\left[\phi_1^-(\xi_r^+)-\frac{i\omega}{Z}\phi_2^-(\xi_r^+)\right]=L_-(\xi_r^+)\sum_{p=1}^{\infty}\frac{d_p^+}{\xi_r^+-\xi_p^-} \tag{64}$$

$$M_+(\xi_r^-)\phi_1^+(\xi_r^-)=Q_+(\xi_r^-)+\sum_{p=1}^{\infty}\frac{d_p^-}{\xi_r^--\xi_p^+} \tag{65}$$

and from equations (64) and (65), we obtain for $r=1,2,\dots$

$$\frac{1}{M_-(\xi_r^+)\lim\limits_{u\to\xi_r^+}\dfrac{J_0(\lambda\omega h)}{\frac{d}{du}J(Z,u)}}d_r^-=L_-(\xi_r^+)\sum_{p=1}^{\infty}\frac{d_p^+}{\xi_r^+-\xi_p^-} \tag{66}$$

$$\frac{M_+(\xi_r^-)}{L_+(\xi_r^-)\lim\limits_{u\to\xi_r^-}\dfrac{\lambda J_1(\lambda\omega h)}{\frac{d}{du}J(Z,u)}}d_r^+=Q_+(\xi_r^-)+\sum_{p=1}^{\infty}\frac{d_p^-}{\xi_r^--\xi_p^+} \tag{67}$$

where

$$\begin{aligned}\frac{d}{du}J(Z,u)&=iu\omega h\frac{J_1(\lambda\omega h)}{\lambda Z}-u\omega hJ_0(\lambda\omega h)\\&=iu\omega h\left[iJ_0(\lambda\omega h)+\frac{1}{\lambda Z}J_1(\lambda\omega h)\right]\end{aligned} \tag{68}$$

$$\frac{\frac{d}{du}J(Z,\xi_p^-)}{\lambda(\xi_p^-)J_1(\lambda(\xi_p^-)\omega h)} = i\xi_p^-\omega h\left[\frac{1}{Z\lambda^2(\xi_p^-)} - Z\right] \quad (69)$$

$$\frac{\frac{d}{du}J(Z,\xi_p^+)}{J_0(\lambda(\xi_p^+)\omega h)} = -\xi_p^+\omega h\left[1 - \frac{1}{Z^2\lambda^2(\xi_p^+)}\right] \quad (70)$$

The infinite sums in the equations converge very rapidly so they can be truncated quickly.

III. Far Field

From (7) and (35), the total field can be expressed as the following integral

$$\psi_1(r,z) = \frac{\omega}{2\pi}\int_L \left[\frac{e^{i\omega ul}\phi_1^+(u)}{\omega H(Z,u)} - \frac{\pi}{2}\frac{Y_0(\lambda\omega)J(Z,u) - J_0(\lambda\omega)Y(Z,u)}{H(Z,u)}\right] \times H_0^{(2)}(\lambda\omega r)e^{-i\omega uz}du \quad (71)$$

where L is the inverse Fourier transform contour.

For the far field, while $\omega r \gg 1$, we can use the asymptotic formula for the Hankel function

$$H_0^{(2)}(\lambda\omega r) \sim \sqrt{\frac{2}{\pi\lambda\omega r}}e^{-i(\lambda\omega r - \pi/4)} \quad (72)$$

and replace $H_0^{(2)}(\lambda\omega r)$ in (71) to get

$$\psi_1(r,z) \sim \frac{\omega}{2\pi}\int_L \left[\frac{e^{i\omega ul}\phi_1^+(u)}{\omega H(Z,u)} - \frac{\pi}{2}\frac{Y_0(\lambda\omega)J(Z,u) - J_0(\lambda\omega)Y(Z,u)}{H(Z,u)}\right] \times \sqrt{\frac{2}{\pi\lambda\omega r}}e^{-i(\lambda\omega r - \pi/4)}e^{-i\omega uz}du \quad (73)$$

After transforming the free variables

$$r = R_1 sin\theta_1 \ , \ z - l = R_1 cos\theta_1 \quad (74)$$

and

$$r = R_2 sin\theta_2 \ , \ z = R_2 cos\theta_2 \quad (75)$$

we can evaluate the integral by the method of saddle point technique [8]

This means that the major contribution to the integral comes from the vicinity of "saddle point" τ_s. This point can be determined via the function $q(\tau)$

$$q(\tau) = -icosh\tau \ , \ q'(\tau) = -isinh\tau \ , \ q'(0) = 0 \ , \ \tau_s = 0 \quad (75)$$

where $q(\tau)$ is an analytic function. Hence the diffracted wave can be approximated by

$$\psi_1(r,z) \sim \frac{i}{\pi}\frac{\phi_1^+(\cos\theta_1)}{H(Z,\cos\theta_1)}\frac{exp(-i\omega R_1)}{\omega R_1} - \frac{i\omega}{2H(Z,\cos\theta_2)}\left[Y_0(\sin\theta_2\omega)J(Z,\cos\theta_2) - J_0(\sin\theta_2\omega)Y(Z,\cos\theta_2)\right]\frac{exp(-i\omega R_2)}{\omega R_2} \quad (76)$$

IV. Conclusion

In this article, diffraction of sound waves emanating from a ring source by a rigid cylindrical pipe with external impedance surface has been investigated by using the Wiener-Hopf technique. The problem was modeled two-dimensional due to symmetry of the geometry and then reduced to Wiener-Hopf equation by application of the Fourier transform, boundary and continuity condition. A solution was found analytically for this problem by solving the Wiener-Hopf equation.

In our solution we also present the factorization and decomposition of the functions $L(u)$, $M(u)$ and $Q(u)$ which are usually required in problems concerning diffraction by semi-infinite cylindrical impedance pipes.

In forthcoming paper, we shall present the effect of absorbent lining on the diffracted field with graphics.

Appendix A

The complex function $M(u)$ has poles and zeros in the complex plane, in particular also along the real axis. We need to evaluate $M(u)$, written as a quotient of two function that are analytic in the upper and lower half plane, along the real axis.

$$M(u) = \frac{J_0(\lambda\omega h)}{J(Z,u)} - \frac{H_0^{(2)}(\lambda\omega h)}{H(Z,u)} = \frac{M_+(u)}{M_-(u)} \quad (77)$$

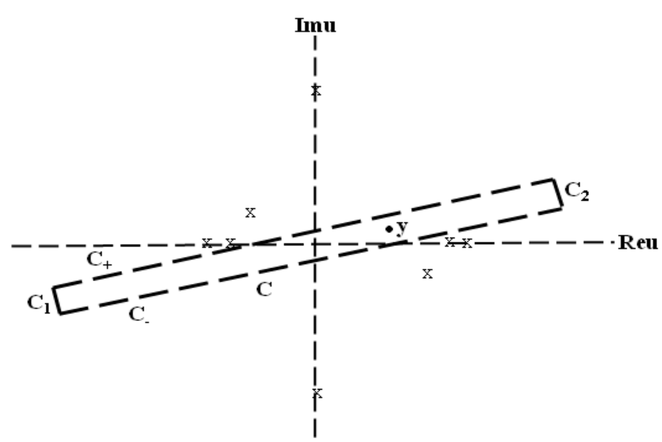

Fig. 3 Integration contour of $M(u)$ in the complex u - plane

Choose a point y within $C = C_+ \cup C_-$. If C is taken in positive direction, then according to Cauchy,

$$lnM(y) = \frac{1}{2\pi i}\oint_C \frac{lnM(u)}{u-y}du \tag{78}$$

letting the ends of C go to infinity simultaneously, the contributions of the end parts will vanish. Hence we can write

$$lnM(y) = \frac{1}{2\pi i}P\oint_{C_-} \frac{lnM(u)}{u-y}du - \frac{1}{2\pi i}P\oint_{C_+} \frac{lnM(u)}{u-y}du \tag{79}$$

where P indicates the Cauchy principal value of the integral and C_+ and C_- denote the upper and lower part of C respectively. The desired splitting can now be achieved by defining

$$M_+(y) = exp\left[\frac{1}{2\pi i}P\int_{C_-} \frac{lnM(u)}{u-y}du\right] \tag{80}$$

$$M_-(y) = exp\left[\frac{1}{2\pi i}P\int_{C_+} \frac{lnM(u)}{u-y}du\right] \tag{81}$$

To avoid the singularities at the branch cuts we will deform the integration contour slightly [9].

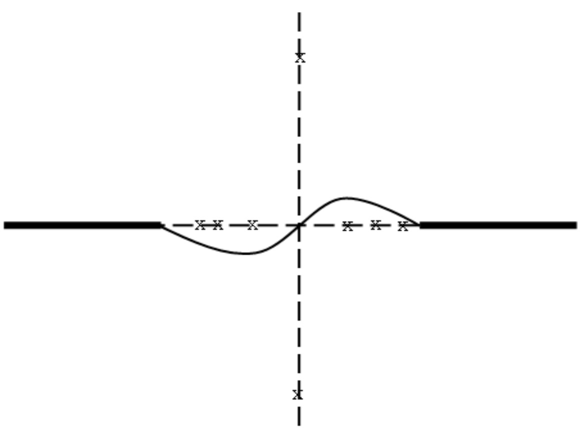

Fig. 4 Sketch of deformed integration contour in complex u - plane

We will use the deformed contour parametrized by

$$u = \zeta(t)\ ,\zeta(t) = t + id\frac{4t/q}{3+(t/q)^4}\ ,\ 0 \le t < \infty \tag{82}$$

$$t = \eta(s)\ ,\eta(s) = \frac{s}{(1-s)^2}\ ,\ 0 \le s \le 1 \tag{83}$$

The parameters d and q determine the height and width of the contour respectively and should not be chosen arbitrarily. Using (84) we find

$$\frac{1}{2\pi i}P\int_{C_+} \frac{lnM(u)}{u-y}du = \int_0^1 h(s,y)\zeta'(\eta(s))\eta'(s)ds \tag{84}$$

where

$$h(s,y) = f(\zeta(\eta(s)),y) + f(-\zeta(\eta(s)),y) \tag{85}$$

$$f(u,y) = \frac{1}{2\pi i}\frac{lnM(u)}{u-y} \tag{86}$$

If the function M_s defined by

$$M_s(y) = exp\left[\int_0^1 h(s,y)\zeta'(\eta(s))\eta'(s)ds\right] \tag{87}$$

can be used to calculate both M_+ and M_-

$$M_+(y) = \begin{cases} M_s & Imy > 0\ or\ Imy = 0\ \&\ Rey < 0 \\ M_s M & else \end{cases} \tag{88}$$

$$M_-(y) = \begin{cases} M_s & Imy < 0\ or\ Imy = 0\quad Rey \& 0 \\ M_s / M & else \end{cases} \tag{89}$$

Appendix B

The function $Q(u)$ is given by

$$Q(u) = -\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)}{H(Z,u)}e^{-i\omega ul}M_-(u) = Q_+(u) + Q_-(u) \tag{90}$$

Applying the Cauchy theorem

$$Q_+(u) = -\frac{1}{2\pi i}\int_{L_+}\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)e^{-i\omega\tau l}M_-(\tau)}{H(Z,\tau)(\tau-u)}d\tau \tag{91}$$

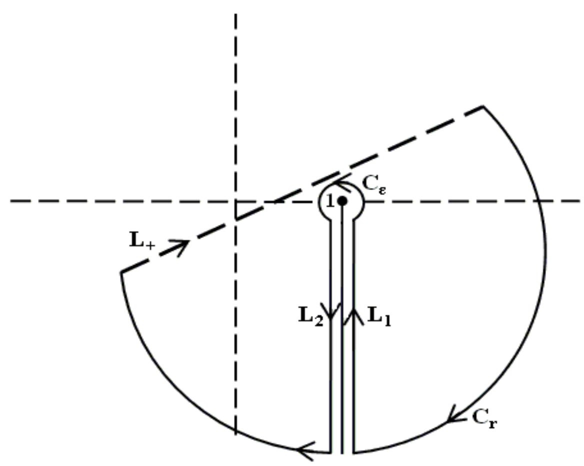

Fig. 5 Integration contour around the branch cut

$$\begin{aligned}
&-1+\tau = te^{-i\pi/2} \quad, L_1: \text{rhs of branch cut}\\
&-1+\tau = te^{i3\pi/2} \quad, L_2: \text{lhs of branch cut}\\
&-1+\tau = \varepsilon e^{i\theta} \quad, C_\varepsilon: \text{around the branch point}\\
&-1+\tau = re^{i\theta} \quad, C_r: \text{on the semi circle}
\end{aligned} \tag{92}$$

$Q_+(u)$ have branch point at $\tau = 1$. We can write the integrals in the following form

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{L_+} f(\tau)d\tau + \frac{1}{2\pi i}\int_{C_r} f(\tau)d\tau + \frac{1}{2\pi i}\int_{L_1} f(\tau)d\tau\\
&\quad + \frac{1}{2\pi i}\int_{C_\varepsilon} f(\tau)d\tau + \frac{1}{2\pi i}\int_{L_2} f(\tau)d\tau = 0
\end{aligned} \tag{93}$$

where

$$f(\tau) = -\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)e^{-i\omega\tau l}M_-(\tau)}{H(Z,\tau)(\tau-u)} \tag{94}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C_r} f(1+re^{i\theta})ire^{i\theta}d\theta \to 0, (r\to\infty)\\
&\frac{1}{2\pi i}\int_{C_\varepsilon} f(1+\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta \to 0, (\varepsilon\to 0)
\end{aligned} \tag{95}$$

after rearranging

$$\frac{1}{2\pi i}\left[\int_{L_+} f(\tau)d\tau + \int_{L_1} f(\tau)d\tau + \int_{L_2} f(\tau)d\tau\right] = 0 \tag{96}$$

and

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_{L_2}\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)e^{-i\omega\tau l}M_-(\tau)}{H(Z,\tau)(\tau-u)}d\tau\\
&\quad = -\frac{1}{2\pi i}\int_0^\infty\left[\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)+2J_0(\lambda\omega)}{H(Z,1-it)+2J(Z,1-it)}\right.\\
&\qquad\left.\times\frac{e^{-i\omega(1-it)l}M_-(1-it)}{1-it-u}(-idt)\right]
\end{aligned} \tag{97}$$

with

$$\begin{aligned}
&H_0^{(2)}(-\lambda\omega h) = H_0^{(2)}(\lambda\omega h) + 2J_0(\lambda\omega h)\\
&(-\lambda)H_1^{(2)}(-\lambda\omega h) = \lambda H_1^{(2)}(\lambda\omega h) + 2\lambda J_1(\lambda\omega h)
\end{aligned} \tag{98}$$

after these operations, we find

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{L_1}\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)e^{-i\omega\tau l}M_-(\tau)}{H(Z,\tau)(\tau-u)}d\tau\\
&\quad + \frac{1}{2\pi i}\int_{L_2}\frac{1}{h}\frac{H_0^{(2)}(\lambda\omega)e^{-i\omega\tau l}M_-(\tau)}{H(Z,\tau)(\tau-u)}d\tau\\
&\quad = -\frac{1}{2\pi i}\int_0^\infty\frac{g(t)}{(1-it-u)}dt
\end{aligned} \tag{99}$$

where

$$\begin{aligned}
g(t) = &-\frac{2i}{h}e^{-i\omega(1-it)l}M_-(1-it)\\
&\times\frac{J_0(\lambda\omega)H(Z,1-it) - H_0^{(2)}(\lambda\omega)J(Z,1-it)}{H(Z,1-it)(H(Z,1-it)+2J(Z,1-it))}
\end{aligned} \tag{100}$$

We obtain half plane analytic function:

$$Q_+(u) = \frac{1}{2\pi i}\int_0^\infty\frac{g(t)}{(1-it-u)}dt \tag{101}$$

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